A Method of Determining the Mass of the Muon's Neutrino

Summary

A proposal for measuring the mass of the muon's neutrino ($m$) is described. It consists of measuring the dynamics of pion decays in flight. From measurements of the momenta of the pion and the muon ($p_\pi$ and $p_\mu$) and the decay angle ($\theta$) the mass of the neutrino can be derived. Suitable conditions are with a pion momentum of between 1 and 2 GeV/c and a decay angle $\theta$ up to 4 mrad.

It is proposed to measure the momenta and angles by means of spark chambers and bending magnets. A simple system of this type is described; this is limited in accuracy by Coulomb scattering in the spark chambers, but by the introduction of focussing magnets and quadrupoles the scattering errors can be virtually eliminated.

The quantity determined is $m^2$, and an improvement of a factor of about 50 over present knowledge should be obtainable with the simple system. This corresponds to a minimum detectable $m$ of 0.35 MeV. With the focussing system a further improvement should be obtainable; the limit to the accuracy would be set by the performance of the bending magnets and quadrupoles.

Introduction

Evidence is now very strong that the neutrino associated with the muon ($\nu_\mu$) is not the neutrino of $\beta$ decay. This has emphasised that at present our knowledge as to the mass of $\nu_\mu$ is only that it is less than 2.5 MeV. There is no firm prediction as to what the mass might be, but a non-zero value would certainly be interesting from the point of view of weak interaction theory.

The requirement for measuring the mass of a particle such as $\nu_\mu$ is to compare sufficiently accurately the total energy $U$ with the momentum $P$, whence
\[ m^2 = U^2 - p^2. \] The comparison enters directly in measurements based on kinematics; via phase space in lifetime or energy spectrum calculations; and in the wave functions in polarisation predictions. In most reactions involving \( \mu \nu \), the upper limit of 2.5 MeV for the neutrino's mass (denoted from now on by \( m \)) is much less than the momentum. In such cases we can expand the expression for \( U \) in terms of \( \frac{m}{p} \), giving \[ \frac{U - p}{p} = \frac{1}{2} \left( \frac{m^2}{p^2} \right), \] showing that the effect of a possible non-zero mass is small and that \( m^2 \) rather than \( m \) is the significant quantity. This result can be illustrated by considering the pion decay at rest. We have \[ U = m - \mu, \quad P = \mu. \] Therefore \[ m^2 = m^2 - 2m \mu + m\mu^2. \]

An uncertainty in \( m_\pi \) of \( \delta m_\pi \) is equivalent to a change in \( m^2 \) of \( 2(m_\pi - \mu) \).

\[ \delta m_\pi = 2U \delta m_\pi. \quad U \mu \] is about 30 MeV, so a \( \delta m_\pi \) of 0.1 MeV is equivalent to \( m^2 = 6 \text{ MeV}^2 \), \( m = 2.4 \text{ MeV} \).

It appears that for an accurate measurement of \( m \) we should look for processes in which low energy neutrinos are produced. One way to achieve this is to study the \( \mu \rightarrow e + \nu_e + \nu \) decay near the end point of the electron energy spectrum. Another way, which forms the basis of the present method, is to study the \( \pi \rightarrow \mu + \nu \) decay using high energy pions and to select those events in which the neutrino is emitted backwards in the c.m.s. system and consequently has only a low energy in the laboratory.

The experiment requires accurate measurements of the pion momentum, the muon momentum and the decay angle \( \theta \) when a relativistic pion decays in flight. In its simplest form, the 'non-focussing' arrangement Fig. 1a, the measurements are made using two bending magnets with a system of spark chambers to define the
tracks. The decays of interest are those for which the neutrino is emitted almost exactly backwards in the c.m.s., about 1 in $10^3$ of all pion decays being useful. This requirement also implies that the two momenta are similar and the decay angle small. A convenient pion momentum is in the range $1 - 2$ GeV/c. With the technique suggested it should be possible to measure the relative momentum of two particles to 1 part in $10^3$ of the particle momentum over a small range.

Suppose we assume again $m = 2.5$ MeV. In the c.m.s. the neutrino has a velocity given by $\gamma = 12 \left( \frac{1}{1 - \beta^2} \right)$. Thus for a pion of $\gamma = 12$, 1,680 MeV/c, such a neutrino can be at rest in the laboratory. Consequently $P_\mu = P_\pi$. Comparing this with a neutrino of $m = 0$, we would have $P_\mu \approx 1681.25$ MeV/c ($U_\mu = 1P_\mu/l = 1.25$ MeV). A 2.5 MeV mass for $m$ reduces $P_\mu$ by 1 part in 1400. This result turns out to be independent of $P_\pi$ as long as $P_\pi \gg m_\pi$.

The following points are vital in the experiment. Firstly, the neutrino mass is derived for each event in terms of the difference $P_\mu - P_\pi$, a difference never greater than a few MeV. Systematic errors in measuring this difference are virtually eliminated by calibrating the system using pions which do not decay. Secondly, as we are studying a two body decay we are concerned only with the mean value of the momentum difference $P_\mu - P_\pi$. Thirdly, the effect of uncertainty in $m_\pi$ can be made negligible. This is because for $P_\pi \gg m_\pi, m_\pi$ itself is becoming less significant. This consideration is one of the main factors in suggesting that $P_\pi$ be in the GeV/c range.

**Kinematics**

We look first at the relationship between $P_\mu$ and $P_\pi$ for zero angle $\theta$. 


and zero $m$, then at the change in $P_{\mu}$ for small $\Theta$ and $m$,

$$
(P_{\mu} - P_\pi)_{\Theta = 0, m = 0} = \frac{m_{\pi}^2 - m_{\mu}^2}{2(U_{\pi} + P_{\pi})},
$$

$$
(\delta P_{\mu})_{\Theta, m} \approx \frac{-U_{\mu} \left( m_{\pi}^2 + P_\pi P_{\mu} \Theta^2 \right)}{m_{\mu}^2 - m_{\pi}^2}, \text{valid for } \Theta, \frac{m^2}{(m_{\pi} - m_{\mu})^2} \ll 1.
$$

So

$$
(P_{\mu} - P_\pi)_{\Theta, m} = \frac{m_{\pi}^2 - m_{\mu}^2}{2(U_{\pi} + P_{\pi})} - \frac{U_{\mu}}{m_{\mu}^2 - m_{\pi}^2} (m^2 + P_\pi P_{\mu} \Theta^2) \quad (1)
$$

Equation (1) is demonstrated in fig. 2 for $P_\pi = 1,000$ MeV/c, where it is considered as generating a family of parabolas of $(P_{\mu} - P_\pi)$ with $\Theta$, $m$ being a parameter. For the small $\Theta$ and $m$ involved, the shape of the parabolas is independent of $m$, so that a non-zero $m$ produces a displacement proportional to $m^2$. We therefore have to measure the position of the parabola along the $P_{\mu} - P_\pi$ axis. In practice this would probably mean regarding each event as giving a value for $(P_{\mu} - P_\pi)_\Theta = 0$, so that we would finally find an accurate mean value for this quantity. From this point of view the calibration events (pions that do not decay in the system) give accurately the position of the origin $(P_{\mu} - P_\pi) = 0$, $\Theta = 0$.

Figure 2 also illustrates the relative importance of the errors in measuring momentum and angle in determining $(P_{\mu} - P_\pi)_\Theta = 0$, showing that only decays up to a certain decay angle are useful. The absolute values of the errors drawn are those set by Coulomb scattering in the 'non-focusing' system.

An error $\delta m_\pi$ in $m_\pi$ will shift the position of the parabola by $\frac{m_{\pi}}{U_{\pi} + P_{\pi}} \delta m_\pi$, and will give rise to systematic error in $m^2$ equal to $\frac{m_{\pi} (m_{\pi}^2 - m_{\mu}^2)}{2P_{\pi}^2} \delta m_\pi$. This is down on the 'pion decay at rest' by a factor
so that for instance at 1 Gev/c, $\delta m = 0.1 \text{ MeV}$ is kinematically equivalent to $m = 0.24 \text{ MeV}$, at 2 Gev/c to 0.12 MeV and so on.

Measurements of momenta and angle.

Two possible systems are discussed for making the measurements of momenta and decay angle. The first of these, the 'non-focussing' system, uses standard components and one can be fairly confident about estimating its performance, this being limited by multiple scattering in the spark chambers. The second, the 'focussing' system, avoids this limitation. While several aspects of this approach have been investigated and are so far all promising, specialist work on the focussing magnets and quadrupoles is needed before a final evaluation is possible.

The 'non-focussing' system is shown in fig. 1a. It consists of two similar 2m. bending magnets having rectangular pole faces and 8 single or double gap spark chambers. In a suitable event, a pion crosses chambers 1 to 4, decays between 4 and 5, and the muon crosses chambers 5 to 8. Chambers 1 to 4 define the particle trajectory, and in particular its angle, before and after the first bending magnet; a detailed knowledge of the field in this magnet then allows $P_{\pi}$ to be found. Similarly $P_{\mu}$ is found using the second bending magnet and chambers 5 to 8. The decay angle $\theta$ is measured using chambers 3 to 6.

Vacuum piping must be used throughout.

It seems likely that the accuracy of this system, both for the momentum and the angle measurement, is set by multiple coulomb scattering in the spark chambers. A double gap spark chamber unit consisting of 25$\mu$ aluminium foils, two 50$\mu$ melinex (mylar) windows, and a few cms. of a Neon-Helium gas has a total thickness
of \(1.1 \times 10^{-3}\) radiation lengths. This leads to an r.m.s. scattering angle at 1 GeV/c of 0.7 mrad, and a projected r.m.s. angle \(\theta_p\) of 0.5 mrad. If the angle of deflection through the bending magnet is \(\alpha\), the limiting accuracy to which this angle can be measured is \(\frac{\theta_p \sqrt{2}}{\alpha}\). This assumes that the uncertainty in determining the position of the track in a chamber does not contribute to the uncertainty in angle; a position scatter of \(\frac{5}{2}\) mm then requires a separation between pairs of chambers such as 1 and 2 to be at least 2 m. The angle \(\alpha\) is approximately 1 radian, so that scattering leads to an error in measuring momentum of 7 parts in \(10^4\), and consequently to an error in \(P\mu - P_\tau\) of 1 part in \(10^3\). It is interesting to note that this result is approximately independent of momentum as long as the bending magnet field is fixed.

There are two approaches to the problem of calibrating the bending magnets to determine the momentum. The first relies solely on the accurate measurement of the field at many points and makes at least relative integrations over the range of input trajectories and momenta in the beam. The second approach would exploit the calibration events in which single particles traverse the entire system to calibrate one magnet against the other. A probe based on the Hall effect can be used to measure the field at all points to 1 part in 1,000, so that the first approach becomes mainly a matter of sufficient care, aided by the small range of input angles and momenta involved and the lack of importance in obtaining a high accuracy of the absolute momentum. The second approach is probably not necessary with the non-focussing bending magnets and so becomes mainly a check on the magnet calibration. But in either case it should be possible to write a relationship between bending angle, input co-ordinates and
momentum with respect to an optic axis over a sufficient range of these parameters to make the error in the magnet calibrations unimportant with respect to errors of multiple scattering.

The measurement of the decay angle $\theta$ is limited by multiple scattering at chambers 4 and 5 to $\sqrt{2}$ times the r.m.s. scattering angle, or 1 mrad. at 1 GeV/c. Calibration events give the mean angle corresponding to $\theta = 0$ and also an experimental value for the error in the angle measurement.

It is possible to design a magnet system that should largely remove the errors introduced by multiple scattering, both in the momenta and in the angle measurements. An apparatus embodying this approach is sketched in fig. 1b. The bending magnets make a 'wedge' type of field and therefore produce horizontal focusing. Spark chambers 2 and 3, 6 and 7, are near the conjugate planes for the two magnets - about 4m from the magnets at 1 GeV/c using a wedge angle of 60°. With this arrangement, the final position of a particle at chamber 3 for a particle starting from a given position and angle at chamber 2, while still dependent on momentum as for a rectangular magnet, becomes almost independent of the initial angle. Consequently uncertainty in this angle becomes much less significant. Preliminary work on this system indicates that an improvement of the order of 5 times on the non-focussing system may be possible, but it is not clear where the ultimate error would lie. It should be emphasised that the focussing requirement for the magnet is not stringent, in fact as far as scattering errors are concerned, any wedge is better than a rectangle as it introduces some compensation and thereby reduces the dependence on initial angle. It is almost certain that with this arrangement the final
calibration of the two magnets would be in terms of each other using the calibration particles.

Errors in measuring $\theta$ due to scattering can be avoided in the same way by using quadrupole triplets, spark chambers 4 and 5 being in the focal planes. For a perfect lens, position definition to $\frac{1}{2}$ mm. and a focal length of 5m. gives the direction to 1 part in $10^4$. The known aberrations of a real lens can be largely eliminated as the momentum and either the input or the output trajectories are well known.

**Determination of $m$ and estimate of performance.**

For each event selected we would measure $P_\mu$, $P_\pi$ and $\theta$ with respect to some reference axis. We would then remove the $\theta$ dependence and so derive a value for $(P_\mu - P_\pi) \theta = 0$. After each event we would calibrate the apparatus with a non decaying particle and again measure $P_\mu$, $P_\pi$ and $\theta$ with respect to the same reference axis, finding $(P_\mu - P_\pi)_{\text{calibration}}$ and plot the two resulting distributions in $(P_\mu - P_\pi)$. The problem is then to measure the separation of these two distributions. Depending on the range of decay angles selected, the $\pi$ decay distribution should be somewhat broader than the calibration distribution but $\theta$ will be limited so that this effect is small. There will also be some background events in the $\pi$ decay distribution from $\pi$ decays outside the wanted decay path. But the main error will be the statistical error set by the width of the distributions and the numbers of events.

The probability of a pion decaying in a decay path of 10m within a decay angle of 2 mrad at 1 Gev/c is $2 \times 10^{-4}$. A beam having a $\frac{2}{3}$ or $\frac{2}{3}$ momentum resolution should be able to give 1 useful event every few machine pulses fairly
easily. Assuming that the widths of the \((P_{\mu} - P_{\pi})\) distributions can be kept down to 1 MeV/c at 1 GeV/c, 10,000 events should give us the mean \((P_{\mu} - P_{\pi})\) moments to 0.01 MeV/c and consequently the separation to 0.014 MeV/c. This corresponds to a minimum detectable mass \(m\) of 0.35 MeV.

What accuracy could be achieved with the 'focussing' system? We would almost certainly have to work at about 2 GeV/c to keep the error due to \(m\) sufficiently low. The width of the \((P_{\mu} - P_{\pi})\) distributions would depend mainly on the magnets and quadrupoles. An improvement of a factor of 5 on the 'non-focussed' system seems possible, and it is not certain that a result somewhat better than this could not be achieved. The flux position is very dependent on the useful apertures of the magnets and quadrupoles. An encouraging feature here is that the decay process itself leads to a spread of only about 2 cm at the far quadrupole (2 mrad. at 10 m, or 1 mrad. at 20 m). Again roughly 2 x 10\(^{-4}\) of the pion decays would be potentially useful. So the problem becomes mainly one of determining, given the acceptable apertures, the intensity of the pion beam at the final wedge. If this were as high as 500 pions/pulse, so approximately 1 event per 10 pulses, \(10^4\) events would still be quite reasonable and a sensitivity corresponding to a minimum detectable mass of 0.16 MeV would be possible. An error in \(m\) of 0.05 MeV is equivalent to an \(m\) of 0.06 MeV at 2 GeV/c.

Other aspects of the experiment.

Selection system. As the momenta of the pions and decay muons are similar and the useful decay angles are small, the selection system could conveniently be based on two gas Cerenkov counters in coincidence, one before the measuring system to select pions, and one after to select muons as sketched in figure 1.
Almost all the unwanted pion decays (those at larger angles) would miss the final counter.

**Beam.** The beam momentum used would probably be between 1 and 2 GeV/c. With the exception of the significance of $m_\pi$ and the design of the selection system, the main aspects of the experiment are unaltered by changing the momentum, provided that the length of the apparatus is scaled in proportion. As useful decays give rise to relatively little change in momentum or angle, the useful momentum bite and angular divergence of the beam can be considered from the point of view of beam transport through the apparatus. A momentum bite of $\frac{1}{2}$ to $1\%$ and an angular acceptance of about 5 mrad, looks reasonable.

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**Reference**

\[ \pi^+ \rightarrow \mu^+ + \nu_\mu \]

Fig. 1a. 'Non-Focussing' System

Fig. 1b. 'Focussing' System

Measurement of $P_\pi$, $P_\mu$, $\Theta$, at 1 GeV/c.
(not to scale)
This illustrates the momentum change on decay, $\Delta p_{e}-p_{\nu}$, as a function of the decay angle $\theta$ and neutrino mass $m$. The parabola is drawn for $m=0$; a non-zero mass results in a downward displacement of $m^2$. Calibration events give the position $(\Delta p_{e}-p_{\nu})=0, \theta = 0$. The experiment is then aimed at finding the value of $(\Delta p_{e}-p_{\nu})$ corrected for angle with respect to the origin. The errors are those expected for a single measurement using the 'non-focussing' system.