Appendix A
Renormalization Properties of Wilson Loops

Wilson lines and loops are very fundamental objects in gauge theories. A Wilson line $W[C_{x_1,x_2}]$ depends on a contour $C$ having endpoints $x_1$ and $x_2$. It is explicitly given by

$$W[C] = \frac{1}{N} P e^{ig \int_C ds \dot{x}^\mu A_\mu(x(s))},$$

where $P$ stands for path ordering along the contour $C$, and $A_\mu = A^a_\mu T^a$, and $T^a$ are the generators of the gauge group. For definiteness, we are going to take the gauge group to be $SU(N)$, the generators to be in the fundamental, normalized by $\text{Tr}(T^a T^b) = \delta^{ab}$.

By definition, under gauge transformations the Wilson line transforms covariantly at the endpoints. It can therefore be used for example to make a non-local configuration of elementary fields sitting at points $x_1$ and $x_2$ gauge invariant. Such non-local objects can be used, for example, as generating functions of operators with spin, as one lets $x_2$ approach $x_1$ in a light-like direction. One can also form gauge invariant non-local quantities from the Wilson line itself, by identifying the two endpoints, and taking a trace, thereby defining a Wilson loop, which is a functional of the contour $C$. This is the case we will discuss from now on. It turns out that Wilson loops having specific contours describe many interesting physical effects. For example, certain configurations of Wilson lines/loops describe the infrared physics of (massive) scattering amplitudes \[1, 2\].

In general, in a quantum field theory we have to regularize and renormalize the wave functions of the elementary fields and the coupling constant. Once this has been carried out, there can be additional ultraviolet divergences coming from specific operators introduced into the theory, for example for composite operators. This is well understood and gives rise to a renormalization of the composite operators, described by renormalization group equations, operator mixing, and anomalous dimensions. We refer the reader to \[3\] for a general treatment, and to \[4\] for examples in the context of conformal field theory.

A natural question is what the renormalization properties of the non-local Wilson loop operators defined above are, depending on the shape of the contour $C$. In
other words, we will be interested in the renormalization properties of the vacuum expectation value
\[
\langle W[C] \rangle := \frac{1}{N} \text{Tr} P \left[ e^{ig \int_C ds i^\mu A_\mu(x(s))} \right].
\] (A.2)

The answer turns out to be very similar to that for local operators.

There are several cases depending on the shape of the contour. We will begin by discussing smooth contours, followed by contours having one or more cusps, and self-intersections, and finally we will discuss the case where the contour contains light-like segments. More details can be found in the original papers, see e.g. [5–8].

We will assume in the following that the renormalization of the Yang-Mills theory has already been carried out, and only discuss the intrinsic divergences associated to the Wilson loop operators. In this case, Wilson loops defined in smooth, non-intersecting contours only have a linear divergence proportional to the length of the contour [6]. This divergence can be removed by a multiplicative renormalization. One can think of this as the mass renormalization of the Wilson line when it is viewed as the effective description of a heavy particle. See Fig. A.1. In dimensional regularization, such a divergence is absent, as it is powerlike.

In the case where the contour has discontinuities, there are specific divergences associated to the cusp points. See Fig. A.1. Such Wilson loops with cusps can be multiplicatively renormalized, i.e.
\[
\langle W_R(C) \rangle = Z(\phi) \langle W(C) \rangle,
\] (A.3)

where the renormalization factor depends only locally on the contour $C$, through the cusp angle $\phi$ and on the (dimensional) regularization parameter $\epsilon$, and scale $\mu^2$, and on the Yang-Mills coupling constant. The locality of the counterterms is an important feature of ultraviolet divergences. For more than one cusp, it implies that the renormalization takes a factorized form, e.g.
\[
Z(\phi_1, \ldots, \phi_n) = Z(\phi_1) \cdots Z(\phi_n).
\] (A.4)

From $Z$ one can define an anomalous dimension in the usual way,
\[
\Gamma(\phi, g_R) = \lim_{\epsilon \to 0} Z \mu \frac{\partial}{\partial \mu} Z^{-1},
\] (A.5)
Fig. A.2 Two Wilson loops which mix under renormalization. Loop (a) self-intersects, whereas (b) consists of two cusped loops.

where $g_R$ is the renormalized coupling constant. The form of $Z$ is restricted by a renormalization group equation.

In the case of self-intersections, the analysis of UV divergences is very similar, except that one now gets nontrivial color dependence. In fact, one sees that in analogy with local operators, there are Wilson loops that mix under renormalization. For example, in the case shown in Fig. A.2, one has mixing between the two Wilson loops shown in (a) and (b). Denoting the doublet by $W = (W[C_\alpha], W[C_\beta])$, one now has a renormalization matrix $Z$,

$$
\langle W_R \rangle = Z(\phi) \langle W \rangle. \quad (A.6)
$$

If a Wilson loop contains light-like segments this leads to additional logarithmic divergences. One can think about this as a limit of the general case. For example, consider a Wilson loop having a cusp with Euclidean cusp angle $\phi = p \cdot q / \sqrt{p^2 q^2}$, where $p^\mu$ and $q^\mu$ are the momenta forming the cusp (cf. Fig. A.1). We can analytically continue to $\phi \rightarrow i \theta$, and consider the limit $\theta \rightarrow \infty$. This corresponds to the limit where one (or both) of the segments $p^\mu$ or $q^\mu$ becomes light-like. In this limit, the cusp anomalous dimension has the behavior [8]

$$
\Gamma_{\text{cusp}}(\phi, g) \sim \lim_{\theta \rightarrow \infty} \theta \Gamma_{\text{cusp}}(g) + O(1). \quad (A.7)
$$

A case of physical interest will be the contour formed by $n$ light-like segments, with cusp points $x_1, \ldots, x_n$. What is the general structure of such a correlation function? Let us choose dimensional regularization to regularize UV divergences (for renormalization of the Lagrangian, as well as for the Wilson loop). In a conformal theory with beta function $\beta = -\varepsilon g$, the renormalization group equations can be solved, leading to the simple general form [5]

$$
\log \langle W \rangle = -\frac{1}{4} \sum_{\ell \geq 1} a^\ell \sum_{i=1}^{n} (-x_{i-1,i+1} \mu)^{\ell \varepsilon} \left( \frac{\Gamma_{\text{cusp}}(\ell \varepsilon)}{(\ell \varepsilon)^2} + \frac{\Gamma_{\text{R}}(\ell)}{\ell \varepsilon} \right), \quad a = \frac{g^2 N}{8\pi^2}. \quad (A.8)
$$

References

Appendix B
Conventions and Useful Formulae

• Index and metric conventions:

\[ \eta_{\mu\nu} = \text{diag}(+,-,-,-), \quad p_\mu p^\mu = p_0^2 - \mathbf{p}^2, \]

\[ \epsilon^{12} = \epsilon^{12} = \epsilon_{21} = +1, \quad \epsilon^{21} = \epsilon^{21} = \epsilon_{12} = -1, \]

\[ (\tilde{\sigma}^{\mu})^{\hat{\alpha}\alpha} = (1, -\sigma), \quad (\sigma^{\mu})_{\alpha\hat{\alpha}} = \epsilon_{\alpha\beta\hat{\beta}} \epsilon (\tilde{\sigma}^{\mu})_{\hat{\beta}\beta} = (1, \sigma), \]

\[ (\bar{\sigma}_\mu)^{\hat{\alpha}\alpha} = (1, \sigma), \quad (\sigma_\mu)_{\alpha\hat{\alpha}} = (1, -\sigma), \]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ \chi^\alpha = \epsilon^{\alpha\beta} \chi^\beta, \quad \bar{\chi}^{\hat{\alpha}} = \epsilon^{\hat{\alpha}\hat{\beta}} \bar{\chi}^{\hat{\beta}}, \quad (\chi^\alpha)^* = \bar{\chi}^{\hat{\alpha}}, \]

\[ (\chi^\alpha \psi^\beta)^\dagger = (\psi^\beta)^\dagger (\chi^\alpha)^\dagger. \]

(B.1)

• Spinor helicity relations:

\[ p^{\alpha\hat{\alpha}} = \lambda^\alpha \bar{\lambda}^{\hat{\alpha}}, \quad p_{\hat{\alpha}\alpha} = \epsilon_{\hat{\alpha}\beta} \epsilon^{\alpha\beta} \lambda^\beta \bar{\lambda}^{\hat{\beta}}, \]

\[ \lambda^\alpha = \epsilon_{\alpha\beta} \lambda^\beta, \quad \bar{\lambda}^{\hat{\alpha}} = \epsilon^{\hat{\alpha}\hat{\beta}} \bar{\lambda}^{\hat{\beta}}, \]

\[ u_+(p) = v_-(p) = \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix} = : |p \rangle, \quad u_-(p) = v_+(p) = \begin{pmatrix} 0 \\ \bar{\lambda}^{\hat{\alpha}} \end{pmatrix} = : \langle \bar{p} |. \]

(B.2)

\[ \bar{u}_+(p) = \bar{v}_-(p) = \begin{pmatrix} 0 \\ \bar{\lambda}^{\hat{\alpha}} \end{pmatrix} = : \langle \bar{p} |, \quad \bar{u}_-(p) = \bar{v}_+(p) = \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix} = : |p \rangle, \]

\[ \langle \lambda^\alpha \lambda^\beta \rangle := \lambda^\alpha \lambda^\beta, \quad [\lambda^\alpha \lambda^\beta] := \lambda^\alpha \lambda^\beta - \lambda^\beta \lambda^\alpha \]

using the chiral representation of the Dirac matrices

\[ \gamma^\mu = p_\mu \Gamma^\mu = \begin{pmatrix} 0 & p^{\alpha\hat{\alpha}} \\ p_{\hat{\alpha}\alpha} & 0 \end{pmatrix} \]

using \( p_{\alpha\hat{\alpha}} := p_\mu (\sigma_\mu)_{\alpha\hat{\alpha}}, \quad p^{\hat{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\hat{\beta}\hat{\alpha}} p_{\beta\hat{\beta}} = p_\mu (\tilde{\sigma}^{\mu})_{\hat{\alpha}\alpha}. \)

(B.3)
We furthermore note
\[ [i|\Gamma^\mu|j] = \langle j|\Gamma^\mu|i \rangle \]
\[ \langle p|\Gamma^\mu|p \rangle = \lambda^\alpha \sigma^\mu_{\dot{a}\dot{a}} \lambda_{\dot{a}} = 2p^\mu, \]
\[ [i|\Gamma^\mu|j]\langle l|\Gamma_\mu|k \rangle = 2[ik\rangle\langle lj]. \]

- The generators of the Lorentz group in the spinor representation are given by
\[
(\sigma^{\mu\nu})_{\alpha}^\beta = \frac{1}{4}(\sigma^{\mu})_{\alpha}^\dot{\alpha}(\bar{\sigma}^{\nu})_{\dot{\alpha}}^{\beta} - (\sigma^{\nu})_{\alpha}^\dot{\alpha}(\bar{\sigma}^{\mu})_{\dot{\alpha}}^{\beta},
\]
\[
(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\beta} = \frac{1}{4}(\bar{\sigma}^{\mu})_{\dot{\alpha}}^{\dot{\beta}}(\sigma^{\nu})_{\alpha}^\dot{\alpha} - (\bar{\sigma}^{\nu})_{\dot{\alpha}}^{\dot{\beta}}(\sigma^{\mu})_{\alpha}^\dot{\alpha}. \tag{B.5}
\]

- Complex conjugation properties:
\[
(\lambda^\alpha)^* = \bar{\lambda}^{\dot{\alpha}}, \quad \langle ij \rangle^* = (\lambda^\alpha_i \lambda_j^\alpha)^* = (\bar{\lambda}^{\dot{\alpha}}_i \bar{\lambda}^{\dot{\alpha}}_j)^* = -[ij]. \tag{B.6}
\]

- Useful trace identities
\[
\text{Tr}[\hat{\alpha}\hat{\beta}\hat{c}\hat{d}] = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)],
\]
\[
\text{Tr}[\hat{\alpha}\hat{\beta}] = 4(a \cdot b). \tag{B.7}
\]
Solutions to the Exercises

Exercise 1.1

Given $(\tilde{\sigma}^\mu)_{\dot{\alpha}\alpha} = (1, -\sigma)$ and $(\sigma^\mu)_{\alpha\dot{\alpha}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}^\mu)^{\dot{\beta}\beta}$ we have with

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

the relations

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i(\sigma_2)_{\alpha\beta}, \quad \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i(\sigma_2)_{\dot{\alpha}\dot{\beta}}. \quad \text{(B.8)}$$

Then we have

$$\begin{aligned}
(\sigma^\mu)^{\dot{\alpha}\alpha} &= -\varepsilon_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}^\mu)^{\dot{\beta}\beta} \varepsilon_{\beta\alpha} = -i^2(\sigma_2\tilde{\sigma}^\mu\sigma_2)^{\dot{\alpha}\alpha} = (\sigma_2\tilde{\sigma}^\mu\sigma_2)^{\alpha\dot{\alpha}}, \\
\sigma_2\tilde{\sigma}^\mu\sigma_2 &= \sigma_2(1, -\sigma)\sigma_2 \\
&= (\sigma_2^2, -\sigma_2\sigma_1\sigma_2, -\sigma_2\sigma_2\sigma_2, -\sigma_2\sigma_3\sigma_2) = (1, \sigma_1, -\sigma_2, \sigma_3), \\
(\sigma_2\tilde{\sigma}^\mu\sigma_2)^T &= (1, \sigma_1, \sigma_2, \sigma_3),
\end{aligned}$$

as $\sigma_2$ is antisymmetric whereas the other Pauli matrices are symmetric. We thus have $(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \sigma)$. The relation $(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, -\sigma)$ follows trivially from $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. To prove the third relation in Exercise 1.1 $\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\mu_{\mu\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}$ we look at the LHS of this relation for fixed values of $\alpha$ and $\beta$:

$$\begin{aligned}
\sigma^\mu_{1\dot{\alpha}}\sigma_{\mu1\dot{\beta}} &= \sigma^\mu_{1\dot{\alpha}}\sigma^\mu_{1\dot{\beta}} - \delta_{1\dot{\alpha}}\delta_{1\dot{\beta}} \\
&= \delta_{1\dot{\alpha}}\delta_{1\dot{\beta}} - \delta_{1\dot{\alpha}}\delta_{2\dot{\beta}} - (i)\delta_{\dot{\alpha}2}(i)\delta_{\dot{\beta}2} - \delta_{\dot{\alpha}1}\delta_{\dot{\beta}1} = 0, \\
\sigma^\mu_{2\dot{\alpha}}\sigma_{\mu2\dot{\beta}} &= \sigma^\mu_{2\dot{alpha}}\sigma^\mu_{2\dot{beta}} - \delta_{2\dot{alpha}}\delta_{2\dot{beta}} \\
&= \delta_{2\dot{alpha}}\delta_{2\dot{beta}} - \delta_{\dot{alpha}1}\delta_{\dot{beta}1} - (i)\delta_{\dot{alpha}1}(i)\delta_{\dot{beta}1} - (1)\delta_{\dot{alpha}2}(1)\delta_{\dot{beta}2} = 0.
\end{aligned}$$
\[
\sigma_{1\dot{\alpha}}^{\mu} \sigma_{2\dot{\beta}} = \sigma_{1\dot{\alpha}}^{0} \sigma_{2\dot{\beta}}^{0} - \sigma_{1\dot{\alpha}}^{i} \sigma_{2\dot{\beta}}^{i}
\]
\[
= \delta_{1\dot{\alpha}} \delta_{2\dot{\beta}} - \delta_{1\dot{\alpha}} (-1) \delta_{\dot{\beta}1} - (-i) \delta_{\dot{\alpha}2} (i) \delta_{\dot{\beta}1} - \delta_{\dot{\alpha}1} (-1) \delta_{\dot{\beta}2}
\]
\[
= 2(\delta_{1\dot{\alpha}} \delta_{2\dot{\beta}} - \delta_{2\dot{\alpha}} \delta_{1\dot{\beta}}) = 2\epsilon_{\dot{\alpha}\dot{\beta}}.
\]
Note that the last equation implies \( \sigma_{1\dot{\alpha}}^{\mu} \sigma_{2\dot{\beta}} = -2\epsilon_{\dot{\alpha}\dot{\beta}} \) by antisymmetry in \( \dot{\alpha}\dot{\beta} \). The final relation \( \epsilon^{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} = 2\eta^{\mu\nu} \) follows by taking the trace on the LHS

\[
\text{Tr}(-i\sigma_2 \sigma^i (i\sigma_2)(\sigma^\mu)^T) = \text{Tr}(\sigma_2 \sigma^v \sigma_2 \sigma^\mu T) = \begin{cases}
-\text{Tr}(\sigma^v \sigma^\mu T) & \text{for } v = 1, 3 \\
+\text{Tr}(\sigma^v \sigma^\mu T) & \text{for } v = 0, 2
\end{cases}
\]
\[
= \begin{cases}
-2\delta^{\mu\nu} & \text{for } v = 1, 3 \\
+2\eta^{\mu\nu} & \text{for } v = 0, 2
\end{cases} = 2\eta^{\mu\nu}.
\]

**Exercise 1.2**

(a) We have

\[
\Gamma^\mu k_\mu = \begin{pmatrix}
k_0 & -\sigma \cdot k \\
+\sigma \cdot k & -k_0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
k_0 & 0 & -k^3 & -\sqrt{k+k} - e^{-i\phi} \\
0 & k_0 & \sqrt{k+k} - e^{-i\phi} & k^3 \\
k^3 & \sqrt{k+k} - e^{-i\phi} & -k_0 & 0 \\
\sqrt{k+k} - e^{-i\phi} & -k^3 & 0 & -k_0
\end{pmatrix},
\]
which upon multiplication with \( u_\pm(k) \) or \( u^-_\pm(k) \) is easily seen to vanish. For the helicity relations we first note

\[
P_\pm = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 \\
\pm 1 & 1
\end{pmatrix}.
\]
Then as \( u_\pm(k) = \begin{pmatrix} \xi \\ \pm \xi \end{pmatrix} \) we have

\[
P_\pm u_\pm = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 \\
\pm 1 & 1
\end{pmatrix} \begin{pmatrix} \xi \\ \pm \xi \end{pmatrix} = \begin{pmatrix} \xi \\ \pm \xi \end{pmatrix} = u_\pm,
\]
\[
P_\pm u_\mp = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 \\
\pm 1 & 1
\end{pmatrix} \begin{pmatrix} \xi \\ \mp \xi \end{pmatrix} = 0.
\]

(b) For an arbitrary Dirac-spinor \( \chi \) we have

\[
\tilde{\chi} P_\pm = \chi^\dagger \Gamma^0 P_\pm = (\Gamma^0 P_\pm)^\dagger = (P_\pm \Gamma^0 \chi)^\dagger = (\Gamma^0 P_\mp \chi)^\dagger.
\]
From this it follows that \( \tilde{u}_\pm P_\mp = \tilde{u}_\pm \) and \( \tilde{u}_\pm P_\pm = 0 \).
(c) Through explicit matrix multiplication one verifies that \( U = \frac{1}{\sqrt{2}}(1 - i \Gamma^1 \Gamma^2 \Gamma^3) \). It is indeed a unitary matrix. The transformed Dirac matrices then take the form

\[
\Gamma^0 \rightarrow \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\Gamma \rightarrow \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}.
\]

Thus the Dirac-matrices in the transformed basis read

\[
\Gamma_{\mu}^{\text{ch}} = \begin{pmatrix} 0 & \sigma_{\mu}^{\hat{\alpha} \hat{\alpha}} \\ \bar{\sigma}_{\mu}^{\hat{\alpha} \hat{\alpha}} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_{5}^{\text{ch}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Expressing the solutions to the Dirac equation in the chiral basis it follows that

\[
U u_+ = \begin{pmatrix} 0 \\ \sqrt{k^+} e^{-i\phi} \end{pmatrix} \quad \text{and} \quad U u_- = \begin{pmatrix} \sqrt{k^-} e^{i\phi} \\ 0 \end{pmatrix}.
\]

Note the appearance of the helicity spinors \( \lambda^\alpha \) and \( \bar{\lambda}^\alpha \) discussed in Sect. 1.6.

**Exercise 1.3**

The prove of Eq. (1.93) is straightforward. With

\[
[i | \Gamma^{\mu} | j] = \tilde{\lambda}_i \hat{\lambda}_j \tilde{\sigma}^{\mu\hat{\alpha} \hat{\alpha}}
\]

we have

\[
[i | \Gamma^{\mu} | j][k | \Gamma^{\nu} | l] \eta_{\mu \nu} = \tilde{\lambda}_i \hat{\lambda}_j \tilde{\lambda}_k \hat{\lambda}_l \tilde{\sigma}^{\mu\hat{\alpha} \hat{\alpha}} \bar{\sigma}^{\nu\hat{\beta} \hat{\beta}} \eta_{\mu \nu} = 2i k^i (l^j).
\]

**Exercise 1.4**

We start from the full Feynman rule four-point vertex contracted with dummy polarization vectors \( \epsilon_i \)

\[
V_4 = -i g^2 f^{abc} f^{cde} (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) + \text{cyclic}
\]

and use \( f^{abc} f^{cde} = -\frac{1}{2} \text{Tr}([T^a, T^b][T^c, T^d]) \) which is obtained from Eq. (1.96). Note that the \( U(1) \) piece cancels out here. Expanding out the commutators in the traces and collecting terms of identical color ordering one finds
\[ V_4 = \frac{ig^2}{2} \text{Tr}(T^a T^b T^c T^d) \left[ 2(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \right] \]

which is the result quoted in the color ordered Feynman rules.

**Exercise 1.5**

(a) Taking parity and cyclicity into account we have the independent 4-gluon amplitudes

\[ A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+), \quad A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+), \]
\[ A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+), \quad A_4^{\text{tree}}(1^-, 2^-, 3^-, 4^+). \]

The last two are related via the \( U(1) \) decoupling theorem as

\[ A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) = -A_4^{\text{tree}}(1^-, 2^+, 4^+, 3^-) - A_4^{\text{tree}}(1^-, 4^+, 2^+, 3^-) \]
\[ = -A_4^{\text{tree}}(3^-, 1^-, 2^+, 4^+) - A_4^{\text{tree}}(3^-, 1^-, 4^+, 2^+). \]

Hence only the three amplitudes \( A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+) \), \( A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+) \) and \( A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \) are independent. In fact the first two of this list vanish, so there is only one independent 4-gluon amplitude at tree-level to be computed.

(b) Moving on two the 5-gluon trees we have the four cyclic and parity independent amplitudes

\[ A_5^{\text{tree}}(1^+, 2^+, 3^+, 4^+, 5^+), \quad A_5^{\text{tree}}(1^-, 2^+, 3^+, 4^+, 5^+), \]
\[ A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+), \quad A_5^{\text{tree}}(1^-, 2^-, 3^-, 4^+, 5^+). \]

Looking at the following \( U(1) \) decoupling relation we may again relate the last amplitude in the above list to the third one

\[ A_5^{\text{tree}}(2^+, 3^-, 4^+, 5^+, 1^-) \]
\[ = -A_5^{\text{tree}}(3^-, 2^+, 4^+, 5^+, 1^-) - A_5^{\text{tree}}(3^-, 4^+, 2^+, 5^-, 1^-) \]
\[ - A_5^{\text{tree}}(3^-, 4^+, 5^+, 2^+, 1^-) \]
\[ = -A_5^{\text{tree}}(1^-, 3^-, 2^+, 4^+, 5^+) - A_5^{\text{tree}}(1^-, 3^-, 4^+, 2^+, 5^+) \]
\[ - A_5^{\text{tree}}(1^-, 3^-, 4^+, 5^+, 2^+). \]

Hence also for the 5-gluon case there are only three independent amplitudes: \( A_5^{\text{tree}}(1^+, 2^+, 3^+, 4^+, 5^+) \), \( A_5^{\text{tree}}(1^-, 2^+, 3^+, 4^+, 5^+) \) and \( A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) \). The first two in this list vanish leaving us with one independent and non-trivial 5-gluon tree-level amplitude of MHV type.
Exercise 1.6

There are two color ordered amplitudes contributing to $A(1_q^- , 2_q^+, 3^- , 4^+) = p_1 + p_2$.

Looking at the first graph (I) we see that it is proportional to $[2|\epsilon_{\frac{i}{p_4}}^\gamma |1\rangle]$. The reference vector choice $\mu_4 \sim \mu_4$ then annihilates this graph

$$(I) \sim [2|\epsilon_{\frac{i}{p_4}}^\gamma |1\rangle] = 0 \text{ for } \mu_4 \sim \mu_4.$$

This is so as $\epsilon_4^\gamma = -\sqrt{2}[4|\mu_4\rangle + |\mu_4\rangle|4\rangle] \Rightarrow \epsilon_4^\gamma |1\rangle = -\sqrt{2}[4|\mu_4\rangle]_{\mu_4 = \lambda_1}$.

Evaluating the second graph (II) with the color-ordered Feynman rules we are led to the following expression

$$(II) = \left(-\frac{i}{\sqrt{2}}\right)^2 [2]|\Gamma\mu_4|1\rangle \frac{-i}{q^2} \times \left[\lambda_{3}^{\lambda_{4}}(p_{34})^{\mu} + (\epsilon_{4}^{\mu})(p_{5q} \cdot \epsilon_{3}^{-}) + (\epsilon_{3}^{-})(p_{q3} \cdot \epsilon_{4}^{\mu})\right]. \quad \text{(B.9)}$$

giving rise to three terms. One sees that term (2) vanishes for our choice $\mu_4 \sim \mu_4 = p_4$

$$(2) \sim [2|\epsilon_{\frac{i}{p_4}}^\gamma |1\rangle]_{\mu_4 \sim \mu_4} = 0.$$

For the term (3) we note $\epsilon_{3}^{-} = \sqrt{2}[3|\mu_3\rangle + |\mu_3\rangle|3\rangle]$ to find

$$(3) \sim [2|\epsilon_{\frac{i}{p_4}}^\gamma |1\rangle] = \sqrt{2}[3|\mu_3\rangle|2\mu_3\rangle].$$

We now make the choice $\mu_3 \sim \mu_3 = p_2$ for the remaining reference vector of leg 3 which also kills this term. Hence, for these two choice of reference vectors only the term (1) in the above Eq. (B.9) contributes. One has

$$\epsilon_{3}^{-} \cdot \epsilon_{4}^{\mu} = \frac{\langle \mu_4 | 3 | \mu_3 \rangle}{\langle 4 | \mu_4 \rangle 3 | \mu_3 \rangle} \Rightarrow \frac{\langle \mu_4 | 3 | \mu_3 \rangle}{\langle 4 | \mu_4 \rangle 3 | \mu_3 \rangle} = \frac{\langle 13 | 24 \rangle}{\langle 41 | 32 \rangle}.$$
Inserting this into the term (1) of Eq. (B.9) and using $q^2 = \langle 12 \rangle \langle 21 \rangle$ yields

\[
(\text{II}) = \frac{i}{2q^2} \left( -\langle 13 \rangle \langle 24 \rangle \langle 41 \rangle \langle 32 \rangle \right) [2(p_3^\dagger - p_4^\dagger)]
\]

\[
= -i \frac{1}{2\langle 12 \rangle \langle 21 \rangle \langle 23 \rangle \langle 31 \rangle} \left[ \langle 23 \rangle \langle 31 \rangle - \langle 24 \rangle \langle 41 \rangle \right]
\]

\[
= -i \frac{\langle 13 \rangle^2}{\langle 12 \rangle \langle 41 \rangle \langle 34 \rangle \langle 41 \rangle} = -i \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.
\]

as claimed. The helicity count of our result $A_{qg^2}^{\text{tree}} = -i \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$ is straightforward and correct

\[
h_1[A_{qg^2}^{\text{tree}}] = -\frac{1}{2}(3 - 1 - 1) = -\frac{1}{2}, \quad h_2[A_{qg^2}^{\text{tree}}] = -\frac{1}{2}(1 - 1 - 1) = +\frac{1}{2},
\]

\[
h_3[A_{qg^2}^{\text{tree}}] = -\frac{1}{2}(4 - 1 - 1) = -1, \quad h_4[A_{qg^2}^{\text{tree}}] = -\frac{1}{2}(0 - 1 - 1) = -1.
\]

Exercise 2.1

(a) We begin with the MHV case $A(1^- , 2^- , 3^+)$:

\[
\begin{array}{c}
2^- \\
\end{array}
\begin{array}{c}
3^+
\end{array}
\begin{array}{c}
= \left( -\frac{gl}{\sqrt{2}} \right) \left[ (\varepsilon_{-1} \cdot \varepsilon_{-2})(p_{12} \cdot \varepsilon_{+3}) + (\varepsilon_{-2} \cdot \varepsilon_{+3})(p_{23} \cdot \varepsilon_{-1}) + (\varepsilon_{+3} \cdot \varepsilon_{-1})(p_{31} \cdot \varepsilon_{-2}) \right].
\end{array}
\]

The gauge choice $\mu_1 = \mu_2 = \mu_3 = \mu$ annihilates the first term as then $\varepsilon_{-1} \cdot \varepsilon_{-2} = 0$. We note using Eq. (1.82)

\[
\varepsilon_{-2} \cdot \varepsilon_{+3} = -\frac{\langle \mu_2 \rangle \langle \mu_3 \rangle}{\langle 3\mu \rangle \langle 2\mu \rangle}, \quad p_{23} \cdot \varepsilon_{-1} = -\sqrt{2} \frac{\langle \mu_3 \rangle \langle 31 \rangle}{\langle 1 \mu \rangle}.
\]

Inserting this into the amplitude (whilst dropping the factor $-ig$ to go from the graph to the amplitude) yields
\[ A(1^-, 2^-, 3^+) = -\frac{\langle \mu_2 \rangle [\mu_3] [\mu_3](31)}{\langle 3\mu \rangle [\mu_2]} - (1 \leftrightarrow 2) \]
\[ = \frac{\langle \mu_3 \rangle^2}{\langle 1\mu \rangle [\mu_2]} \langle \mu_2 \rangle(31) - \langle \mu_1 \rangle(32) \]
\[ = \frac{\langle \mu_3 \rangle^2}{\langle 1\mu \rangle [\mu_2]} \langle 1\mu \rangle(2) \]
\[ = \frac{\langle \mu_3 \rangle^2}{\langle 1\mu \rangle [\mu_2]} (12). \]

The MHV\textsubscript{3} kinematics implies \( \tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3 \) or \( \tilde{\lambda}_2 = a\tilde{\lambda}_1 \) and \( \tilde{\lambda}_3 = b\tilde{\lambda}_1 \) leading to
\[ A(1^-, 2^-, 3^+) = \frac{b^2}{a} (12). \]

Momentum conservation then implies for this parametrization
\[ (\lambda_1 + a\lambda_2 + b\lambda_3)\tilde{\lambda}_1 = 0, \]
which tells us that
\[ a = \frac{\langle 31 \rangle}{\langle 23 \rangle} \quad \text{and} \quad b = \frac{\langle 12 \rangle}{\langle 23 \rangle}. \]

Plugging this into the above yields the final result
\[ A(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}. \]

(b) The MHV\textsubscript{3} case works analogously. We have for \( A(1^+, 2^+, 3^-) \)
\[
\begin{align*}
2^+ & \quad = \left( \frac{-gi}{\sqrt{2}} \right) \left[ (\varepsilon_{+,1} \cdot \varepsilon_{+,2})(p_{12} \cdot \varepsilon_{-,3}) \\
& \quad + (\varepsilon_{+,2} \cdot \varepsilon_{-,3})(p_{23} \cdot \varepsilon_{+,1}) \\
& \quad + (\varepsilon_{-,3} \cdot \varepsilon_{+,1})(p_{31} \cdot \varepsilon_{+,2}) \right].
\end{align*}
\]

Now we note for the gauge \( \mu_1 = \mu_2 = \mu_3 = \mu \) that
\[ \varepsilon_{+,1} \cdot \varepsilon_{+,2} = 0, \quad \varepsilon_{+,2} \cdot \varepsilon_{-,3} = -\frac{\langle \mu_3 \rangle [\mu_2]}{\langle 2\mu \rangle [3\mu]}, \quad p_{23} \cdot \varepsilon_{+,1} = \sqrt{2} \frac{\langle 12 \rangle \langle \mu_2 \rangle}{\langle 1\mu \rangle}. \]

Inserting these into the above amplitude again dropping the overall factor \(-ig\) yields
\[ A(1^+, 2^+, 3^-) = -\frac{\langle \mu_3 \rangle^2}{\langle 1\mu \rangle \langle 2\mu \rangle} [12]. \]
The same argument as before lets us set \( \lambda_2 = a \lambda_1 \) and \( \lambda_3 = b \lambda_1 \) and from momentum conservation \( \lambda_1 (\tilde{\lambda}_1 + a \tilde{\lambda}_2 + b \tilde{\lambda}_3) = 0 \) we deduce

\[
a = \begin{bmatrix} 31 \\ 23 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 23 \end{bmatrix}.
\]

Hence we have shown that

\[
A(1^+, 2^+, 3^-) = -\begin{bmatrix} 12 \\ 23 \end{bmatrix} \begin{bmatrix} 31 \\ 23 \end{bmatrix},
\]

as claimed.

**Exercise 2.2**

We want to determine the NMHV gluon amplitude \( A_{\text{tree}}^6(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) \). A shift in \( 1^+ \) and \( 6^- \) leads to the BCFW recursion relation

\[
A_{\text{tree}}^6(1^+, \ldots, 6^-) = \sum_{i,s} A_i(\hat{1}, 2, \ldots, i-1, -\hat{P}_i^s(z_{P_i}), 1 \ldots, n-1, \hat{n}).
\]

Diagrammatically we have two on-shell diagrams contributing

\[
\begin{align*}
\text{Diagram (1):} & \quad \begin{array}{c}
\text{MHV}_3 \\
\end{array} \\
\text{Diagram (2):} & \quad \begin{array}{c}
\text{MHV}_5 \\
\end{array}
\end{align*}
\]

which we denote by (1) and (2). Using the minus sign convention \( |\!| \!-P\!| = -|P| \) and \( |\!|-P\!| = |P| \) we have for the first diagram (1)

\[
(1) = \frac{[\hat{1} \! 2]^3}{[2\hat{P}_{12}][\hat{P}_{12} \! \hat{1}]} \times \frac{1}{\langle 12 \rangle [21]} \times \frac{[\hat{P}_{21}3]}{[34][45][56][\hat{6}\hat{P}_{12}]}.
\]
Writing \( P_{ij} = P_i + P_j \) we have \( z_P = \frac{P_i^2}{\langle P_{12}[1]\rangle} = \frac{\langle 12 \rangle \langle 21 \rangle}{\langle 62 \rangle \langle 21 \rangle} = \frac{\langle 12 \rangle}{\langle 62 \rangle} \) and hence

\[
\begin{align*}
|\hat{1}\rangle &= |1\rangle, \\
|\hat{1}\rangle &= |1\rangle - \frac{\langle 12 \rangle}{\langle 62 \rangle} |6\rangle, \\
|\hat{6}\rangle &= |6\rangle, \\
|\hat{6}\rangle &= |6\rangle + \frac{\langle 12 \rangle}{\langle 62 \rangle} |1\rangle.
\end{align*}
\]

(B.11)

Furthermore one has

\[
\hat{P}_{12} = \hat{P}_1 + \hat{P}_2 = \lambda_2 \tilde{\lambda}_2 + \tilde{\lambda}_1 \left( \lambda_1 - \frac{\langle 12 \rangle}{\langle 62 \rangle} \tilde{\lambda}_1 \right)
\]

\[
= \lambda_2 \tilde{\lambda}_2 + (\langle 62 \rangle)^{-1} (\langle 62 \rangle \lambda_1 + \langle 21 \rangle \lambda_6) = \lambda_2 \left( \tilde{\lambda}_2 + \frac{\langle 61 \rangle}{\langle 62 \rangle} \lambda_1 \right),
\]

where we used the Fierz identity in the last step. Hence \( |\hat{P}_{12}\rangle = |2\rangle \) and \( |\hat{P}_{12}\rangle = |2\rangle + \frac{\langle 61 \rangle}{\langle 62 \rangle} |1\rangle \). Combining the above we deduce (again using \( P_{ij} = P_i + P_j \))

\[
[2 \hat{P}_{12}] = \frac{\langle 61 \rangle}{\langle 62 \rangle} [21], \\
[\hat{P}_{12} \hat{1}] = [21], \\
[56] = \frac{\langle 5| P_{16}[2] \rangle}{\langle 62 \rangle},
\]

\[
[\hat{P}_{12} 3] = \frac{\langle 6| P_{12}[3] \rangle}{\langle 62 \rangle}, \\
[\hat{6} \hat{P}_{12}] = -\frac{P^2_{26} + P^2_{12} + P^2_{16}}{\langle 62 \rangle}.
\]

Then we find for the on-shell diagram (1) of Eq. (B.10) the total contribution

\[
(1) = \frac{\langle 6| P_{12}[3] \rangle^3}{\langle 61 \rangle \langle 12 \rangle [34][45][5| P_{16}[2] \rangle} \frac{1}{P^2_{26} + P^2_{12} + P^2_{16}}.
\]

(B.12)

Moving on to the second contribution (2) we are led to consider

\[
(2) = \frac{(4 \hat{P}_{56})^3}{\langle \hat{P}_{56} \rangle \langle 12 \rangle [23] [34]} \times \frac{1}{\langle 56 \rangle [65]} \times \frac{\langle 56 \rangle^3}{\langle 6 \hat{P}_{56} \hat{56} \rangle}. 
\]

(B.13)

Now the shift parameter \( z_P \) takes the value \( z_P = \frac{[65]}{[31]} \) and we may deduce in analogy to the considerations above that

\[
|\hat{1}\rangle = |1\rangle, \\
|\hat{1}\rangle = |1\rangle + \frac{[56]}{[51]} [6], \\
|\hat{56}\rangle = |5\rangle + \frac{[16]}{[15]} [6].
\]

This entails the relations

\[
\langle 4 \hat{P}_{56} \rangle = \frac{\langle 4| P_{56}[1] \rangle}{[51]}, \\
\langle \hat{P}_{56} \hat{1} \rangle = \frac{P^2_{15} + P^2_{56} + P^2_{16}}{[15]}, \\
\langle 56 \rangle = \langle 56 \rangle,
\]

\[
\langle 6 \hat{P}_{56} \rangle = \langle 65 \rangle, \\
\langle \hat{P}_{56} 5 \rangle = \frac{[16]}{[15]} \langle 65 \rangle, \\
\langle \hat{12} \rangle = \frac{5| P_{16}[2] \rangle}{[51]}.
\]
Plugging these into (2) and simplifying terms we arrive at

\[
(2) = \frac{\langle 4|P_{56}|1 \rangle^3}{\langle 23|34|16|65|5|P_{16}|2 \rangle} \frac{1}{P_{15}^2 + P_{56}^2 + P_{16}^2}.
\]

(B.14)

Combining the contributions (1) and (2) we finally conclude

\[
A_{6}^{\text{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)
\]

\[
= \frac{\langle 6|P_{12}|3 \rangle^3}{\langle 61|12|34|45|5|P_{16}|2 \rangle} \frac{1}{P_{26}^2 + P_{12}^2 + P_{16}^2}
+ \frac{\langle 4|P_{56}|1 \rangle^3}{\langle 23|34|16|65|5|P_{16}|2 \rangle} \frac{1}{P_{15}^2 + P_{56}^2 + P_{16}^2}.
\]

(B.15)

Exercise 2.3

As suggested we consider the collinear limit in the (++)-channel with \( \lambda_5 = \sqrt{z} \lambda_p \) and \( \lambda_6 = \sqrt{1-z} \lambda_p \)

\[
A_{6}^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)
\]

\[
\xrightarrow{\lambda_5|6}
\]

\[
\frac{1}{\sqrt{z(1-z)^3}(56)} \frac{i}{\langle 12 \langle 23 \langle 34 \langle 4P \rangle \langle P1 \rangle \rangle^4}
\]

\[
\overset{!}{=} \text{Split}^{\text{tree}}_{\lambda} (\lambda, 5^+, 6^+) A_{5}^{\text{tree}} (1^-, 2^-, 3^+, 4^+, 5^+, 6^+)
+ \text{Split}^{\text{tree}}_{\lambda} (\lambda, 5^+, 6^+) A_{5}^{\text{tree}} (1^-, 2^-, 3^+, 4^+, 5^-, 6^-)
\]

By comparing the limiting expression on the top line to the terms of the lower lines we see that the last term is absent in the limiting expression. As

\[
A_{5}^{\text{tree}} (1^-, 2^-, 3^+, 4^+, 5^-, 6^-) \neq 0,
\]

we deduce that

\[
\text{Split}^{\text{tree}}_{\lambda} (\lambda, a^+, b^+) = 0,
\]

in agreement with Eq. (2.41).

Exercise 2.4

Taking the leg \( 5^- \) of Eq. (2.58) to the soft limit we have the reduced total momentum conservation condition \( p_1 + p_2 + p_3 + p_4 + p_6 = 0 \). We pull out the pole term \((|45||56|)^{-1}\) and find in the limit \( p_5 \to 0 \) using \( p_{15}^2 + p_{12}^2 + p_{16}^2 = p_{34}^2 \)

\[
A_{6}^{\text{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)
\]

\[
\xrightarrow{p_5 \to 0}
\]

\[
\frac{1}{[5|P_{16}|2|45|56]} \left( \frac{\langle 6|P_{12} \rangle[3|56]}{\langle 61|12|34|34|43 \rangle} + \frac{\langle 46 \rangle[61][54]}{\langle 23|34|16|16|61 \rangle} \right)
\]
Plugging this into the above we indeed find

\[ \frac{1}{[5|p_{16}|2][45][56]} \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 61 \rangle ([34][56]|23) + [61][54]|(13) \].

The two terms in the bracket may be simplified as follows

\[ (\cdots) = \frac{[56][43]}{[54][61]} (32) + [54][61]|(12) = [54][61]|p_1\langle 2 \rangle + [53][46]|(32)
\]

\[ = [46]|(54|42) + [53]|(32) = [46]|5|p_{34}|2 = -[46]|5|p_{16}|2 . \]

Plugging this into the above we indeed find

\[ A^{\text{tree}}_6(1^+, 2^+, 3^+, 4^-, 5^-) \xrightarrow{\gamma_2 \rightarrow 0} \frac{[46]}{[45][56]} \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 61 \rangle \langle 46 \rangle^3_3 = \text{Soft}(4, 5^-, 6) A^{\text{tree}}_5(1^+, 2^+, 3^+, 4^-, 6^-) . \]

which is the expected result consistent with factorization. As both terms in the result contribute.

**Exercise 2.5**

Let us consider the collinear limit \( 3^- \parallel 4^+ \) with

\[ \lambda_3 \rightarrow \sqrt{z} \lambda_P , \quad \lambda_4 \rightarrow \sqrt{1 - z} \lambda_P \]

of the quark-gluon amplitude \( A^{\text{tree}}_5(1^-_q, 2^+_q, 3^-, 4^+, 5^+) \) to wit

\[ A^{\text{tree}}_5(1^-_q, 2^+_q, 3^-, 4^+, 5^+) \xrightarrow{\gamma_3 \parallel 4^+} \text{Split}_+^{\text{tree}}(z, 3^-, 4^+) A^{\text{tree}}_4(1^-_q, 2^+_q, P^-, 5^+)
\]

\[ + \text{Split}_-^{\text{tree}}(z, 3^-, 4^+) A^{\text{tree}}_4(1^-_q, 2^+_q, P^+, 5^+) \xrightarrow{\gamma_3 \parallel 4^+} = 0 \]

\[ \frac{z^2}{\sqrt{z(1 - z)}} \frac{\langle P \rangle^3 \langle 2P \rangle}{\langle 12 \rangle \langle 2P \rangle \langle P5 \rangle \langle 51 \rangle} . \]

(B.16)

where the 4-point amplitude of Eq. (2.62) was inserted. The limiting form of Eq. (B.16) suggests the original amplitude before the limit to take the form

\[ A^{\text{tree}}_5(1^-_q, 2^+_q, 3^-, 4^+, 5^+) = \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} . \]

In fact the form above lets one conjecture the multiplicity \( n \) form

\[ A^{\text{tree}}_n(1^-_q, 2^+_q, 3^-, \ldots, n^+) = \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \cdots \langle n1 \rangle} . \]

(B.17)

By analogy to Eq. (B.16) one easily convinces oneself that the conjectured form of the \( n \)-point amplitude Eq. (B.17) is consistent with the collinear \( 3^- \parallel 4^+ \) and \( i^+ \parallel (i + 1)^+ \) for \( i = 4, \ldots, n - 1 \) limits. Let us also study two soft limits of Eq. (B.17).
First we take $\lambda_3 \to 0$. Then we immediately see that
\[
A_n^{\text{tree}}(1_{\bar{q}}^-, 2^+_q, 3^-, 4^+, \ldots, n^+) \quad \xrightarrow{3^- \to 0} \quad 0 = \text{Soft}^{\text{tree}}(2, 3^-, 4) A_n^{\text{tree}}(1_{\bar{q}}^-, 2^+_q, 4^+, \ldots, n^+).
\]
This results implies that
\[
A_n^{\text{tree}}(1_{\bar{q}}^-, 2^+_q, 3^+, \ldots, n^+) = 0, \quad \text{(B.18)}
\]
being consistent with Eq. (2.63). Taking the soft limit $4^+ \to 0$ (or any other positive helicity gluon leg) on the other hand again checks the consistency of Eq. (B.17)
\[
A_n^{\text{tree}}(1_{\bar{q}}^-, 2^+_q, 3^-, 4^+, \ldots, n^+) \quad \xrightarrow{4^+ \to 0} \quad \langle 35 \rangle \langle 34 \rangle \langle 45 \rangle \xrightarrow{\text{Soft}^{\text{tree}}(3, 4^+, 5)} A_n^{\text{tree}}(1_{\bar{q}}^-, 2^+_q, 3^-, 5^+, \ldots, n^+).
\]

**Exercise 2.6**

Using the on-shell recursion and the form of the three-point scalar-gluon amplitudes of Sect. 2.5 one has
\[
A_4(1^+, 2\phi, 3\bar{\phi}, 4^-) = A_3(\hat{1}^+, 2\phi, -\hat{P}\phi) \frac{1}{p^2 - m^2} A_3(\hat{P}\phi, 3\bar{\phi}, 4^-)
= -\frac{\langle q_1|\hat{P}|\hat{1} \rangle}{\langle q_1|\hat{1} \rangle} \frac{1}{p^2 - m^2} \frac{\langle \hat{4}|p_3|q_2 \rangle}{[\hat{4} q_2]}
\]
With the gauge choice $q_1 = \hat{p}_4$ and $q_2 = \hat{p}_1$ along with the identities $[\hat{4}] = [4]$ and $[\hat{1}] = [1]$ one has
\[
\langle q_1|\hat{1} \rangle = 0, \quad [\hat{4}] = [41] = [41],
\langle q_1|\hat{P}|\hat{1} \rangle = [\hat{4}] [\hat{P}|1] = -\langle 4|p_3 + \hat{p}_4|1 \rangle = -\langle 4|p_3|1 \rangle,
\langle \hat{4}|p_3|q_2 \rangle = [\hat{4}] [p_3|\hat{1} \rangle = [4|p_3|1 \rangle.
\]
Plugging these into the above yields the final compact result
\[
A_4(1^+, 2\phi, 3\bar{\phi}, 4^-) = -\frac{\langle 4|p_3|1 \rangle^2}{(41)[14][(p_1 + p_2)^2 - m^2]}.
\]

**Exercise 2.7**

It suffices to consider only the single-particle representation quoted in Eq. (2.105). The commutation relations with $d$
\[
[d, p^{\alpha\dot{\alpha}}] = p^{\alpha\dot{\alpha}}, \quad [d, k_{\alpha\hat{\alpha}}] = k_{\alpha\hat{\alpha}}, \quad [d, m_{\alpha\beta}] = 0 = [d, \bar{m}_{\hat{\alpha}\hat{\beta}}].
\]
are manifest from simple counting by noting the commutators \([d, \lambda^a] = + \lambda^a\) and \([d, \partial_a] = - \partial_a\). It remains to compute the commutator \([k_{a\dot{a}}, \rho^{\dot{\beta}}\dot{\gamma}]\). One easily establishes
\[
[k_{a\dot{a}}, \rho^{\dot{\beta}}\dot{\gamma}] = [\partial_a, \lambda^\beta \lambda^\dot{\beta}] \partial_{\dot{a}} + \partial_{\dot{a}}[\partial_a, \lambda^\beta \lambda^\dot{\beta}] = \delta^\beta_{\dot{a}} \lambda^\dot{\beta} \partial_\dot{a} + \delta^\dot{\beta}_a \lambda^\beta \partial_a + \delta^\beta_{\dot{a}} \delta^\dot{\beta}_a.
\]

Using Eq. (2.98) with raised index
\[
\varepsilon^{\alpha \Gamma} \lambda_{\Gamma \partial} \partial_{\beta} = \varepsilon^{\alpha \Gamma} \lambda_{(\alpha} \partial_{\beta \Gamma \partial)} + \frac{1}{2} \varepsilon^{\alpha \Gamma} \varepsilon_{\Gamma \partial} \lambda_{\beta} \partial_{\Gamma \partial},
\]
and the sister equation with dotted indices one easily concludes
\[
[k_{a\dot{a}}, \rho^{\dot{\beta}}\dot{\gamma}] = \delta^\beta_{\dot{a}} \varepsilon^{\dot{\beta} \dot{\gamma} \Gamma} \partial_{\dot{a}} \Gamma + \delta^\dot{\beta}_a \varepsilon^{\beta \gamma \Gamma} m_{\alpha \Gamma} + \delta^\beta_{\dot{a}} \delta^\dot{\beta}_a \left( \frac{1}{2} \lambda_{\Gamma \partial} \partial_{\Gamma \partial} + \frac{1}{2} \lambda_{\dot{\gamma} \partial} \partial_{\dot{\gamma} \partial} + 1 \right),
\]
which proves Eq. (2.103).

**Exercise 2.8**

Using the inversion transformation \(I \cdot x^\mu = \frac{x^\mu}{x^2}\) and the translation transformation \(P^\mu \cdot x^\mu = x^\mu - a^\mu\) we have with \(K^\mu = I \cdot P^\mu \cdot I\)
\[
K^\mu \cdot x^\mu = I \cdot P^\mu \cdot \frac{x^\mu}{x^2} = I \cdot \frac{x^\mu - a^\mu}{(x - a)^2} = \frac{x^\mu - a^\mu}{x^2 - a^2} = \frac{x^\mu - a^\mu}{(x^\mu - a^\mu)^2} x^2 \frac{x^2}{1 - 2 a \cdot x + a^2 x^2},
\]
as claimed.

**Exercise 2.9**

(a) We start with \(\delta^{(2)}(\lambda^\alpha a + \mu^a b)\) for Grassmann even \(a\) and \(b\). Then we have
\[
\delta^{(2)}(\lambda^\alpha a + \mu^a b) = \delta(\lambda^1 a + \mu^1 b) \delta(\lambda^2 a + \mu^2 b) = \frac{\delta(a + \mu^1 b)}{|\lambda^1|} \delta \left( b \left( \mu^2 - \frac{\mu^1 \lambda^2}{\lambda^1} \right) \right) = \frac{\delta(a) \delta(b)}{|\lambda^1| |\mu^2 - \mu^1 \lambda^2| / |\lambda^1|}, \tag{B.19}
\]
where we used \(|\lambda^1 \mu^2 - \mu^1 \lambda^2| = |(\lambda, \mu)|\) in the last step.

(b) In the fermionic case one arrives at the result by simple multiplication
\[
\delta^{(2)}(\lambda^\alpha a + \mu^a b) = (\lambda^1 a + \mu^1 b)(\lambda^2 a + \mu^2 b) = ab(\lambda^1 \mu^2 - \lambda^2 \mu^1) = \delta(a) \delta(b) \langle \lambda, \mu \rangle.
\]
(c) Finally we turn to the evaluation of the fermionic delta-function Eq. (2.153). For the three-point case we have

$$\delta^{(8)}(q^\alpha A) = \prod_{A=1}^{4} \delta^{(2)}(\lambda_1^\alpha \eta_1^A + \lambda_2^\alpha \eta_2^A + \lambda_3^\alpha \eta_3^A).$$  \hspace{1cm} (B.20)$$

The spinor $\lambda_3^\alpha$ may be expressed in the basis of the non-collinearly assumed $\lambda_1^\alpha$ and $\lambda_2^\alpha$:

$$\lambda_3^\alpha = x_1 \lambda_1^\alpha + x_2 \lambda_2^\alpha \Rightarrow \langle 13 \rangle = x_2 \langle 12 \rangle, \quad \langle 23 \rangle = x_1 \langle 21 \rangle.$$ 

Inserting this into Eq. (B.20) we obtain

$$\delta^{(8)}(q^\alpha A) = \prod_{A=1}^{4} \delta^{(2)}\left(\lambda_1^\alpha \left(\eta_1^A + \langle 23 \rangle \langle 21 \rangle \eta_3^A \right) + \lambda_2^\alpha \left(\eta_2^A + \langle 31 \rangle \langle 21 \rangle \eta_3^A \right)\right) = \langle 12 \rangle^{4} \delta^{(4)}(\eta_1^A + \langle 23 \rangle \langle 21 \rangle \eta_3^A) \delta^{(4)}(\eta_2^A + \langle 31 \rangle \langle 21 \rangle \eta_3^A),$$

where we have used Eq. (B.19) in the last step.

**Exercise 2.10**

The generalization of the discussion in Sect. 2.7.6 to the $n$-point case proceed in great analogy. We start with the recursion formula of Eq. (2.144) for $p = 0$

$$A_n^{\text{MHV}} = \int \frac{d^4 \eta_P}{P^2} A_3^{\text{MHV}}(z_P) A_{n-1}^{\text{MHV}}(z_P)$$

$$= - \int \frac{d^4 \eta_P}{P^2} \left( \delta^{(4)}(\eta_1[2, \hat{P}] + \eta_2[\hat{P} 1] + \eta_P[12]) \delta^{(8)}(\hat{\lambda}_P \eta_P + \lambda_3 \eta_3 + \cdots + \lambda_n \hat{n}_n) \right)$$

$$\times \frac{1}{[12][2 \hat{P}][\hat{P} 1](\hat{P} 3)(34) \cdots \langle n - 1n \rangle \langle n \hat{P} \rangle},$$

where we assume the super-MHV formula to hold for $(n - 1)$-points. Localizing $\eta_P$ through the fermionic $\delta^{(4)}$-function as in Eq. (2.159) and inserting this into the remaining fermioniv $\delta^{(8)}$ function yields the total supermomentum conservation in analogy to Eq. (2.160)

$$\delta^{(8)}\left(- \frac{\lambda_P}{[12]}(\eta_1[2 \hat{P}] + \eta_2[\hat{P} 1] + \lambda_3 \eta_3 + \cdots + \lambda_n \hat{n}_n)\right)$$

$$= \delta^{(8)}\left(- \frac{\hat{n}_1 \eta_1[21]}{[12]} + \lambda_2 \eta_2[21] + \lambda_3 \eta_3 + \cdots + \lambda_n \hat{n}_n \right) = \delta^{(8)}(q).$$
Then all that remains is the consideration of the bosonic factors in Eq. (B.21). We have with \( \tilde{P} = \hat{P}_1 + p_2 \)

\[
- \frac{1}{\tilde{P}^2} [12]^4 \frac{1}{[12][2\tilde{P}][\hat{P}_1][\hat{P}3](34) \cdots (n-1n)\langle n\hat{P} \rangle} \\
= [21](23) \frac{1}{\langle n1 \rangle(12)(23)(34) \cdots (n-1n)}.
\]

which completes the proof.

**Exercise 3.1**

(a) We begin with the computation of \( d_{23} \). Using the labeling of Eq. (3.157) the three on-shell constraints for \( l_4, l_3 \) and \( l_5 \) are

\[
l_4^2 = 0, \quad l_3^2 = (l_4 - p_4)^2 = 0, \quad l_5^2 = (l_4 + p_5)^2 = 0
\]

in combination with the MHV3 and \( \text{MHV}_3 \) to the left and right of \( l_4 \) stating that \( \lambda_{l_4} \sim \lambda_4 \) and \( \tilde{\lambda}_{l_4} \sim \tilde{\lambda}_5 \) we then find the solution to the above three conditions in the form

\[
l_{4}^{\alpha \dot{\alpha}} = \xi \lambda_{4}^{\alpha} \hat{\lambda}_{5}^{\dot{\alpha}}.
\]

The remaining constraint \( l_1 \bigp{2} = (l_4 + p_1 + p_5)^2 \) then fixes the constant \( \xi \) to the value \( \xi = \langle 15 \rangle / \langle 41 \rangle \). Again there is only one consistent solution to the quadruple cut conditions due to the three-point kinematics. For the coefficient \( d_{23} \) we have the product of the MHV4 amplitude with two \( \text{MHV}_3 \) and one \( \text{MHV}_3 \) amplitudes

\[
d_{23} = \frac{1}{2} \frac{\langle 12 \rangle^3}{\langle 23 \rangle(3l_3)(l_3l_1)} \frac{[4l_4]^3}{[l_4l_3][l_34]} \frac{\langle l_3l_5 \rangle^3}{(l_45)(5l_5)} \frac{\langle l_1l_5 \rangle^3}{[l_51][l_1]} \\
= \frac{1}{2} \frac{[4][l_4l_5l_1][2]^3}{\langle 23 \rangle(5l_51)[3][3][4][5][l_4l_3l_1]}.
\]  

(B.22)

Using the relations \( l_3 = l_4 - p_4, l_5 = l_4 + p_5 \) and \( l_1 = l_4 + p_1 + p_5 \) one computes

\[
[4][l_4l_5l_1][2] = -s_{51}s_{45} \langle 12 \rangle \langle 41 \rangle, \quad \langle 5l_51 \rangle = s_{51} \langle 15 \rangle \langle 41 \rangle, \\
\langle 3][4]\rangle = \frac{\langle 15 \rangle}{\langle 41 \rangle} \langle 34 \rangle[54], \quad \langle 5l_4l_3l_1 \rangle = s_{51}s_{45} \langle 45 \rangle \langle 41 \rangle.
\]

Plugging these results into Eq. (B.22) yields

\[
d_{23} = -\frac{1}{2} s_{51}s_{45} \frac{\langle 12 \rangle^3}{\langle 23 \rangle(34)(45)(51)} = \frac{i}{2} s_{51}s_{45} A_{5}^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+),
\]

as claimed in Eq. (3.157).
(b) For the evaluation of $d_{34}$ using the labeling of Eq. (3.157) the three on-shell constraints for $l_1, l_5$ and $l_2$ read

$$l_1^2 = 0, \quad l_5^2 = (l_1 - p_1)^2 = 0, \quad l_2^2 = (l_1 + p_2)^2 = 0,$$

together with the three-vertex kinematics based conditions $\lambda_{l_1} \sim \lambda_2$ and $\bar{\lambda}_{l_1} \sim \bar{\lambda}_1$ implies the ansatz for $l_1$

$$l_1^{\alpha \dot{\alpha}} = \xi \lambda_2 \bar{\lambda}_1.$$

The remaining condition $l_2^2 = (l_1 - p_1 - p_5)^2 = 0$ is obeyed for $\xi = \langle 15 \rangle / \langle 25 \rangle$. The coefficient $d_{34}$ then follows from the product of the four tree-level amplitudes depicted in Eq. (3.157)

$$d_{34} = \frac{1}{2} \frac{\langle l_4 l_2 \rangle^3}{\langle l_23 \rangle \langle 34 \rangle \langle 4l_4 \rangle \langle 5l_5 \rangle} \frac{[l_4 5]^3}{[l_5 4][l_5 4]} \frac{\langle 1l_1 \rangle}{\langle l_1 5 \rangle \langle l_5 1 \rangle} \frac{[l_2 1]^3}{[l_2 1][2l_2]}$$

$$= \frac{1}{2} \frac{\langle 1l_1 l_2 4 \rangle}{\langle 34 \rangle \langle 4l_4 l_5 l_1 \rangle} \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle.$$

Evaluating this using $l_5 = l_1 - p_1, l_2 = l_1 + p_2$ and $l_4 = l_1 - p_1 - p_5$ one has

$$\langle 1l_1 l_2 4 \rangle = s_{12} s_{51} \langle 12 \rangle / \langle 25 \rangle, \quad \langle 4l_4 l_5 l_1 \rangle = s_{12} s_{51} \langle 45 \rangle / \langle 25 \rangle,$$

$$\langle 3l_2 2 \rangle = \langle 15 \rangle / \langle 25 \rangle (32)[12], \quad \langle 1l_5 5 \rangle = - \langle 12 \rangle / \langle 25 \rangle s_{51}.$$

Plugging this into Eq. (B.23) one finds

$$d_{34} = -\frac{1}{2} s_{12} s_{51} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} = \frac{i}{2} s_{12} s_{51} A_5^{\text{tree}} (1^-, 2^-, 3^+, 4^+, 5^+).$$

This proves the result quoted in Eq. (3.157).

**Exercise 3.2**

(a) $F_{4,m}$ can only depend on the Lorentz invariants $s, t, m^2$. Moreover, it has mass dimension zero, therefore it can only depend on dimensionless variables, hence

$$F_{4,m} = F_{4,m}(-m^2/s, -m^2/t).$$

(b) Switching to dual coordinates, this becomes

$$F_{4,m} = \chi_{13}^2 \chi_{24}^2 \int \frac{d^4x_0}{i\pi^2} \prod_{j=1}^n \frac{1}{-x_{0j}^2 + m^2}.$$
Equation (3.199) gives the following Feynman parametrization

\[ F_{4,m} = x_{13}^2 x_{24}^2 \int_0^\infty \prod_{i=1}^4 d\alpha_i \frac{\delta(c_i\alpha_i - 1)}{[-(\alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_4 x_{24}^2) + (\sum_{i=1}^4 \alpha_i)^2 m^2]^2}. \]

We can choose \( c_i = 1 \) in order to simplify the mass term.

(c) One then sees that two Mellin-Barnes parameters are sufficient to factorize the integrand. Indeed,

\[
(a + b + c)^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int \frac{dz_1 dz_2}{(2\pi i)^2} \Gamma(-z_1) \Gamma(-z_2) \Gamma(\lambda + z_1 + z_2) \alpha_{z_1} \beta_{z_2} \gamma^{-\lambda-z_1-z_2}. 
\]

Carrying out the \( \alpha \) integrals using Eq. (3.202) one finds

\[
F_{4,m} = \int \frac{dz_{1,2}}{(2\pi i)^2} \left( \frac{-s}{m^2} \right)^{1+z_1} \left( \frac{-t}{m^2} \right)^{1+z_2} \times \Gamma(-z_1) \Gamma(-z_2) \Gamma(2 + z_1 + z_2) \frac{\Gamma^2(1 + z_1) \Gamma^2(1 + z_2)}{\Gamma(4 + 2z_1 + 2z_2)}. 
\]

(d) Here the real part of the integration variables is to be chosen such that

\[-1 < \text{Re}(z_i) < 0,\]

which assures that all intermediate steps leading to this expression are well-defined.

(e) We have \( F_{4,m} \propto \int dz_i (m^2)^{-2-z_1-z_2} \). In order to take the limit \( m \to 0 \), one needs to analytically continue in the \( z_i \) variables until the exponent of \( m^2 \) becomes positive. (The latter terms vanish as \( m \to 0 \).) This can be done either by hand, or using the Mathematica implementation \textit{MBasymptotics.m}. The result is

\[
F_{4,m} = 2 \log(-m^2/s) \log(-m^2/t) - \pi^2 + O(m^2),
\]

or, equivalently,

\[
F_{4,m} = \log^2 \left( \frac{m^2}{-s} \right) + \log^2 \left( \frac{m^2}{-t} \right) - \log^2 \left( \frac{-s}{-t} \right) - \pi^2 + O(m^2).
\]

**Exercise 3.3**

(a) Higher orders in \( \varepsilon \). This problem is solved in the main text in Sect. 3.8.3.
(b) Massive triangle integral. We start by generalizing the integral to arbitrary powers of the propagators,

\[ F_{3,m}(a_1, a_2, a_3) := \int \frac{d^Dk}{i\pi^{D/2}} \frac{1}{(-k^2 + m^2)^{a_2}(-(k + p_1)^2 + m^2)^{a_1}(-(k + p_2)^2 + m^2)^{a_3}}. \]

Moreover, it is convenient to use the notation

\[ 1^\pm F_{3,m}(a_1, a_2, a_3) = F_{3,m}(a_1 \pm 1, a_2, a_3), \]

and similarly for 2^\pm and 3^\pm. Then, we derive IBP identities \(0 = \int \partial_k \mu \, v^\mu\), with \(v^\mu = k^\mu\). (Other equations can be derived for \(v^\mu = p_1^\mu\) or \(v^\mu = p_2^\mu\), but we will not need them here.) Specifying to \(a_i = 1\) and \(D = 4\), we have

\[ 0 = [2m^2(1^+ + 2^+ + 3^+) - 1^+2^- + 3^+2^-]1^\pm F_{3,m}(1, 1, 1). \]

On the other hand, differentiating w.r.t. \(m^2\), we have

\[ m^2\partial_{m^2} F_{3,m}(1, 1, 1) = -m^2(1^+ + 2^+ + 3^+). \]

Combining these two equations, we find a differential equation for \(F_{3,m}\) in terms of the bubble integral \(J(2, 1)\) considered in the main text, with \(q^2 = s\),

\[ m^2\partial_{m^2} F_{3,m}(1, 1, 1) = -J(2, 1). \]

For dimensional reasons, we have \(F_{3,m}(1, 1, 1) = 1/s f_{3,m}(y)\), where we introduced the variable \(y = -m^2/s\). Using Eq. (3.233), we find

\[ y \partial_y f_{3,m}(y) = -\frac{1}{\sqrt{1 + 4y}} \log \left( \frac{\sqrt{1 + 4y} - 1}{\sqrt{1 + 4y} + 1} \right), \]

with the boundary condition \(f_3(\infty) = 0\). Integrating this equation leads us to Eq. (3.235).

(c) Finite propagator integral from the IBP relations for the triangle subintegral it follows that one can rewrite the original integral \(Q\) in terms of simple one-loop propagator integrals. Up to a trivial scale dependence, the latter give rise to factors

\[ B(a_1, a_2; D) = \frac{\Gamma(a_1 + a_2 - D/2) \Gamma(D/2 - a_2) \Gamma(D/2 - a_1)}{\Gamma(D - a_1 - a_2) \Gamma(a_1) \Gamma(a_2)}. \]

We have

\[ Q(1, 1, 1, 1; D) = \frac{-2}{D-4} B(1, 1; D) \left[ B(3 - D/2, 2; D) - B(1, 2; D) \right] \frac{1}{(-q^2)^{5-D}}. \]
The occurring \( \Gamma \) functions can be expanded in \( \varepsilon \), cf. Eq. (3.204). We find

\[
Q(1, 1, 1, 1; 4 - 2\varepsilon) = 6\xi_3 \frac{1}{-q^2} + \mathcal{O}(\varepsilon),
\]

and higher orders in \( \varepsilon \) can also be generated without difficulty.

**Exercise 4.1**

(a) Here we verify that \( F_4 \) of Eq. (4.29) satisfies the conformal Ward identity (4.27). We start by showing that

\[
K^\mu x_{ab}^2 = 2(x_\mu^a + x_\mu^b)x_{ab}^2,
\]

and hence

\[
K^\mu \log x_{ab}^2 = 2(x_\mu^a + x_\mu^b).
\]

From this it follows that

\[
K^\mu F_4 = \Gamma_{\text{cusp}}(a)(x_\mu^1 + x_\mu^3 - x_\mu^2 - x_\mu^4) \log \left( \frac{x_{13}^2}{x_{24}^2} \right),
\]

in complete agreement with Eq. (4.27).

(b) A particular solution to the Ward identities (4.27) was given in Ref. [1]. For four and five points, the homogeneous solution of the differential equation is a constant, i.e. no conformal invariants can be built from four or five light-like separated points. For more points, one can build conformal cross-ratios, see Eq. (4.30). Using Eq. (B.24) it is easy to see that these are indeed invariant,

\[
K^\mu \frac{x_{ij}x_{kl}}{x_{ik}^2 x_{jl}^2} = 0.
\]

Another way to see the invariance is to perform inversions. At six points, there are three independent cross-ratios \( u, v, w \), see Eq. (4.31), and therefore the homogeneous solution is an a priori unconstrained function of \( u, v, w \) (and the coupling).

**References**