Picturing perturbative parton cascades in QCD matter

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A B S T R A C T

Based on parametric reasoning, we provide a simple dynamical picture of how a perturbative parton cascade, in interaction with a QCD medium, fills phase space as a function of time.

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1. Introduction

There are essentially two motivations for studying how a perturbative parton shower, embedded in QCD matter, evolves on all energy scales and on all angular scales. First, this process has been recognized since long as a useful set-up for elaborating the dynamics of thermalization in QCD [1]. This is so, since the parton distribution characterizing a jet is initially far from equilibrium, but in a thermal QCD medium it will evolve at late times into a distribution that is indistinguishable from a thermal one. In this context, recent detailed analyses suggest for instance that the thermalization time obtained from a weak coupling treatment [2–4] is comparable to that established for non-abelian gauge theories in the strong coupling limit [5–8]. Secondly, studying the medium-modifications of perturbative parton showers is motivated, of course, by the recent measurements of reconstructed jets in heavy ion collisions at the LHC [9–16] that characterize in unprecedented detail the medium-induced redistribution of jet energy and jet quanta in longitudinal (i.e. along the jet axis) and transverse (i.e. orthogonal to the jet axis) phase space. While it is conceivable that strong coupling techniques are needed to explain these data [17], the observation of modified but vacuum-like fragmentation patterns [11,12] in high energy jets at the LHC gives support to approaches that formulate medium effects within a perturbative framework [18]. Recent approaches extend the early perturbative formulation of medium-induced parton energy loss [19–25] in particular by better analyzing the role of color coherence and transverse broadening in the evolution of the medium-modified parton cascade [26–34], and by developing full Monte Carlo models for medium-modified parton showers [35–37].

Remarkably, the recent developments towards a jet quenching phenomenology applicable to all jet energy and angular scales, as well as the recent studies of the conceptually related thermalization problem, have identified largely independent of each other a set of parametric momentum scales and angular scales that characterize different aspects of the jet quenching phenomenon. The present work grew out of our attempt to combine, what is parametrically known from these studies, into a simple dynamical picture of how a perturbative parton cascade embedded in thermal QCD matter fills phase space as a function of time. In elaborating this picture, we came across parametric estimates of physics effects that we had not seen discussed before, such as parametric estimates for the interplay between vacuum and medium-induced radiation, for the medium-induced cascading of DGLAP vacuum radiation to lower momentum scales, for the late time (i.e. after the jet has left the medium) vacuum cascading of medium-induced radiation, and for the angular scales at which rare large angle scattering dominates over multiple soft angle scattering. Including these effects, we provide in Section 2 a simple, kinematically complete and parametrically correct picture of the evolution of a perturbative parton cascade in QCD matter. We then illustrate the use of this picture by relating parametric estimates of the jet energy fraction of different kinematical regions to characteristic features in the measurements of quenched jets.

2. Jet evolution in the log p–log θ-plane

In this section, we motivate Fig. 1 that illustrates what happens parametrically in perturbation theory when a jet propagates through a thermal cloud of temperature T. We consider a set-up in which, at very early time t = 1/Q ~ 0, a hard parton distribution \( \delta(\theta) \delta(p - Q) \) is embedded in the thermal bath. This ‘jet’ is localized in angle \( \theta \), and it lives at time t = 0 on a momentum scale...
one finds with a probability $O(\alpha_s)$ per logarithmic phase space quanta due to vacuum radiation,
\begin{equation}
\frac{dP_{\text{find}}}{d \log p \ d \log \theta} \sim \alpha_s.
\end{equation}

The region where this primary splitting gives the dominant contribution in the log $p$–log $\theta$–plane is marked as the DGLAP-region in Fig. 1.

2.2. LPM region

In contrast, medium-induced parton branching fills the log $p$–log $\theta$–plane from the bottom up (in $p$) and from the inside out (in $\theta$) [19]. This is so since transverse momentum is acquired by Brownian motion in the medium, $k_{t\perp}^2 \sim \langle q t \rangle$; the formation time constraint $t > p/k_{t\perp}^2 \sim p/\langle q t \rangle$ implies then that medium-induced quanta can be formed in the region $p < k_{\text{form}}$ where
\begin{equation}
k_{\text{form}}(t) \equiv \langle q t \rangle^2.
\end{equation}

or, alternatively, that quanta at a scale $p$ can be formed at times $t > t_{\text{form}}(p)$ where
\begin{equation}
t_{\text{form}}(p) \equiv \sqrt{\frac{p}{\langle q \rangle^2}}.
\end{equation}

These quanta are created at small angles $\theta \ll \sqrt{\frac{\langle q \rangle^2}{p}}$, and to our purposes we can treat the emitted quanta as being collinear with respect to the emitter. Their angular distribution will be determined by reinteractions with the medium, discussed in Section 2.6.

A quantum can be formed at the scale $k_{\text{form}}$, but in a weakly coupled theory, it is formed only with a probability $\alpha_s$. Therefore, at the scale $k_{\text{form}}$, there are $O(\alpha_s)$ quanta per logarithmic phase space due to medium-induced parton branching. At scales below $k_{\text{form}}$ (denoted as the LPM-region in Fig. 1) the formation time is faster $t_{\text{form}} < t$, and as a result of this the medium-induced splittings become more and more abundant as one moves from the scale $k_{\text{form}}$ to an increasingly softer scale $p < k_{\text{form}}$. There is an $O(\alpha_s)$ probability of emitting a splittee at the scale $p$ at every $t_{\text{form}}(p)$ and thus the probability of finding a splittee with a momentum $p < k_{\text{form}}$ is parametrically
\begin{equation}
\frac{dP_{\text{find}}(t)}{d \log p} \sim \alpha_s \frac{t}{t_{\text{form}}(p)} \sim \alpha_s \langle q t \rangle^{1/2} p^{-1/2} t.
\end{equation}

While $t_{\text{form}}(p)$ determines the minimal duration for a quantum to be created with probability $O(\alpha_s)$, the parametrically longer time
\begin{equation}
t_{\text{split}}(p) \sim t_{\text{form}}(p)/\alpha_s
\end{equation}
is needed to create this quantum with probability $\sim 1$. At fixed time $t$, the quanta that are created thus with $O(1)$ probability live at the scale $p < k_{\text{split}}$
\begin{equation}
k_{\text{split}}(t) \sim \alpha_s^2 k_{\text{form}}(t) \sim \alpha_s^2 \langle q t \rangle^2,
\end{equation}
which marks the end of the LPM-region in Fig. 1.

\footnote{The angle at which a quantum is created is $\theta^2 \sim \langle q t \rangle^2$, and which for a perturbative medium $\sim \langle q t \rangle$ reads $\theta^2 \sim \alpha_s(T/p)^{3/2}$.}

\footnote{At leading order, for $p < Q$, the prefactor can be extracted, e.g., from the splitting function of [138], and in numerical form the spectrum reads
\begin{equation}
\frac{dP_{\text{find}}(t)}{d \log p} \approx \frac{1}{\pi} C_2 \alpha_s p^{-1/2} \langle q t \rangle^{1/2} (p/t).
\end{equation}}
2.3. Medium cascade

Not all quanta that are created will stay where they were created. Those modes that have time to lose a significant fraction of their energy will cascade to a significantly lower scale $p$. For LPM-type radiation, the splitting that degrades energy the most is the hardest splitting and the hardest splitting available is the quasi-democratic one.\(^3\)

The timescale for a quasi-democratic splitting is of the same order of magnitude as the splitting time $t_{\text{split}}(p)$ at the same scale. This is so because, in a non-abelian theory, the parametric emission time for LPM-type radiation is independent of the momentum of the parent and is set by the momentum of the softer splitter, which for a quasi-democratic splitting is parametrically of the same order as the parent momentum.

A quasi-democratic splitting moves the energy deposited at scale $p$ to a lower scale of $p/2$. Parametrically the lower scale, $p/2$, is of the same order as the original $p$, and therefore even a democratic splitting is a local process in the logarithmic $p$-space. However, if a given mode has had time to undergo a quasi-democratic splitting, then a successive quasi-democratic splitting of its daughters will take place on a shorter time scale. Therefore, the scales which have had time to undergo a quasi-democratic splitting with $O(1)$ probability can cascade all the way to the scale $T$ of the thermal bath within the same epoch $t$ within which the first democratic splitting occurred [25,31] in the context of thermalization, the analogous process was discussed in [1–3].

To quantify these considerations, we introduce the time $t_{\text{res}}(p)$ that a quantum resides at scale $p$ before splitting further. For $p > k_{\text{split}}$, this residence time must equal the lifetime $t$ of the system, and for lower scales $p$ it must scale like the formation time. We therefore have

$$t_{\text{res}}(p) \sim \begin{cases} t, & \text{for } k_{\text{form}} > p > k_{\text{split}}, \\ \frac{p}{k_{\text{split}}} t, & \text{for } p < k_{\text{split}}. \end{cases}$$

(9)

The energy $\epsilon$ in this cascade is dominated by the hardest scale that can cascade, $\epsilon = m(k_{\text{split}})k_{\text{split}} \approx k_{\text{split}}$. This energy will move through all scales $p$ down to $T$ via quasi-democratic splittings, and since quanta spend a shorter time $t_{\text{res}}(p)/t$ at lower momentum scale, only the fraction $t_{\text{res}}(p)/t$ of quanta that arrived within the residence time will not have left already and will therefore contribute to the energy at the scale $p < k_{\text{split}}(p)$.

$$\frac{d\epsilon}{d\log p} = p \frac{d\alpha}{d\log p} \sim \frac{k_{\text{split}}}{t} t_{\text{res}}(p) \sim \alpha_s \sqrt{q} p t.$$  

(10)

Here, the middle equality is obviously true for $p = k_{\text{split}}$, and – since the entire energy contained in quanta at scale $k_{\text{split}}$ will flow to the scale $T$ in the same epoch – the $p$-dependence for $p < k_{\text{split}}$ is set entirely by the residence time at the lower scale $p$.

By coincidence, the energy distribution (10) in the region of the medium cascade matches the distribution in the LPM region since $k_{\text{split}} \sim \frac{\log p}{t} \sim \frac{t}{t_{\text{split}}(p)}$, so that

$$\frac{d\epsilon}{d\log p} \sim \frac{p}{t} t_{\text{split}}(p).$$

(11)

Accordingly, also the number of quanta per log $p$ shows the same $1/\sqrt{p}$-dependence as in the LPM region,

$$\frac{dn}{d\log p} = \frac{1}{p} \frac{d\epsilon}{d\log p} \sim \alpha_s \sqrt{q} t/\sqrt{p}.$$  

(12)

This similarity is particular to the $\sqrt{p}$ power law. We emphasize that despite these similarities, the physics in the region of the medium cascade and in the LPM region are somewhat different.

2.4. Medium cascade of the vacuum quanta

The quanta produced in the DGLAP region will also undergo medium interactions that will make the quanta radiate and split. The distribution of radiation from the vacuum quanta is the same as from any other mode. Therefore the distribution of the daughters originating from the vacuum quanta above $k_{\text{split}}$ is

$$\frac{dP_{\text{find}}}{d\log p d\log \theta} \sim \alpha_s \frac{t}{t_{\text{split}}(p)},$$  

(13)

where $\alpha_s$ is just the number of vacuum quanta per double logarithmic unit of phase space. Again, the ratio $t/t_{\text{split}}(p)$ is smaller than 1 for modes above $k_{\text{split}}$, and therefore the number of daughters is smaller than the number of vacuum split quanta.

Below $k_{\text{split}}$, however, both $t/t_{\text{split}}(p) > 1$ and $t/t_{\text{res}}(p) > 1$, leading to a cascade that is similar to the medium cascade discussed above. With the same arguments the number of quanta becomes

$$\frac{dn}{d\log p d\log \theta} \sim \alpha_s \frac{t}{t_{\text{split}}(p)} \quad \text{for } p < k_{\text{split}}(p).$$

(14)

This contribution dominates the distribution in the triangular region at large angles in Fig. 1 marked as “Medium cascade of vacuum induced quanta”.

2.5. Absence of Bethe–Heitler region

We have argued that from $k_{\text{form}}$ down to the momentum scale $p$, particle radiation is LPM suppressed. Given that formation times decrease with decreasing $p$, one may wonder why there is not a softer momentum scale at which individual quanta in the heat bath are resolved and an un-suppressed Bethe–Heitler radiation pattern $n(p) \propto 1/p$ off individual scattering centers results. To discuss this possibility, we need to introduce microscopic characteristics of the perturbative medium that we have in mind. The small angle scattering time in this medium will be $t_{\text{scatt}} \sim 1/\alpha_s T$, and $q \sim \alpha_s^2 T^2$ [39–41]. The cross-over to a Bethe–Heitler region is then expected to take place at the scale $k_{\text{BH}}$ determined by

$$t_{\text{form}}(k_{\text{BH}}) \sim t_{\text{scatt}},$$

(15)

which implies

$$k_{\text{BH}} \sim T.$$  

(16)

Therefore, only the infrared tail of the medium cascade has an $O(1)$ Bethe–Heitler correction whereas the LPM-suppressed splitting gives a good description of the radiation at all higher scales $p > k_{\text{BH}}$. We discuss now the angular distribution of quanta on the different momentum scales $p$. There are two mechanisms. First, multiple soft scattering gives rise to transverse Brownian motion and determines the distribution at small angles. Second, rare large angle scattering leads to deviations from Brownian motion that were first described by Molière for QED [42]. This Molière scattering will put quanta at large angular scales that cannot be reached by Brownian motion in time $t$.

\(^3\) Splittings where both splitters carry $O(1)$ of the parent momentum will be referred to as ‘quasi-democratic’ in the following.
2.6.1. Angular distribution at small $\theta$

In general, transverse Brownian motion moves quanta by an angle
\[ \theta_{\text{BR}}^2 \sim \frac{k_{\text{BR}}^2}{p^2} \sim \frac{\hat{q}_{t_{\text{res}}}(p)}{p^2}, \]
that is set by the time $t_{\text{res}}(p)$ that the quantum resides and thus broadens at scale $p$. Quanta with $p > k_{\text{split}}$ will have spent a time of order of the duration of the jet evolution at $p$, that means, $t_{\text{res}}(p) \sim t$. Therefore, these quanta reach a typical angle
\[ \theta_{\text{BR}}(p) \sim \sqrt{\frac{q_{t_{\text{res}}}(p)}{p}} \quad \text{for } k_{\text{form}} > p > k_{\text{split}}. \]
This is the limiting angle up to which the LPM region extends in Fig. 1. This angle reaches from
\[ \theta_{\text{BR}}(k_{\text{form}}) \sim \frac{1}{\sqrt{k_{\text{form}}t}} \]
at the upper bound of the LPM region to a parametrically larger angle
\[ \theta_{\text{BR}}(k_{\text{split}}) \sim \frac{1}{\sqrt{\alpha_s^2 k_{\text{split}}t}} \sim \frac{1}{\alpha_s^2 \sqrt{k_{\text{form}}t}} \]
at the lower bound. For $p < k_{\text{split}}$, where resplitting happens, the time $t_{\text{res}}(p)$ a quantum stays at the scale $p$ before leaving by undergoing a quasi-democratic splitting is shorter than the lifetime of the system, $t_{\text{res}}(p) < t$, and this shortens the typical angle reached by transverse Brownian motion in the region of the medium cascade,
\[ \theta_{\text{BR}}^2(p) \sim \frac{\hat{q}_{t_{\text{res}}}(p)}{p^2} \sim \frac{\hat{q}_{t_{\text{res}}}}{p^{1/2} k_{\text{split}}^{1/2}} \quad \text{for } p < k_{\text{split}}. \]
It is remarkable that in the region of the medium cascade this angle does not change with evolution time $t$, although many other scales in Fig. 1 do. In fact, if one adopts the perturbative estimate $\hat{q} \sim \alpha_s^2 T^3$, one finds that this angle is set by the simple ratio of momentum scales,
\[ \theta_{\text{BR}}(p) \sim \left( \frac{T}{p} \right)^{3/4}. \]
This way of rewriting Eq. (21) makes it also clear that $p \sim T$ is the largest scale that isotropizes.

Since Brownian motion leads to a Gaussian distribution in $d^2 \theta$, which is parametrically flat for $\theta < \theta_{\text{BR}}$, the energy contained in this region in a double logarithmic unit of phase space can be written using (11)
\[ \frac{d\epsilon}{d \log p \, d \log \theta} \sim p \frac{t}{t_{\text{split}}(p)} \theta^2 \theta_{\text{BR}}^2. \]

Before concluding this subsection, we point out as an aside that there is a logarithmic enhancement at small $\theta$. The reason is that the particles created later have had a shorter time to broaden and this effect accumulates quanta in the collinear region. To be specific, consider the contribution of some last small time interval $\alpha_s^2 t$ to the radiation. It is
\[ \frac{dP_{\text{create}}}{d \log(p)} \sim \frac{\alpha_s^b}{t_{\text{split}}(p)} \]
and these radiated quanta appear under the angle
\[ \theta_{b}^2 \sim \frac{\hat{q}_{t_{\text{res}}}}{p^2}. \]
Therefore in the differential probability distribution, the $\alpha_s^b$’s cancel, and all logarithmic times make an equally large contribution to all logarithmic bins they can reach. Therefore, one expects to see a logarithmic enhancement of Eq. (23) in the collinear region.

2.6.2. Angular distribution in the Molière scattering region

So far, our estimates did not rely on assumptions about the microscopic structure of the medium. If hard scatters resolve partonic constituents in the medium, then these rare occurrences can move a quanta to angles $\theta > \theta_{\text{BR}}$ [43]. At scale $p \in [T, k_{\text{form}})$, there are $t/t_{\text{split}}(p)$ quanta per log $p$, and each of them will undergo a hard momentum transfer $q_{\perp} = p \theta$ during their stay at the momentum scale $p$ with probability [39–41]
\[ \frac{dP_{\text{kick}}}{d \log \theta} \sim \frac{\alpha_s^2 n_T}{p^2 \theta^2} t_{\text{res}}(p) \]
where $n_T \sim T^3$ is the number density of resolvable scattering centers. Multiplying these numbers, we get the probability to find a quantum at large angles\(^4\)
\[ \frac{dP_{\text{find}}}{d \log p \, d \log \theta} \sim \left( \frac{\alpha_s^2 n_T T^{1/2} q_1^{1/2} / (p^{5/4} \theta^2)}{\alpha_s^2 n_T T(p/\theta)^2} \right)^2 \]
for $p > k_{\text{split}}$. We find that along lines $p \propto \theta^{-4/5}$ for $p > k_{\text{split}}$ (or $p \propto \theta^{-1}$ for $p < k_{\text{split}}$), there is a fixed number of quanta per double logarithmic phase space. Since this is a power-law dependence in $\theta$ while Brownian motion dies out exponentially above $\theta_{\text{BR}}$, rare hard Molière scattering, if allowed for by the microscopic structure of the medium, will dominate the distribution of medium-induced quanta above $\theta_{\text{BR}}$.

Comparing the probability distribution (27) for Molière scattered quanta at large angle $\theta > \theta_{\text{BR}}$ (p) to that of vacuum quanta (Eq. (2) for $p > k_{\text{split}}$) and to that of vacuum quanta undergoing the medium cascade (Eq. (13) for $p < k_{\text{split}}$), we find that outside a relatively narrow region of logarithmic phase space, the DGLAP vacuum radiation (thin dash-dotted line in Fig. 1), or the cascade of the vacuum quanta (thick dash-dotted line in Fig. 1), will overshadow the contribution from Molière scattering. The quanta sensitive to the microscopic structure give a dominant contribution to the energy only for $\theta_{\text{BR}} \ll \theta < \theta_{\text{Mol}}$ with
\[ \theta_{\text{Mol}} \sim \left( \frac{\alpha_s^2 T^{1/2} q_1^{1/4} p^{-5/4}}{n_T^{3/4} q_1^{1/2} p^{-3/4}} \right) \]
for $p > k_{\text{split}}$. Here, the expression for $p < k_{\text{split}}$ can be written in a form resembling Eq. (22)
\[ \theta_{\text{Mol}} \sim \frac{1}{\alpha_s^{1/2}} \left( \frac{T}{p} \right)^{3/4} \quad \text{for } p < k_{\text{split}}. \]
As already observed for $\theta_{\text{BR}}$, also the angular limit $\theta_{\text{Mol}}$ for Molière scattering is independent of time within the region of the medium cascade.

2.7. Extending Fig. 1 to $t > t_{\text{form}}(Q)$

So far in our consideration, we have made the assumption that $Q$ is the hardest scale in the system. The scales $k_{\text{form}}$ and $k_{\text{split}}$, however, grow fast $\propto t^2$, and eventually they will reach the
\[ \theta_{\text{Mol}} \sim \frac{1}{\alpha_s^{1/2}} \left( \frac{T}{p} \right)^{3/4} \]
\[ \theta_{\text{Mol}} \sim \frac{1}{\alpha_s^{1/2}} \left( \frac{T}{p} \right)^{3/4} \]
scale $Q$. In the following we discuss how the dynamics changes after $k_{\text{form}}$ and $k_{\text{split}}$ reach $Q$.

When $k_{\text{form}} \sim Q$ (or equivalently $t > t_{\text{form}}(Q)$) the quasi-democratic splitting of the original jet becomes allowed, but it happens only with probability $\alpha_s$; successive quasi-democratic splitting is also suppressed by further powers of $\alpha_s$. Therefore, for a typical jet, no qualitative change happens at $t_{\text{form}}(Q)$. What changes, however, is the average rate at which the leading parton is losing energy $d\langle \sigma_q \rangle/dt$; here the brackets are to be understood as an average over an ensemble of independent jets. The energy lost by the leading parton is dominated by the hardest splitting possible, which before $t_{\text{form}}(Q)$ is $k_{\text{form}}$ but stays at $Q$ after $t_{\text{form}}(Q)$. The average rate for losing energy depends on the probability of the hardest emission $t/t_{\text{split}}$, and it depends on the energy lost in the event of a hard emission taking place. For $t < t_{\text{form}}(Q)$ the average rate reads [19]

$$d\langle \sigma_q \rangle/dt \sim (t/t_{\text{split}}(k_{\text{form}})) k_{\text{form}}/t \sim \alpha_s \langle \sigma_q \rangle t,$$

whereas the rate saturates for $t > t_{\text{form}}(Q)$

$$d\langle \sigma_q \rangle/dt \sim Q t/t_{\text{split}}(Q) \sim \alpha_s Q^{1/2} q_1^{1/2}.$$

When $k_{\text{split}} \sim Q$ – corresponding to the jet stopping time of $t_{\text{split}}(Q) \sim (Q/T)^{1/2}/\alpha_s^2 T$ [19,25] – the probability for a democratic splitting of the jet becomes $O(1)$, and the successive democratic splittings will happen in a time scale faster than the time it took for the original splitting. As argued, e.g., in Refs. [1–4,25,31], the jet will therefore become quenched and lose all of its energy to the thermal bath in a timescale it takes to undergo the first quasi-democratic splitting.

3. Jet quenching in a finite medium

The picture Fig. 1 discussed in Section 2 is a snapshot at fixed time of the logarithmic phase space distribution of partonic fragments. Some of the phase space boundaries will evolve in time as indicated in the figure, while others will stay put. Therefore, this picture informs us also about how the jet quenching process occurs dynamically as a function of time.

In the phenomenologically realized situation, the jet propagates over a finite path-length $L$ in QCD matter, and (for sufficiently short $t < t_{\text{form}}(Q)$, see details below) Fig. 1 thus represents the distribution of partonic jet fragments at moment $t \sim L$ when the jet escapes the medium. In the following, we discuss in more detail the energy distribution of the quenched parton shower at that moment, and the physics that modifies this distribution at later times and for longer path lengths.

3.1. Vacuum radiation for $t > L$

The typical virtuality of quanta at time $t \sim L$ is $O(\langle \sigma_q t_{\text{res}}(p) \rangle)$. The dominant mechanism that further degrades this virtuality in the subsequent vacuum evolution is soft and collinear splitting. On the one hand, this mechanism does not move appreciably the emitter in $log \theta$ and $log p$, on the other hand, it puts $O(\alpha_s)$ quanta in the double logarithmic phase space below any emitter. Therefore, late time fragmentation affects the distributions of Fig. 1 only in those regions of phase space in which there are less quanta than $O(\alpha_s)$ times the number of quanta at higher momentum scales in the corresponding angular scale at time $t \sim L$. As we explain now, this is only the case in the collinear region of sufficiently small $log \theta$, where momentum broadening and smallness of angular phase space have resulted in a small density of medium-induced quanta,$^5$ and in a sufficiently soft region at large angle where there are sufficiently many medium-induced quanta at higher momentum scale that can split further at late times.

3.1.1. Radiation of vacuum quanta at $t > L$

At small angular scales $\theta < \theta_{\text{BR}}(k_{\text{split}}) \sim (\alpha_s^2 k_{\text{split}}^{-1/2})^{1/2}$, there are less than 1 medium induced quanta integrated over $p$ and therefore late vacuum fragmentation is dominated by the original single parton at the scale $Q$. This becomes dominant over the medium-induced distribution when there are less than $\theta(\alpha_s)$ medium induced quanta per double logarithmic unit of phase space. This small-angle region $\theta(p) \ll \theta_{\text{BR}}(p)$, in which vacuum radiation dominates is obtained by requiring that (23) is larger than $\alpha_s p$,

$$\theta_{\text{BR}}(p) \sim \sqrt{\alpha_s \theta_{\text{BR}}(p)} \frac{t_{\text{split}}(p)}{t}\text{.}$$

This region is delineated by the thin dashed line between $k_{\text{split}}$ and $k_{\text{form}}$ in Fig. 1. We note that the contribution of this late time vacuum splitting to jet observables will be that of a vacuum jet, but one of degraded energy. For an ensemble average, the reduced energy of this vacuum jet contribution is given by Eqs. (30) and (31),

$$\langle Q^l \rangle = Q - \langle \sigma_q \rangle \sim - \alpha_s k_{\text{form}} \frac{Q}{\alpha_s Q^{1/2} q_1^{1/2} t_{\text{split}}(Q)}.$$

In principle, late time splitting of vacuum quanta contributes also to $\theta > \theta_{\text{BR}}(k_{\text{split}})$. In this region of the medium-cascade, however, there is also a contribution from the late time vacuum splitting of medium-induced quanta to which we shall turn next. Since this latter contribution dominates, we do not continue in Fig. 1 the line of $\theta_{\text{BR}}(p)$ into the region $\theta > \theta_{\text{BR}}(k_{\text{split}})$ corresponding to $p < k_{\text{split}}$.

3.1.2. Radiation of medium-induced quanta at $t > L$

Most of the medium-induced quanta reside on the angular scale $\theta_{\text{BR}}(p)$. Correspondingly, for a given angular scale $\theta > \theta_{\text{BR}}(k_{\text{split}})$, the number of medium-induced quanta is dominated by the momentum scale

$$p_{\text{BR}}(\theta) \sim \left( \frac{\theta_{\text{BR}}(\theta)}{\theta} \right)^{4/3} p\text{.}$$

For these angular scales, the number of the medium-induced quanta along $\theta_{\text{BR}}(p) > O(1)$ per double logarithmic phase space. Therefore, the late time vacuum splitting of these more than $O(1)$ medium-induced quanta dominates over the vacuum splitting of the $O(1)$ vacuum quantum described in Section 3.1.1 above.

For a fixed $\theta$, the number of quanta at the scale $p_{\text{BR}}$ reads

$$\frac{dn}{d\log p d\log \theta} \sim \frac{t_{\text{split}}(p)}{t_{\text{split}}(p) p_{\text{BR}}} \sim \frac{t}{t_{\text{split}}(p)} \left( \frac{\theta}{\theta_{\text{BR}}} \right)^{2/3}\text{.}$$

Then the number of emitted quanta by the medium induced fragments at the scale $p_{\text{BR}}$ to all scales below is

$$\frac{dn_{\text{late vac}}}{d\log p d\log \theta} \sim \alpha_s \frac{t}{t_{\text{split}}(p)} \left( \frac{\theta}{\theta_{\text{BR}}} \right)^{2/3}\text{.}$$

$^5$ One might wonder if there is a subtlety concerning whether the angle $\theta$ is measured relative to the initial direction of the original parton or it’s direction $\tilde{\theta}_0 \sim \sqrt{q^2/Q}$ after leaving the medium at time $t \sim L$. However, as $Q$ is by assumption the hardest scale in the system, the angle $\tilde{\theta}_0$ is respectively the smallest angular scale in the system, and the small angular broadening experienced by the original parton does not affect our estimates.
This dominates over the medium-induced quanta for 
\[ \theta < \theta_\alpha \sim \alpha_s^{3/4} \theta_{\text{BR}} \] for \( \theta > \theta_{\text{BR}}(k_{\text{split}}) \). \tag{37} 

This condition extends the \( \theta_\alpha \)-line to larger angular scales \( \theta_\alpha > \theta_{\text{BR}}(k_{\text{split}}) \). We have denoted this extension by a thicker dashed purple line in Fig. 1 to emphasize that there are more than \( O(\alpha_s) \) quanta along this line for \( \theta_\alpha > \theta_{\text{BR}}(k_{\text{split}}) \).

3.2. Difference in energy degradation of leading hadrons and jets

Since leading hadrons are the leading fragments of leading partons, the total energy \( \alpha_s k_{\text{form}} \) radiated away from \( Q \) up to time \( t \) due to medium effects sets the scale for the suppression of leading hadrons.\(^6\) In contrast, what is radiated away from the leading parton is not necessarily radiated outside the phase space within which the jet is reconstructed. It is only the energy in scales \( p < k_{\text{split}} \) that has had time to undergo the medium cascade and that therefore escapes for sure to sufficiently large angles [1,2,31].

The energy missing from the reconstructed jet is thus given by
\[ \epsilon_{\text{jet}} \sim k_{\text{split}} \sim \alpha_s^2 q_t. \tag{38} \]

Fig. 1 therefore implies that for all in-medium path lengths \( L \leq t_{\text{split}}(Q) \), \( d\epsilon(Q)/dt > d\epsilon_{\text{jet}}/dt \), and thus leading hadrons are more suppressed than reconstructed jets. It is only at the time \( t \sim t_{\text{split}}(Q) \) when \( k_{\text{split}} = Q \) that \( \epsilon(Q) \sim \epsilon_{\text{jet}} \).

3.3. Angular jet energy distribution

The parametric estimates for the angular distribution of jet energy obtained in this note can be written in a compact form for \( p < k_{\text{form}} \)
\[ \frac{d\epsilon}{d\log p d\log \theta} \sim \frac{t}{t_{\text{split}}(p)} \left[ \begin{array}{c} \alpha_s(t_{\text{split}}(p)/t) & \text{for } \theta < \theta_{\text{BR}}(k_{\text{split}}), \\ \alpha_s(\theta)/\theta_{\text{BR}}^{2/3} & \text{for } \theta_{\text{BR}}(k_{\text{split}}) < \theta < \theta_\alpha, \\ \alpha_s(\theta)/\theta_{\text{BR}}^{2} & \text{for } \theta_\alpha < \theta < \theta_{\text{BR}}, \\ \alpha_s t_{\text{split}}(p)/t_{\text{mol}} & \text{for } \theta > \theta_{\text{mol}}. \end{array} \right. \tag{39} \]

The expression (39) is valid for both, the LPM region, \( k_{\text{split}} < p < k_{\text{form}} \), and the region of the medium cascade, \( p < k_{\text{split}} \). The second of the five conditions listed in (39) is realized only when \( \theta_{\text{BR}}(k_{\text{split}}) < \theta_\alpha \), which corresponds to \( p < \alpha_s^{3/2} k_{\text{split}} \). For \( \theta_\alpha < \theta_{\text{BR}}(k_{\text{split}}) \), the region dominated by late energy degraded vacuum radiation (first line in Eq. (39)) extends up to \( \theta_\alpha \) and connects directly to the region dominated by Brownian motion of medium quanta.

The angular jet energy distribution (39) is depicted in Figs. 2 (for the LPM-region) and 3 (for the medium cascade region). Figs. 2 and 3 illustrate that indeed, late time vacuum radiation of \( \alpha_s \) quanta dominates at angles \( \theta < \theta_{\text{BR}}(p) \). It is only in the range \( \theta_{\text{BR}}(p) < \theta < \theta_{\text{mol}} \) that purely medium-induced contributions dominate. In the regions \( \theta_{\text{BR}}(k_{\text{split}}) < \theta_\alpha \) and \( \theta > \theta_{\text{mol}} \) the energy is dominated neither by pure vacuum radiation nor by medium-induced radiation, but is a result of an interplay of both types of radiation. While one may naively have thought that the energy from large angle Molière scattering appears at much larger angles than that from small angle Brownian motion, Figs. 2 and 3 demonstrate that the dominant energy from both contributions pile up at the same angular scale \( \theta_{\text{BR}} \). It is then only the shape of the angular dependence between \( \theta_{\text{BR}} \) and \( \theta_{\text{mol}} \) that may give access to microscopic details of jet-medium interaction [43].

We further emphasize that \( \theta_{\text{BR}}, \theta_\alpha, \theta_{\text{mol}} \) are independent of evolution time below \( k_{\text{split}} \), and that a characteristic medium-induced enhancement is seen at this angular scale. This angular scale is set directly by the temperature, while the only medium-dependent information entering the size of the peak is \( \hat{q} \). If this structure would be experimentally accessible, it would thus give direct access to the temperature dependence of \( \hat{q} \). More generally, Figs. 2 and 3 illustrate that the energy per unit of double logarithmic phase space will peak for all momentum scales \( p \) on the characteristic scale \( 2\theta_{\text{BR}}(p) \) that is a medium-induced scale. This is a robust expectation for perturbative mechanisms of jet quenching. These parametric considerations may provide a motivation to

\(^6\) As explained in Ref. [44], the typical energy shift seen in leading hadron spectra can be significantly smaller than \( \alpha_s k_{\text{form}} \) due to trigger bias effects.
characterize experimental data on the angular jet energy distribution in \( \log \theta \) and to search for such an enhancement.

4. Conclusions

The main deliverables of this paper are Figs. 1, 2 and 3 which provide a unified view of the physics underlying jet quenching. As discussed, this view is consistent with many results on jet quenching in the (parametrically recent) literature. We note that all the physics phenomena invoked in our discussion are at least in principle implemented in some of the documented jet quenching models. The contribution of our note is not so much to point to novel physics effects, but to provide a map of the phase space regions in which specific known physics is expected to dominate. For instance, the characterization of the angular distribution of jet energy in Figs. 2 and 3 points to the interest in searching in \( \log \theta \) plots (within Monte Carlo studies and within data) for characteristic medium-induced enhancements at small- and small-angle scattering properties (\( q \)) of the medium, but also (if one can identify the region of Molière scattering) about its quasi-particle content [43]. We hope that in this and in other ways, the simple Figs. 1, 2 and 3 will be of use in the further discussion of jet quenching phenomena.

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References