EXPLICIT EXAMPLE FOR THE CALCULATION OF TWO-DIMENSIONAL INVARIANTS

Summary and introduction.

According to CERN-PS/RH 9 (called I) and CERN-PS/RH 10 (called II) the invariants can be calculated explicitly, once the Hamiltonian is given. With the help of CERN-PS/A.Sch/RH.1 (called III), the invariants can be evaluated further nearly without calculations in terms of resonance curves or maximum amplitudes, or tolerances on non-linearities. For this purpose, however, the invariants must be known numerically.

The aim of the present paper is to use the formulae of I, II, III in an explicit example.

This example is chosen in such a way that it applies directly to the electromechanical analogue model of CERN-PS and thus serves to compare theory and experiment in a well defined case.

However, the example applies at least qualitatively, also to the machine itself. It seems hopeless to prepare and to handle a case, which represents the real synchrotron to a degree as we now know it to be. Therefore one has to idealize, wherever one tries to get theoretical insight. For the present example we choose a quadratic and a cubic non-linearity, both combined of a constant part and a varying part, the latter being purely sinusoidal. The mentioned idealization is therefore: neglect of other non-linearities and neglectation (in the used non-linearities) of other harmonics than those which just will excite the considered non-linear resonances. The structure of the theory shows that this idealization does not lead too far from the reality, because indeed for a particular non-linear resonance one non-linearity and of this one Fourier component plays a dominating role. Quantitatively something may be changed if there are other non-linearities and harmonics present. Qualitatively everything remains the same and even the quantitative changes may remain so small that numerical results are still good estimates. It is, of course, possible to work out examples in just the same way, which are still more realistic, e.g., a sequence of kicks with phases and amplitudes given but at random. This would represent one single member of a statistical ensemble of macroscopically equal machines. Or, one could consider such an ensemble and calculate the expectation values of the coefficients of the invariants.
The present example is not the simplest one. It covers already a class of cases which are not too unrealistic and it has the advantage to remain very simple.

As to the electromechanical analogue model, this example is not idealized, since there one just produces some definite non-linearity and excites the resonance with a sin-wave. Therefore one should expect quantitative agreement within the accuracy of measurements and the range of validity of the theory.

Whereas III discusses only the 3rd order sub-resonance, it seemed worthwhile to prepare this example in a more general way and to include the 4th order too. It is easily specialized to simpler cases by putting some coefficients equal to zero.

In applying it to the real machine, we assume the field to be ideally symmetric with respect to the xG-plane and the closed orbit to coincide with the ideal orbit. If we do this, we can interpret the used non-linear coefficients $n'$ and $n''$ directly as those measured by our magnet people (or found from their measurements by numerical methods).

If, however, we assume a non-vanishing closed orbit (which may originate from a momentum error and thus has no y-component) $x_0 = x_0(\theta)$, $y_0 \equiv 0$ then in the $n'$ and $n''$, as used here, parts of this closed orbit are contained (II, 34-38) and they do no longer represent our knowledge on magnets. In the case of an unsymmetric field this example does not apply at all.
1) The Hamiltonian in complex notation.

From (II, 30a) we have

\[ h_{jklm}^{(N)} (\ell + m \text{ even}) = (-1)^{\frac{\ell + m}{2}} \frac{1}{i^{2N-1} jikflim} \left[ -i \frac{k}{2} Q_x^2 - \frac{l + m}{2} \frac{B_0}{2} y^{(N-1)} \right] \]

\[ h_{jklm}^{(N)} (\ell + m \text{ odd}) = (-1)^{\frac{\ell + m + 1}{2}} \frac{1}{i^{2N-1} jikflim} \left[ -i \frac{k}{2} Q_x^2 - \frac{l + m}{2} \frac{B_0}{2} y^{(N-1)} \right] \]

We assume the ideal field symmetry:

\[ B_x \equiv 0 \]

and have, with the common abbreviation (field index)

\[ n^{(r-1)} = \left. \frac{\partial^r B}{\partial x^r} \right|_{x = y = 0} \]

inserted into (a).

\[ h_{jklm}^{(N)} = (-1)^{\frac{l + m}{2}} \frac{1}{2^{N-1} jikflim} \left[ -i \frac{k}{2} Q_x^2 - \frac{l + m}{2} \frac{B_0}{2} x^{(N-2)}(x) \right] \]

where \( \ell + m \) is always even; \( j + k + \ell + m = N \)
2) The assumed model

For the field index we assume

\[ n^2 = \frac{\partial^2}{\partial x^2} |_{x-y=0} = n''_0 + n'_2 \cos p''Q + n'_0 \sin p''Q + n'_2 \cos (p''Q - \phi'') \]

(a)

\[ n'' = \frac{\partial^2}{\partial x^2} |_{x-y=0} = n''_0 + n''_1 \cos p''Q + n''_2 \sin p''Q + n''_2 \cos (p''Q - \phi'') \]

(b)

Thus, from (II 6b) the equations of motion (which, however do not interest in the following) are

\[ x + (1-n)x = - \frac{2\varphi_g}{dx} \]

\[ \dot{y} + ny = - \frac{\varphi_g}{dy} \]

(c)

\[ \varphi_g(xy) = \text{Real part of} \left[ - \frac{n''}{3} \varphi^3 - \frac{n''}{4} \varphi^4 \right], \varphi = x + iy. \]

Since all formulae in I and II refer to an exponential Fourier representation, we introduce \( \cos \theta + i \sin \theta = e^{i \theta} \) and write

\[ n^2 = n^2_0 + \frac{1}{2} (n'_2 - i n'_2) e^{i p''Q} + \frac{1}{2} (n'_2 + i n'_2) e^{-i p''Q} \equiv n^2_0 + \psi^+ \psi^+ - \psi^- \psi^- \]

\[ n'' = n''_0 + \frac{1}{2} (n''_2 - i n''_2) e^{i p''Q} + \frac{1}{2} (n''_2 + i n''_2) e^{-i p''Q} \equiv n''_0 + \psi^+ \psi^+ - \psi^- \psi^- \]

Thus the various symbols mean:

\[ \psi^+ \psi^-, \psi^- \psi^+, \psi^+ \psi^-, \psi^- \psi^- \text{ the exponential Fourier coefficients} \]

\[ n^2_0, n^2_2, \psi^+ \psi^-, \text{ the squares of the amplitudes of the varying parts of the perturbation or} \]

\[ \psi^+ \psi^-, \psi^- \psi^-, \psi^+ \psi^-, \text{ the amplitudes themselves.} \]
3) Explicit Hamiltonian coefficients in Fourier representation.

For the determination remain only those $h_{jklm}^{(N)}$ with $\xi + m$ even and we find immediately from (3c) and (4c) the following table:

$$
\begin{array}{|c|c|c|c|}
\hline
jkln & h_{jklm,-p}^{(4)} & h_{jklm,0}^{(4)} & h_{jklm,p}^{(4)} \\
\hline
1020; 0120 & -\frac{1}{8} Q_x Q_y^{-1} \psi_{-p} & -\frac{1}{8} Q_x Q_y^{-1} n_o & -\frac{1}{8} Q_x Q_y^{-1} \psi_{p} \\
0120; 1002 & -\frac{1}{4} Q_x Q_y^{-1} \psi_{-p} & -\frac{1}{4} Q_x Q_y^{-1} n_o & -\frac{1}{4} Q_x Q_y^{-1} \psi_{p} \\
0111; 1011 & -\frac{1}{8} Q_x^{-3/2} \psi_{-p} & -\frac{1}{8} Q_x^{-3/2} n_o & -\frac{1}{8} Q_x^{-3/2} \psi_{p} \\
1200; 2100 & -\frac{1}{24} Q_x^{-3/2} \psi_{-p} & -\frac{1}{24} Q_x^{-3/2} n_o & -\frac{1}{24} Q_x^{-3/2} \psi_{p} \\
3000; 0300 & -\frac{1}{24} Q_x^{-3/2} \psi_{-p} & -\frac{1}{24} Q_x^{-3/2} n_o & -\frac{1}{24} Q_x^{-3/2} \psi_{p} \\
\hline
\text{other} & 0 & 0 & 0 \\
\hline
\end{array}
$$
Only those $h_{jkln}^{(4)}$ are listed, which are needed for the stabilizing coefficients or contribute to a 4th order resonance. The remaining ones would be needed only if one had to calculate higher orders (5th, 6th, ...).
4) The coefficients $g^{(N)}_{4klm}$ of the invariants: 3rd order resonance.

From (II, 73)

(a) $g_{4klm,p} = \delta_{pp'} g_{4klm,p'}$

and after short and simple algebra

(b) $g^{(4)}_{2000} = \frac{1}{32\xi_x^2} \left\{ \frac{\xi_x^2}{8} \left[ \frac{6}{p_x^2} + \frac{1}{Q_x (1 - \frac{1}{3Q_x - p^2})} + \frac{3}{5Q_x^2} + \frac{5}{3Q_x^2} \right] \right\} + n_o^2 - \frac{5}{3Q_x^2}$

(c) $g^{(4)}_{0022} = \frac{3}{32\xi_y^2} \left\{ \frac{\xi_y^2}{8} \left[ \frac{6}{p_y^2} + \frac{1}{Q_y (1 - \frac{1}{3Q_y - p^2})} + \frac{3}{5Q_y^2} + \frac{5}{3Q_y^2} \right] \right\} + n_o^2 \left( \frac{1}{Q_x - 2Q_y - p^2} + \frac{2}{Q_x^2} \right)$

(d) $g^{(4)}_{1111} = \frac{1}{32\xi_x \xi_y} \left\{ \frac{\xi_x^2}{8} + \frac{n_o^2}{8} \left[ \frac{4}{Q_x^2 - p^2} + \frac{1}{Q_y (1 - \frac{1}{3Q_y - p^2})} + \frac{1}{Q_x + 2Q_y - p^2} \right] \right\} + n_o^2 \left( \frac{2}{4Q_y^2 - Q_x^2} - \frac{1}{Q_x^2} \right)$

In these formulae some evident assumptions have been made:

1) $Q_x \neq Q_y \neq p^o$
2) $Q_x \neq Q_y \neq 0$
3) $Q_x + 2Q_y \neq 0$
4) $2Q_x \neq Q_y \neq 0$
(7b), (7c) and (7d) constitute the stabilizing coefficients, (7a) the exciting coefficient. Having chosen a definite resonance line of 3rd order:

\[ n_1 Q_x + n_2 Q_y = p \quad ; \quad \left| n_1 \times n_2 \right| = 3, \]

one has directly from (a) the exciting coefficient which, however, vanishes unless \( p = p' \). Thus one has to put \( p' = p \) everywhere and to omit the terms with vanishing denominator. (Indicated by dashes at the round brackets).

5) The coefficients \( C_{ijklm}^{(4)} \) of the invariants; 4th order resonance.

From (22, 74)

\[
S_{4000p} = \frac{4}{4! 3! Q_x^2} \left[ \delta_{pp'} P_{pp'}^{mn} - \delta_{pp'} \frac{3y^3 P_{pp'}^{n1} (2Q_x - p)}{Q_x^2 (Q_x - p)} \right]
\]

for the line \( w_{4000} = 4Q_x = p \)

\[
S_{4000p} = \frac{5}{4! 3! Q_y^2} \left[ \delta_{pp'} P_{pp'}^{mn} - \delta_{pp'} \frac{3y^3 P_{pp'}^{n1} (2Q_y - p)}{Q_y^2 (Q_y - p)} \right]
\]

for the line \( w_{4000} = 4Q_y = p \)

\[
S_{2020p} = \frac{1}{3! 2! Q_x^2} \left[ \delta_{pp'} P_{pp'}^{mn} + \delta_{pp'} \frac{y^3 P_{pp'}^{n1} (Q_x - 4Q_y)(2Q_x - p)}{Q_x^2 (Q_x - p)} \right]
\]

for the line \( w_{2020} = 2Q_x + Q_y = p \)
The remaining 4th order lines are not excitable in this example on account of the field symmetry. From the general proof (II.33a) follows, that all $g_{ijkl}$ with $j + m$ odd vanish.

Having chosen a definite resonance line of 4th order $n_1 q_x + n_2 q_y = p; \{n_1 + n_2\} = 4$ one has from (a) or (b) or (c) the $g$-coefficients for the excitation, whereas (7b) (7c) (7d) remain the stabilizing coefficients.

One has to be sure that one is not at the same time on a third order resonance.

In order to excite a 4th order resonance, at least one of the two frequencies $p', p''$ has to be equal to $p$. The terms with the $\delta_{pp'}$-symbol, however, are in general negligibly small.

6) Explicit formulation of the invariants.

For finding amplitude ranges one has to insert the coefficients into the invariants. There, the following symbols occur (I.40a,b):

$\begin{align*}
n_1 q_x + n_2 q_y = p = \varepsilon \quad \text{is a distance from the resonance line} \\
(q_x \text{ and } q_y \text{ are called } \omega_1 \text{ and } \omega_2 \text{ there})
\end{align*}$

$\varepsilon_{ab} \equiv \varepsilon_{2200}$ are the stabilizing coefficients (as e.g. $\varepsilon_{2200}$).

$A = \frac{r_1}{n_1} - \frac{r_2}{n_2}$ and $B = \frac{r_1}{n_1} = - \frac{r_2}{n_2}$

are the quadratic invariants and are calculated using the initial amplitudes $r_{10}$ and $r_{20}$.

These initial amplitudes, however, are not directly the particles amplitudes. Rather

$\begin{align*}
x_1^2 &= y_1^2 = y_2^2 = \gamma_1^2 \gamma_2^2 \gamma_1^2 \gamma_2^2 = q_x x_1^2 + \frac{q_x}{x_1} = q_x \left(x_1^2 + \frac{q_x}{x_1}^2\right) &= q_x A^2
\end{align*}$

(see in the same order: $I(139a)(I,21a)(I,13c)(I,2d)$ with $I(20a)$).
Thus

\[
\begin{align*}
\gamma_1^2 &\approx \frac{Q}{x^2} A_x^2 \\
\gamma_2^2 &\approx \frac{Q}{y^2} A_y^2
\end{align*}
\]

\(A_x, A_y\) are the particles amplitudes

The argument of \(\cos(\delta - n_1 \phi_1 - n_2 \phi_2)\) remains undetermined.

Finally \(\gamma\) follows to be

\[
\gamma = \left| Y_{jklm} \right| = \left| e_{jklmp} \right| = \left| e_{n_1 \sigma_1 \sigma_2 \sigma_3 \sigma_4} \right|
\] (where the equations (I,44b)

(I,37c) (I,34b) have been used in the same order), thus

\[
\gamma = \left| e_{n_1 \sigma_1 \sigma_2 \sigma_3 \sigma_4} \right|
\]

is the modulus of the \(p^{th}\) Fourier component of the exciting \(g\)-coefficient (e.g. \(e_{4000p}\) or \(e_{2020p}\) or \(e_{1020p}\)).

For the further discussion of the invariants see CERN-PS/A.Sch.RH. 1.

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