QCD DUALITY SUM RULES: INTRODUCTION AND SOME RECENT DEVELOPMENTS*)

S. Narison

Laboratoire de Physique Mathématique
U.S.T.L., Montpellier
and
CERN - Geneva

ABSTRACT

We intend to introduce the readers to the method of QCD-duality sum rules. We also give an up-to-date review of some recent developments in this field.

*) Lectures delivered at the Gif Summer School Institute, Gif-Sur-Yvette (Paris, September 1986), to be published in the Proceedings.
TABLE OF CONTENTS

I - PREFACE AND ACKNOWLEDGEMENTS

II - INTRODUCTION

III - THEORY AND HISTORY

1. Weinberg-like current algebra sum rules.
2. SVZ-expansion and QCD vacuum condensates.

IV - PHENOMENOLOGY

1. Vacuum condensates from $e^+e^- \rightarrow I = 1$ Hadrons.
2. Running quark masses.
3. Deviation from kaon PCAC and value of the quark condensates.
5. Light exotic mesons.
6. $K_0 ^- \bar{K}_0$ systems.
7. Heavy quark systems.
8. Three-point functions and form factors.
9. QCD at non-zero temperature.
10. Composite models of weak interactions.

V - CONCLUSIONS
I - AVERTISSEMENT ET REMERCIEMENT

Ce rapport est écrit en anglais dans l'espoir de mieux communiquer son contenu car dans la langue française, un mot mal placé ou (et) mal choisi pourra engendrer des confusions d'interprétation du message qu'on a voulu transmettre.

Je voudrais remercier les organisateurs pour leurs efforts à maintenir l'image traditionnelle de l'école de Gif-sur-Yvette qui s'est déroulée dans un cadre et atmosphère agréables.

Les discussions avec les participants ont été aussi fructueuses.

II - INTRODUCTION

We have, by now, increasing evidence that quantum chromodynamics (QCD) is the most efficient candidate for describing the theory of strong interactions. This evidence is clear at high momentum, say larger than 1 GeV, where perturbative calculations in series of the strong interaction coupling constant give a nice description of various hard processes (deep inelastic scattering, Drell-Yan, jets, high pT ...) thanks to the asymptotic freedom property of the theory where the renormalization group improved coupling falls like 1/(log - q^2/\Lambda^2) at short distance. We have also, by now, a probable experimental indication of the three gluon interaction characteristics of the non-Abelian SU(3)_c-gauge structure of QCD from jet analysis 1).

However, despite such great successes of QCD, we are quite far from understanding the long-distance phenomena governed by confinement because of the peculiar infrared behaviour of the theory. Therefore, the understanding of the hadron dynamics of QCD needs at present the introduction of an approximate scheme, which, in practice, corresponds to various QCD models proposed in the literature, where the connections among them should be studied and understood 2). The most popular models like the potential and bag models have been reviewed in this school 3) whilst different aspects of lattice gauge theories have been intensively discussed 4). My aim is to present an alternative approach to the hadron dynamics. The method of QCD sum rules is based on the duality between the QCD expression of the hadronic Green's function and its
spectral function which can be derived (as we shall see later on) from the analytical properties of such a Green's function. The lectures will be divided into two main parts. In the first one, I recall the classic Weinberg sum rules of current algebra which are the prototype form of sum rules known before the advent of QCD. I also discuss the present status of the operator product expansion (OPE) of the Green's function in terms of the non-vanishing values of the quark and gluon vacuum condensates. This first part is concluded by the presentation of various interesting forms of the QCD-duality sum rules. In the second part of the lectures, I discuss various phenomenological applications of the sum rules.

III- THEORY AND HISTORY

1. Weinberg-like current algebra sum rules

The idea of the spectral function was known a long time before the advent of QCD, if one remembers various (ad hoc) superconvergent sum rules of the 60's, which are based on the belief of the asymptotic realization of chiral and flavour symmetries at short distances. Among these sum rules, those of Weinberg for SU(2)_L x SU(2)_R chiral symmetry and of Das Mathur, Okubo for SU(3)_F flavour symmetry are the most representative ones and will be presented in the following.

a) Control of the realizations of chiral SU(2)_L x SU(2)_R symmetry

For this purpose, let us consider the axial-vector current which brings the pion into the vacuum. We know that within QCD the axial-vector current has two realizations. The first one is the short-distance realization in terms of the quark fields:

\[ < 0 | \gamma^\mu T^5 u | x > = \sqrt{2} f^u \mu, \]

where \( f^u_{\pi} = 93.3 \text{ MeV} \) is the pion decay constant which controls the \( \pi \rightarrow \mu \nu \) decay whilst \( p^\mu \) is its momentum. The long-distance realization of the axial current is obtained via the chiral Lagrangian of the \( \sigma \)-model:
\[ L_0 = - \frac{1}{4} f_\pi^2 \int \frac{d^4 x}{x^4} \partial_\mu U \partial^\mu U^+ \]  

where \( U = e^{i \frac{x \cdot \pi}{f_\pi}} \) is the pion rotation matrix, \( \pi \) the pion field and \( \tau \) the Pauli-matrices. \(^5\) Now, we will show how the Weinberg sum rules connect these short- and long-distance realizations. Therefore, let us study the two-point correlator:

\[ W^{\mu \nu}_{LR} = i \int d^4 x e^{i q x} \langle 0 | J^\mu_L(x)(J^\nu_R(0))^\dagger | 0 \rangle \]

\[ = (g^{\mu \nu} - q^\mu q^\nu) \Pi_{LR}^{(1)}(q^2) + q^\mu q^\nu \Pi_{LR}^{(0)}(q^2) \]  

where in terms of the quark fields:

\[ J^\mu_L = \bar{u}_L \gamma^\mu (1 - \gamma_5) d_L ; \quad J^\mu_R = \bar{u}_R \gamma^\mu (1 + \gamma_5) d_R \]

are the left- and right-handed components of the currents which span the Gell-Mann current algebra. In the chiral limit \( m_{u,d} = 0 \) or \( q^2 \rightarrow \infty \), where \( SU(2)_L \times SU(2)_R \) chiral symmetry is realized, the asymptotic expression of \( W^{\mu \nu}_{LR} \) is zero. Using the well-known spectral representation obeyed by \( \Pi_{LR}^{(1,0)}(q^2) \):

\[ \Pi_{LR}(q^2) = \frac{1}{\pi} \int_{0}^{\infty} \frac{dt}{t - q^2 - i \epsilon} \frac{1}{t} \text{Im} \Pi_{LR}(t) + "\text{subtraction..."}, \]

the vanishing value of \( W^{\mu \nu}_{LR} \) leads to the Weinberg sum rules:

\[ \int_0^\infty \text{dt} \text{Im} \Pi_{LR}^{(1)}(t) + \text{Im} \Pi_{LR}^{(0)}(t) = 0 \quad (6) \]

\[ \int_0^\infty \text{dt} t \text{Im} \Pi_{LR}^{(1)}(t) = 0 \quad (7) \]

where the first sum rule corresponds to the \( q^\mu q^\nu \) component of \( W^{\mu \nu}_{LR} \) and the second one to the \( g^{\mu \nu} \) part. In the massive quarks case, it has been shown \(^7\) that the first sum rule is still superconvergent because the leading effects to the RHS are \( (\alpha_s / \pi)(1 / x^2)(m_i m_j / q^2) \) where \( i,j \) are the quark flavor indices and \( \alpha_s \) is the QCD coupling. However, the superconvergence assumption of the second sum rule is not realized for massive quarks. The RHS of this sum rule behaves as \( (3/2 x^2)(m_i m_j / q^2) (\log q^2 / v^2) \) where \( v \) is the \( \overline{\text{MS}} \)-renormalization scheme scale. Therefore, some improvements of the second sum rule become necessary except in the \( \bar{u}d \) channel where the current quark masses of the QCD Lagrangian are very small \( (m_u = 5 \text{ MeV}, m_d = 10 \text{ MeV}) \). The spectral function entering into the sum rules can be studied using the long-distance behaviour of the axial and vector currents:

\[ A^\mu(x) = - \sqrt{2} f_\pi \bar{u}_L \gamma^\mu \pi + (\sqrt{2})^3 [\bar{u}_L \gamma^\mu \pi] / f_\pi + ... \]

\[ V^\mu(x) = i \bar{u}_L \gamma^\mu \pi + ... \]  

\(^3\) in the unitary limit of the pion field and \( \tau \) the Pauli-matrices. \(^5\) Now, we will show how the Weinberg sum rules connect these short- and long-distance realizations. Therefore, let us study the two-point correlator:

\[ W^{\mu \nu}_{LR} = i \int d^4 x e^{i q x} \langle 0 | J^\mu_L(x)(J^\nu_R(0))^\dagger | 0 \rangle \]

\[ = (g^{\mu \nu} - q^\mu q^\nu) \Pi_{LR}^{(1)}(q^2) + q^\mu q^\nu \Pi_{LR}^{(0)}(q^2) \]

where in terms of the quark fields:

\[ J^\mu_L = \bar{u}_L \gamma^\mu (1 - \gamma_5) d_L ; \quad J^\mu_R = \bar{u}_R \gamma^\mu (1 + \gamma_5) d_R \]

are the left- and right-handed components of the currents which span the Gell-Mann current algebra. In the chiral limit \( m_{u,d} = 0 \) or \( q^2 \rightarrow \infty \), where \( SU(2)_L \times SU(2)_R \) chiral symmetry is realized, the asymptotic expression of \( W^{\mu \nu}_{LR} \) is zero. Using the well-known spectral representation obeyed by \( \Pi_{LR}^{(1,0)}(q^2) \):

\[ \Pi_{LR}(q^2) = \frac{1}{\pi} \int_{0}^{\infty} \frac{dt}{t - q^2 - i \epsilon} \frac{1}{t} \text{Im} \Pi_{LR}(t) + "\text{subtraction..."}, \]

the vanishing value of \( W^{\mu \nu}_{LR} \) leads to the Weinberg sum rules:

\[ \int_0^\infty \text{dt} \text{Im} \Pi_{LR}^{(1)}(t) + \text{Im} \Pi_{LR}^{(0)}(t) = 0 \quad (6) \]

\[ \int_0^\infty \text{dt} t \text{Im} \Pi_{LR}^{(1)}(t) = 0 \quad (7) \]

where the first sum rule corresponds to the \( q^\mu q^\nu \) component of \( W^{\mu \nu}_{LR} \) and the second one to the \( g^{\mu \nu} \) part. In the massive quarks case, it has been shown \(^7\) that the first sum rule is still superconvergent because the leading effects to the RHS are \( (\alpha_s / \pi)(1 / x^2)(m_i m_j / q^2) \) where \( i,j \) are the quark flavor indices and \( \alpha_s \) is the QCD coupling. However, the superconvergence assumption of the second sum rule is not realized for massive quarks. The RHS of this sum rule behaves as \( (3/2 x^2)(m_i m_j / q^2) (\log q^2 / v^2) \) where \( v \) is the \( \overline{\text{MS}} \)-renormalization scheme scale. Therefore, some improvements of the second sum rule become necessary except in the \( \bar{u}d \) channel where the current quark masses of the QCD Lagrangian are very small \( (m_u = 5 \text{ MeV}, m_d = 10 \text{ MeV}) \). The spectral function entering into the sum rules can be studied using the long-distance behaviour of the axial and vector currents:

\[ A^\mu(x) = - \sqrt{2} f_\pi \bar{u}_L \gamma^\mu \pi + (\sqrt{2})^3 [\bar{u}_L \gamma^\mu \pi] / f_\pi + ... \]

\[ V^\mu(x) = i \bar{u}_L \gamma^\mu \pi + ... \]
Possible final state interactions between pseudoscalar particles can lead to the formation of resonances having the quantum numbers \( l^{--}, l^{++}, 0^{--}, 0^{++} \). Using a narrow width approximation and assuming the \( \rho \) and \( A_1 \) dominances to the vector and axial-vector currents, one obtains from the first and second sum rules the constraints \(^{5}\):

\[
\frac{M_\rho^2}{2\gamma_\rho^2} - \frac{M_{A_1}^2}{2\gamma_{A_1}^2} - 2f_\pi^2 = 0 \quad , \\
\frac{M_\rho^4}{2\gamma_\rho^2} - \frac{M_{A_1}^4}{2\gamma_{A_1}^2} = 0 \quad .
\]

\( \gamma_V \) is the \( V \)-meson coupling to the current:

\[
\langle 0 | V^\mu | \rho \rangle = \sqrt{2} \frac{M_\rho^2}{2\gamma_\rho} \varepsilon^\mu \quad .
\]

It is easy to show that in this normalization:

\[
\Gamma_{\rho \rightarrow \epsilon^+ \epsilon^-} = \frac{2}{5} \times \alpha \left( \frac{M_\rho}{2\gamma_\rho^2} \right) \quad .
\]

Weinberg has used the constraints in Eq. (9) in order to predict the \( A_1 \)-mass. In this way, he obtained in this approximation the value:

\[
M_{A_1} = 1.1 \text{ GeV} \quad .
\]

As we have noticed, the sum rules provide a bridge between the short- and long-distance phenomena. There are various improvements\(^ {7,8}\) of the Weinberg sum rules in the literature both on the theoretical and experimental sides, and which lead to a slightly higher value of the \( A_1 \)-mass in good agreement with the data.

b) Control of the realizations of \( SU(3)_F \) symmetry.

In the same way as for \( SU(2)_L \times SU(2)_R \) chiral symmetry, one can also derive Weinberg-like sum rules based on the asymptotic realization of \( SU(n)_F \) symmetry\(^6\). In so doing, let us write the electromagnetic current in terms of its flavour components:

\[
J_{EM}^\mu (x) = \frac{2}{3} v_\mu + \frac{1}{3} v_\mu + \frac{2}{3} v_\mu - \frac{1}{3} v_\mu + \ldots \quad .
\]
where \( \psi_i^\mu = \phi_i^\mu \phi_i \) with \( i = u, d, s \) and \( \phi_i \) is the quark field. By studying the two-point function:

\[
\Pi^{\mu\nu}_i(q) = i \int d^4x e^{iqx} \langle 0 | T \psi_i^\mu(x) (\psi_i^\nu(0)) | 0 \rangle ,
\]

DMO\footnote{See Ref. [6] for details.} found the superconvergent sum rules:

\[
\int_{4m_i^2}^{\infty} \frac{dt}{t} \left( \text{Im} \Pi_{uu}^{33}(t) - \text{Im} \Pi_{dd}^{88}(t) \right) = 0 ,
\]

\[
\int_{4m_i^2}^{\infty} \frac{dt}{t} \left( \text{Im} \Pi_{ud} + \text{Im} \Pi_{du} - 2 \text{Im} \Pi_{is}(t) \right) = 0 ,
\]

(14)

which correspond to the difference of the isovector and isoscalar parts of the electromagnetic current for SU(3)\(_F\). The QCD calculation of the combination of two-point functions concerned here shows that for large \(-q^2\):

\[
\Pi_i - \Pi_j \approx \frac{3}{2} \frac{m_j^2 - m_i^2}{q^2} \left( \frac{1}{q^2} + \frac{\log}{q^2} \right) ,
\]

(15)

which confirms the superconvergence assumption of DMO. The spectral function can be estimated using the vector meson dominance assumption in Eqs (10) and (11). This leads to the well-known phenomenologically successful relation among vector mesons\footnote{See Ref. [10] for details.}:

\[
\frac{\rho_{\rho \to e^+e^-}}{\rho_{\omega \to e^+e^-}} - 3 \left( \frac{\rho_{\phi \to e^+e^-}}{\rho_{\phi \to e^+e^-}} + \frac{\rho_{\Phi \to e^+e^-}}{\rho_{\Phi \to e^+e^-}} \right) = 0 \left( \frac{m_s^2}{m_s^2} \right) .
\]

(16)

One can also relate the spectral function entering into Eq. (14) to the total cross-section \( e^+e^- \) into hadrons via the optical theorem:

\[
\sigma_H(t) = \frac{4\pi^2}{q^2} \frac{2}{\pi} \text{Im} \Pi(t) .
\]

(17)

Therefore the DMO sum rule and the low-energy data from \( e^+e^- \) can lead to interesting predictions. We have shown in these two examples the ability of the sum rules for relating measured and calculable quantities\footnote{See Ref. [9] for details.}. We have also discussed the utility of perturbative QCD for controlling various superconvergence assumptions proposed in the 60's. Now, we come to the non-perturbative effects of the sum rules.
2. SVZ-expansion and QCD vacuum condensates

a) SVZ-expansion

According to SVZ, quark and gluon condensates contribute to the hadronic correlation via an operator product expansion (OPE) à la Wilson. In this way, the two-point correlation:

$$\Pi(q^2) = i \int d^4 x \epsilon^{i qx} \langle 0 | T J_H(x)(J_H(0))^\dagger | 0 \rangle ,$$

(18)
takes the form, for large $q^2$, in the Euclidean region:

$$\Pi(q^2) = C_1 \mathbf{1} + \sum_{n=2} C_{2n} < \mathcal{O}_{2n} > / (q^2)^n ,$$

(19)

where the unit operator $\mathbf{1}$ is the usual perturbative calculation. $C_{2n}$ are Wilson coefficients which come from perturbative calculations of Feynmann diagrams (short wavelength fluctuations) whilst $< \mathcal{O}_{2n} >$ are vacuum condensates which correspond to the large fluctuations. By using arguments based on the dilute gas instanton approximation, SVZ parametrize the condensates in terms of the dilute gas instanton density $d(\rho) = \exp(-2\pi/\rho)$:

$$< \mathcal{O}_{2n} > = \int d(\rho) \rho^{2n+1} d(\rho)$$

(20)

In this way, the lowest dimension condensates $2n \lesssim 11$ are interpreted as large-size instanton effects while the higher dimension condensates are interpreted as small-size instantons. There is no good control of the latter but we assume that at values of $q^2$ larger than 1.2 GeV$^2$, the knowledge of the few lowest dimension vacuum condensates already gives a good description of the hadronic correlator. In practice, truncating the series in Eq. (19) at $n = 3, 4$ appears to give a quite good description of the whole hadron properties as we shall see later on. Up to dimension six, these condensates are:

$$\begin{align*}
0_4^\Phi &= m < \bar{\psi} \psi > \quad \text{quark condensate} , \\
0_4^G &= \alpha_s < G^{\mu \nu} G_{\mu \nu} > \quad \text{gluon} , \\
0_6^\Psi &= < \bar{\psi} \Gamma_1 \psi \Gamma_2 \psi > \quad \text{four-quark} , \\
0_6^G &= g f_{abc} < G^{\mu \nu}_a G_{\nu \mu}^b G_{\mu \nu}^c > \quad \text{triple gluon} , \\
0_6^{G\psi} &= m g < \bar{\psi} \sigma^{\mu \nu} \frac{\lambda_a}{2} \psi G^{a}_{\mu \nu} > \quad \text{mixed} .
\end{align*}$$

(21)
b) Quark condensates

The non-vanishing value of the lowest dimension quark condensate $\langle \bar{\psi} \psi \rangle$ is attributed to the spontaneous breaking of $SU(2)_L \times SU(2)_R$ chiral symmetry realized à la Nambu-Goldstone. The study of the renormalization of this operator shows that $m \langle \bar{\psi} \psi \rangle$ is renormalization group invariant, while it is protected from renormalons because of chiral symmetry (12).

Using the commutator of the axial charge $\int d^3 x \ A_0(x, t)$ with the divergence of the axial current, one can relate this condensate to the pion parameters. In this way, one obtains the well-known PCAC relation:

$$ (m_u + m_d) \langle \bar{u} u + \bar{d} d \rangle = -2 m_f^2 \frac{\alpha_s}{\pi}. $$

For heavy quarks where symmetry is realized à la Wigner-Weyl, Eq. (22) is replaced by the behaviour of $\langle \bar{Q} Q \rangle$ (Q is the heavy-quark field) as (11):

$$ M_Q \langle \bar{Q} Q \rangle = -\frac{1}{12 \pi} \alpha_s \langle G^2 \rangle - \frac{1}{360 \pi} \alpha_s \frac{\langle g G^3 \rangle}{M_Q^2} - \frac{\alpha_s}{30 \pi} \frac{\langle (DG)^2 \rangle}{M_Q^2} $$

where the $1/M_Q^2$ corrections have been obtained in Ref. (13). $M_Q$ is the heavy quark mass; $\alpha_s \langle G^2 \rangle$, $\langle g G^3 \rangle$ and $\langle (DG)^2 \rangle$ are the gluon condensates which we shall discuss later on.

According to SVZ, one can also expect to have condensates of higher dimensions such as $\langle \bar{\psi} \Gamma_1 \psi \bar{\psi} \Gamma_2 \psi \rangle$ (here $\Gamma_i$ are any Dirac matrices). However, contrary to the $\langle \bar{\psi} \psi \rangle$ condensate, one has not a renormalization group invariant quantity here as the four-fermion condensates mix under renormalization with others of the same dimension (14). Contrary to $m \langle \bar{\psi} \psi \rangle$, there is no direct way for estimating the four-fermion condensate. SVZ assume the vacuum saturation of this condensate in order to relate it to $\langle \bar{\psi} \psi \rangle$. Therefore, one would obtain to leading order in the large number of colours $N_c$:

$$ \langle \bar{\psi} \Gamma_1 \psi \bar{\psi} \Gamma_2 \psi \rangle \approx \frac{1}{4N_c^2} \left[ \text{Tr}(\Gamma_1 \Gamma_2) - \text{Tr} \Gamma_1 \text{Tr} \Gamma_2 \right] \langle \bar{\psi} \psi \rangle^2. $$

However, the validity of the estimate in Eq. (24) has been tested phenomenologically in the vector (15), pseudoscalar (16) and baryons channels (17). It has been realized that Eq. (24) might be an underestimate of the real value of the four-quark condensate by a factor two to three. So, it is still desirable to have a control of the $(1/N_c)$ effects to Eq. (24) at finite $N_c = 3$. 
c) Gluon condensates

The existence of gluon condensates is intimately related to the trace of the energy momentum tensor and so to the breaking of scale invariance due to the dilatation anomaly. Sum rule analyses of the $e^+ e^-$ into $I = 1$ Hadrons$^{15,18}$ and charmonium data$^{11,19}$ have led to the "canonical" value:

$$\alpha_s < G^2 > \approx (447 \, \text{MeV})^4,$$  \hspace{1cm} (25)

However, other analyses based on different sum rules might indicate that the result in Eq. (25) is an underestimate of the actual value of the gluon condensate$^{20}$. Taking into account disagreements between various authors, one could consider Eq. (25) within a factor two to three. However, with larger values of the gluon condensate (see section IV.1) one should take care on the convergence of the OPE and study accurately the strength of the high dimension condensates. The strength of the gluon condensate has also been measured from the lattice$^{4b,c,21}$, which has a value compatible with Eq. (25) by taking into account various uncertainties generated by the two methods. However, from the point of view of renormalization group invariance, it is the quantity$^{22}$:

$$\beta(\alpha_s) < G^2 > + \gamma_m m < \bar{q}q >,$$  \hspace{1cm} (26)

which does not get renormalized, i.e. Eq. (26) is a pure number which has no $q^2$-dependence. ($\beta(\alpha_s)$ is the Gell-Mann Low $\beta$-function and $\gamma_m$ is the quark mass anomalous dimension.) However, the absence of renormalons in the operator product expansion (OPE) requires that $\alpha_s < G^2 >$ possesses a perturbative part$^{12}$. In the case of the SVZ-expansion with truncated series, this perturbative component is a real number and so $\alpha_s < G^2 >$ should be modified as:

$$\alpha_s < G^2 > + \lambda^4 \log \frac{\nu}{\lambda},$$  \hspace{1cm} (27)

where $\lambda$ is an infra-red scale whose meaning should be however clarified. The value of the triple gluon condensate $g f_{abc} < G_{a \ b \ c} G >$ is also useful to know because this condensate appears usually in the OPE of heavy quarks and gluonic currents. A crude estimate based on the dilute gas approximation gives the value$^{11}$:

$$g f_{abc} < G_{a \ b \ c} G > \approx 1 \, \text{GeV}^2 < \alpha_s G^2 >,$$  \hspace{1cm} (28)

while a lattice measurement gives a result comparable to Eq. (28) for the absolute value$^{22}$. One should notice that, contrary to $\alpha_s < G^2 >$, the triple gluon condensate has no positivity properties. Therefore, its sign
cannot be directly controlled. The estimate of higher dimension gluon condensates is done through the factorization hypothesis or its variant based on the heavy quark expansion. However, one might expect that an estimate in this way is very rough as \( < G^4 > \) condensates do not factorize at large \( N_c \).

d) **Mixed quark-gluon condensates**

The lowest dimension mixed condensate \( g < \bar{\psi} \sigma^{\mu \nu} \lambda^a \psi G_{\mu \nu}^a > \) has been estimated from baryonic sum rules:

\[
g < \bar{\psi} \sigma^{\mu \nu} \lambda^a \psi G_{\mu \nu}^a > \approx 2M_0^2 < \bar{\psi} \psi > \tag{29}
\]

where the fitted value \( M_0^2 \approx 0.2 \text{ GeV}^2 \) comes from a consistent choice of the spin-isospin 1/2 and 3/2 current. This value of \( M_0^2 \) is in agreement with the upper bound of 0.5 \( \text{GeV}^2 \) deduced from a Schwarz inequality-type analysis if one considers this latter for grants. The mixed condensate might also be related to the triple condensate via analysis based on the OPE. One gets the relation:

\[
m g < \bar{\psi} \sigma^{\mu \nu} \lambda^a \psi G_{\mu \nu}^a > = - g f_{abc} < G_{a} G_{b} G_{c} > \left( \frac{5 \alpha_s}{12 \pi} \right) + \ldots \tag{30}
\]

in the case of light quark fields where we have neglected terms like \( m^2 \log \frac{m^2}{\Lambda^2} \). Then, Eqs (29) and (30) might suggest that the triple condensate in Eq. (28) is positive definite in contrast to the one coming from lattice evaluation.

The clarification of this point is important for further phenomenological applications.

e) **On the separation of short and long wavelength fluctuations**

For a simple pedagogical reason, we study this question in the example of the Higgs model. We start from the bare Lagrangian of \( \lambda \phi^4 \)-theory:

\[
L = \frac{1}{2} (\partial_\mu \phi_B)^2 - \frac{1}{2} m_B^2 \phi_B^2 + \frac{\lambda_B}{4!} \phi_B^4 \tag{31}
\]

where \( \phi \) is the scalar field, \( m \) is its mass and \( \lambda \) its coupling. The index \( B \) corresponds to bare quantities. For simplifying our discussion, let us ignore renormalization effects and work with \( m_B^2 > 0 \), i.e. we have no condensate which breaks spontaneously the symmetry. We study the scalar propagator:

\[
D(q) = i \int d^4 x \ e^{i q x} < 0 | T \phi(x) \phi(0) | 0 > \tag{32}
\]

In the first way, we use the standard perturbative expansion:

\[
\mathcal{D}(q) = \quad + \quad \frac{\psi}{\mathcal{O}}
\]

Using, for instance, a Pauli-Villars regularization (the following conclusion is independent of the choice of regularization), one obtains:

\[
D(q^2) \approx \frac{1}{q^2 - m_B^2} \left( 1 + \frac{\lambda_B}{32\pi^2} \frac{M^2 - m_B^2 \log(M^2/m_B^2)}{q^2 - m_B^2} \right)
\]

\[
\approx \frac{1}{q^2} + \frac{1}{4q^4} \left( m_B^2 + \frac{\lambda_B}{32\pi^2} \frac{(M^2 - m_B^2 \log \frac{M^2}{m_B^2})}{m_B^2} \right)
\]  

(34)

where \( M^2 \) is the U.V. arbitrary scale.

In the second way, one evaluates the propagator using the SVZ expansion for \(-q^2 >> m_B^2\). Therefore:

\[
D(q^2) \approx C_1 \mathbb{1} + C_\phi \langle \varphi^2 \rangle + \ldots
\]  

(35)

One introduces a renormalization point \( \nu \) for separating the long and short wavelength fluctuations\(^{26}\). The Wilson coefficient \( C_1 \) comes from the perturbative graph and corresponds to the short fluctuations (\( p > \nu \)):

\[
C_1 = \frac{1}{2} + \frac{1}{4} \left( m_B^2 + \frac{\lambda_B}{32\pi^2} \frac{(M^2 - \nu^2 - m_B^2 \log \frac{M^2}{\nu^2})}{\nu^2} \right)
\]  

(36)

The Wilson coefficient \( C_\phi \) is obtained from the Feynman graph associated to the \( \varphi^2 \) "condensate":

\[
C_\phi = \frac{\lambda_B}{2q^4}
\]  

(37)

The condensate \( \langle \varphi^2 \rangle \) corresponds to the evaluation of the tadpole-like graph for \( p < \nu \) (large fluctuations). Therefore:

\[
\langle \varphi^2 \rangle \approx \frac{1}{16\pi^2} \left( \nu^2 - m_B^2 \log \frac{\nu^2}{m_B^2} \right)
\]  

(38)

One can easily see that at this level of approximation, the SVZ-expansion in Eqs (35) to (38) and the usual series in Eq (34) coincide. However, it is interesting to check if this coincidence continues to hold in higher orders of the expansion and for some other models\(^{12,26,27}\). This coincidence and so the \( \nu \)-independence of the OPE requires a careful definition of the vacuum condensates which should be non-trivial at higher dimensions. In the case of QCD, Ref. 13) attempts to study the OPE from the example of a vector two-point correlator by focusing on the absence of mass singularities in the OPE. However, a direct connection of their results with the conclusion obtained from QCD-like models is not transparent and still remains to be clarified.
f) Methods for evaluating the Wilson coefficients in QCD

In practice, the evaluation of the Wilson coefficients appearing in the OPE occurs through Feynman diagram calculations. In order to illustrate our discussions, let us study the two-point correlator built from the bilinear massless quark currents \( \bar{\psi} \Gamma \psi \) where \( \Gamma \) are any Dirac matrices. The effects of the unit operator \( 1 \) are due to usual quark-parton diagrams including radiative corrections and quark mass corrections. The Wilson coefficients of the non-perturbative condensates come from the diagrams drawn in the following Figures:

\[
\begin{align*}
\text{a)} \quad & < \bar{\psi} \psi > \\
\text{b)} \quad & + \cdots \alpha_s < G^2 > \\
\text{c)} \quad & < \bar{\psi} \Gamma_1 \psi \Gamma_2 \phi > \\
\text{d)} \quad & : g(\bar{\psi} G^{\mu \nu} a_{\mu \nu} \psi) \\
\text{e)} \quad & : g f^{abc} G^\mu_a G^\rho_c G^\mu_c \phi > \\
\text{f)} \quad & : \bar{\psi} \gamma_\mu \phi \bar{\psi} \gamma^\mu \phi >
\end{align*}
\]

These diagrams follow from the usual old-fashioned Wick product. The computations simplify considerably if one uses the following program:

Write the Taylor expansion of the quark condensate:

\[
< \bar{\psi}^F_{i \alpha}(x) \psi^{F'}_{j \beta}(0) > = < \bar{\psi}^F_{i \alpha}(0) \psi^{F'}_{j \beta}(0) > + x^\mu < \partial^\mu \bar{\psi}^F_{i \alpha} \psi^{F'}_{j \beta} > + \ldots
\]

\[
= \frac{1}{4 N_c} < \bar{\psi} \phi > \delta_{\alpha \beta} \delta_{F,F'} \{ \xi_{i j} + \frac{i}{4} m_F x^\mu (\gamma^\mu \phi)_{i j} + \ldots \} ,
\]

(39)

where \( F, F' \) are flavour indices, \( i, j \) are spinorial and \( \alpha, \beta \) colour indices; \( m_F \) is the quark mass which appears after the use of the Dirac equation.

Express the gluon field in terms of the field strength in order to obtain explicitly the gluon condensate. This can be done in the Schwinger fixed-point gauge \( x^\mu a^\mu (x) = 0 \). So:

\[
B^a_{\mu}(x) = \frac{1}{2} x^\nu G^a_{\mu \nu}(x) + \frac{1}{11} \frac{1}{3} x^\nu \delta^a_{\rho \mu} G^\rho_{\nu}(x) + \ldots + \frac{1}{n!(n+2)} x^\nu x^\nu_{\nu_1} \ldots x^\nu_{\nu_n} a^a_{\nu_1} \ldots a^a_{\nu_n} G^a_{\nu \mu} .
\]

(40a)
Use the fact that:

\[ \begin{align*}
\mathfrak{v}^1 \delta^1_{\mathfrak{v}^1} F_{\mathfrak{v}^1}(0) &= x^1 [D_{\mathfrak{v}^1} F_{\mathfrak{v}^1}] \text{ (gauge condition)}, \\
\mathfrak{v}^1 \mathfrak{v}^2 \mathcal{B}_{\mathfrak{v}^1 \mathfrak{v}^2}(0) &= 0.
\end{align*} \tag{40b} \]

Therefore, we can replace in this gauge the ordinary derivatives by covariant ones. In this way:

\[ B_{\mu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(n+2)} \mathfrak{v}^1 \mathfrak{v}^2 \ldots \mathfrak{v}^n [D_{\mathfrak{v}^1}(0), [D_{\mathfrak{v}^2}(0), \ldots [D_{\mathfrak{v}^n}(F_{\mathfrak{v}^m}(0)] \ldots ]] ] \text{.} \tag{40c} \]

Therefore, one obtains:

\[ \langle B_{\mu}(x) B_{\nu}(y) \rangle = \frac{1}{4} x^\lambda y^\rho < G_{\lambda \mu} G_{\rho \nu} + \ldots \]

\[ = \frac{1}{4D(D-1)} \mathfrak{v}^1 \mathfrak{v}^2 \mathcal{G} [\mathfrak{g}_{\alpha \mu} \mathfrak{g}_{\nu \rho} - \mathfrak{g}_{\lambda \nu} \mathfrak{g}_{\mu \rho}] < G_{\alpha \beta} G_{\alpha \beta} \geq \] \tag{40d}

where \( D = 4 - \varepsilon \) is the space-time dimension. This formula is the one which one uses for evaluating the coefficient of the gluon condensate.

By the same arguments, one obtains for the quark fields:

\[ \psi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{v}^1 \mathfrak{v}^2 \ldots \mathfrak{v}^n [D_{\mathfrak{v}^1}(0), \ldots, D_{\mathfrak{v}^n}(0)] \psi, \tag{41a} \]

\[ \bar{\psi}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{v}^1 \mathfrak{v}^2 \ldots \mathfrak{v}^n [\bar{\psi}(0), D^+_{\mathfrak{v}^1}(0), \ldots, D^+_{\mathfrak{v}^n}(0)] \bar{\psi}, \tag{41b} \]

which leads to the version of Eq (39) including the four-quark condensates:

\[ \langle \bar{\psi}_i^F(x) \psi_j^{F^*}(o) \rangle = \frac{1}{4N_c} \delta_{FF^*} \delta_{\alpha \beta} \left\{ \begin{array}{l}
\delta_{ij} + \frac{i}{4} m_F x^\lambda (\gamma_{\mu})_{ij} < \bar{\psi} \sigma_{\mu \nu} G_{\nu}^{\lambda} \frac{\lambda}{2} \psi > \\
\frac{i}{16} x^2 (\delta_{ij} + \frac{i}{6} m_F x^\mu (\gamma_{\mu})_{ij} < \bar{\psi} \sigma_{\mu \nu} G_{\nu}^{\lambda} \frac{\lambda}{2} \psi > \\
+ \frac{i}{288} x^2 (\gamma_{\mu})_{ij} g^2 < \bar{\psi} \gamma^\rho \frac{\lambda}{2} \phi \Sigma \bar{\psi} \gamma^f \frac{\lambda}{2} \phi^f > \end{array} \right\}. \tag{42} \]

Eq (42) indicates that the "propagation" of the \( \langle \bar{\psi} \psi \rangle \) condensate induces contributions of four-quark and mixed condensates. These effects should be added respectively to the ones of Fig. c and d). The effects of the mixed condensate of Fig. d) can be obtained by using the Taylor expansion:
\[ < \Phi_i(x) B(z) \Phi_j(0) > = \frac{1}{2} z^\mu < \Phi_i^{\mu \nu} \Phi_j > + \frac{1}{2} x^\mu < \Phi_i D_\mu^F \Phi_j > + \ldots \]

\[ = \frac{z^\mu}{g_0} \left( [\sigma_{\mu \rho} - \frac{m_F}{2} \gamma_{\mu \rho} x \gamma_{\mu \rho}] + i \frac{m_F}{2} x^\sigma \sigma_{\mu \rho} x^\nu \right)_{ij}. \]

\[ < \Phi_i \sigma_{\tau \kappa} G^{\tau \kappa} \Phi_j > + i(- \frac{2}{3} \gamma_\rho \gamma_\mu + \frac{2}{3} \gamma_\rho \gamma_\mu + \frac{1}{2} \gamma_\rho \gamma_\mu \sigma_{\mu \rho} )_{ij} \]

\[ = \frac{g^2}{2} < \phi \gamma^\alpha \phi > \Sigma_{\alpha}^{\beta} \phi \phi^{\beta} > \]

which also indicates that the propagation of the mixed condensate also induces a quartic condensate. The evaluation of the four-quark condensate from diagrams in Figs c, f are obtained using Wick's theorem. The total four-quark effects should then take into account the ones from Eqs (42) and (43).

As applications of the above methods one should obtain for the two-point correlator of the vector current \( \bar{\phi} \gamma^\mu \phi \), the transverse part:

\[ \Pi(q^2) = - \frac{1}{16 \pi^2} \left[ 1 - \frac{\alpha_s}{\pi} \right] \log \frac{-q^2}{2} + \frac{1}{24} \left[ m < \bar{\phi} \phi > + \frac{1}{2} \frac{\alpha_s}{\pi} < G^2 > \right] + \frac{g^2}{9 g_0^2} \alpha_s < \bar{\phi} \phi >^2, \]

where the contributions of the mixed and triple gluon condensates vanish at this leading order approximation and where we have used (for convenience) the factorization hypothesis of the four-quark condensate.


For the purpose of our discussions, let us still start from the two-point correlator:

\[ \Pi(q^2) = i \int d^4 x e^{i q x} < 0 \mid T \left( J_{\text{H}}(x) J_{\text{H}}(0) \right)^* \mid 0 >, \]

which obeys the Källen-Lehmann representation:

\[ \Pi(q^2) = \int_{-\infty}^{\infty} \frac{dt}{t - q^2 - i\epsilon} \frac{1}{\pi} \text{Im} \Pi(t) + \text{"subtraction"}. \]

Eq. (46) is a typical global duality sum rule as it equates the theoretical part of \( \Pi(q^2) \) known in the Euclidean region via the OPE to the spectral function which can be experimentally measured in the timelike region. The spectral function involves in principle all intermediate states \( \Gamma \) having the quantum numbers of the hadronic current \( J_{\text{H}}(x) \). The simplest way of parametrizing the spectral function is the duality ansatz "one resonance" + "QCD continuum". The resonance contributes in the narrow width approximation as:

\[ < 0 \mid J_{\text{H}}(x) \mid H > = f_H \frac{m_H^2}{4 \pi^2}, \]
where $f_H$ is the resonance coupling to the current and $M_H$ its mass. The "QCD continuum" starts from a threshold $\sqrt{s}$ which should not be interpreted as a physical threshold but as an average of the effects of all excited states. This continuum contribution comes from the discontinuity of QCD diagrams of the OPE. In practice it is the coefficient of the $\log n^2$ terms of the OPE.

a) Laplace (Borel) global duality

This sum rule is obtained by applying to both sides of Eq. (46), the Laplace-Borel operator $^{11,29}$ : ($Q^2 = -q^2 > 0$)

$$\tilde{L} = \frac{(-Q^2)^N}{(N-1)!} \frac{\delta^N}{(\theta Q^2)^N}$$

$$\lim Q^2, N >> \text{ and } Q^2/N = \frac{1}{\tau} \text{ fixed.}$$

(48a)

Therefore, one gets

$$\tilde{L} \Pi = \tau \int_0^\infty dt e^{-\tau t} \frac{1}{n} \text{Im} \Pi(t)$$

and the moment ratios:

$$R(\tau) = -\frac{d}{d\tau} \log \int_0^\infty dt e^{-\tau t} \frac{1}{n} \text{Im} \Pi(t).$$

(49)

*) The advantage of $\tilde{L} \Pi$ with respect to $\Pi$ is threefold:

- The application of various derivatives in $Q^2$ eliminates the arbitrary subtraction ultraviolet scale which one usually introduces in the QCD evaluation of the Green's function.

- The Laplace transform $\tilde{L} \Pi$ converges faster than the QCD series of $\Pi$ because the Laplace operator $\tilde{L}$ introduces an extra factor $1/n!$ into the coefficient of the condensates $<O_{2n}>$. In this way, $\tilde{L} \Pi$ is less sensitive to the effects of the high dimension condensates than $\Pi$.

- The exponential increases the role of the lowest ground state into the sum rule and so the sum rule is less sensitive to the effects of the high mass continuum states. We shall see later on that the naive duality ansatz gives a quite good parametrization of the spectral function for this type of sum rule.

* The optimal information from the sum rule in Eq. (48) is obtained in the so-called sum rule window where the results present stability in the change of the sum rule variable $\tau$. In this region one should also have the dominance of the lowest ground state over the continuum effects. At the same time, one should have a dominance of the lowest dimension condensates in the OPE in order to trust the QCD series. If one is lucky
enough, this sum rule window can be a large plateau, whilst in some
other cases it can only be an extremum. In practice this stability
region corresponds to the value of the scale \( \tau_c \) of the order of
1.2 GeV\(^2\) characteristic of hadronic mass. However, we have not yet
a deep interpretation of the meaning of this phenomenological value,
except from the one of "minimum sensitivity" of physical observables on
unphysical parameters at this sum rule window. However, according to
arguments based on the analysis of heavy quark systems\(^{2a}\), the
value of \( \tau_c \) corresponds to the level-shift between the ground state
and the first excitation. This value is almost flavour-independent.

*) The moment ratios \( R(\tau) \) have the virtue of being very sensitive to the lowest
ground state mass squared \( M_H^2 \) and not to its residue \( f_H \). Its accuracy
and physical meaning have been tested in Ref. 29 b) from the example of
the three-dimensional harmonic oscillator in quantum mechanics. The non-
relativistic analogue of the Laplace transformed two-point correlator is

\[
F(\tau) = \sum_{n=0,2,4,\ldots} (R^{(o)}_n)^2 E_n^\tau
\]

(50)

where \( R_n^{(o)} \) is the radial wave function for zero angular momentum and \( E_n \)
the corresponding eigenvalue. \( \tau \) is the parameter which regulates the
energy resolution of the sum rules and plays the role of the "imaginary"
time variable. For the case of the harmonic oscillator potential
\( V(r) = \frac{1}{2} m \omega^2 r^2 \), one knows the exact expression of \( F(\tau) \), which implies for
\( \tau \to \infty : \quad E_0 = -\frac{d}{d\tau} \log F(\tau) = R(\tau \to \infty). = \frac{3}{2} \omega \)

(51)

\[ \lim_{\tau \to \infty} \]

Now, in order to make a close analogy with QCD, we can compare the
approximate series of \( R_{\text{approx}}(\tau) \) with the exact form \( R_{\text{exact}}(\tau) \). The
zeroth-order approximation corresponds to the free motion. The next
ones are the Born corrections. This comparison is shown in the
following figure:

![Graph showing R(\tau)/E_0 vs \omega \tau]

- 2.0
- 1.6
- 1.2
- 1.0
- 0.8
- 0.4

0.4 0.8 1.2 1.6 2.0 2.4 2.8 \( \omega \tau \)
for $R(\tau)$ with two and four terms and the exact expression. In the limit of imaginary time variable $\tau \to \infty$, the exact expression of $R(\tau)$ tends asymptotically to the lowest energy eigenvalue $E_0$. The exact form of $R(\tau)$ and the formal limit $\tau \to \infty$ are not reached in QCD. At finite $\tau$ and for a truncated series in $\tau$, one observes that $R(\tau)_{\text{approx.}}$ is above the eigenvalue $E_0$ which is reflected by the $R(\tau)$ positivity properties. The addition of more and more terms in the expansion increases the agreement of $R(\tau)$ with $E_0$. The minimum of $R(\tau)_{\text{approx.}}$ gives an upper bound to the value of $E_0$. At this minimum, the information on $E_0$ is optimal and the continuum contribution to $R(\tau)$ is minimal. The strength of the continuum is controlled by the distance between $R(\tau)_{\text{approx.}}$ and $E_0$. We shall see, later on, explicit examples for the uses of $R(\tau)$ in QCD. In practice, the minimum of $R(\tau)$ separates the QCD region where the approximation is reliable $[0, \tau_{\text{MAX}}]$ from the one where the effects of high dimension condensates are important. In the region $[0, \tau_{\text{MAX}}]$, one confronts $R(\tau)_{\text{QCD}}$ with $R(\tau)_{\text{EXP}}$ using a least-square fit program. Optimal results should be stable against the changes in $\tau$. In principle, the results should also be insensitive to the value of the continuum threshold $\sqrt{\Sigma}$. However, this stability in $\sqrt{\Sigma}$ is not often obtained within the parametrization of the continuum using the discontinuity of QCD diagrams. In this case, the optimal results are the sets of output from the fit procedure with the minimal $\chi^2/\text{NDF}$.

b) **Finite Energy Sum Rule local duality (FESR)**

Another type of sum rules used in QCD is the FESR:

$$
\int_0^\tau \frac{c}{\xi} dt \cdot \frac{t^n}{\xi} \left( \left( H_\tau (t) \right)_{\text{QCD}} - \left( H_\tau (t) \right)_{\text{EXP}} \right) \quad n=0,1,\ldots, \tag{52}
$$

which was known a long time ago. It equates the QCD expression with the experimental one. The derivations of Eq. (50) within QCD can be done in various ways. One can use either a Cauchy theorem applied to a finite radius contour $Q^2$ in the $q^2$-complex plane\(^7,9a\). One can also use in the duality ansatz of the spectral function, the $\tau$-expansion of both sides of Eq. (48) and equate various coefficients having the same $\tau$-behaviour for any $\tau^{30}$. A more formal way for deriving the FESR which brings insights into the meaning of local duality in QCD is the Gaussian sum rule method\(^31\), which we will explain briefly. The Gaussian sum rule centred at a point with finite width resolution $\sqrt{4\sigma}$ is:
\[
G(\xi, \sigma) = \frac{1}{\sqrt{4\pi \sigma}} \int_0^\infty dt e^{-\frac{(t+\xi)^2}{4\sigma}} \int_0^1 \frac{1}{x} \text{Im} \Pi(t) dt
\]

which one can obtain by applying the Laplace operator :

\[
\mathcal{L} = \lim_{\tau^2 \to \infty} \frac{\tau^2}{\tau^2} = \frac{1}{N} \text{Im} \Pi(t)
\]

\[
F(\tau) = e^{-\xi \tau} \int_0^\infty dt e^{-\tau t} \frac{1}{x} \text{Im} \Pi(t).
\]

Eq. (53) shows that the strict local duality :

\[
G(\xi, 0) = \frac{1}{x} \text{Im} \Pi(\xi)
\]

is obtained in the formal limit \( \sigma \to 0 \) which is not unfortunately reached in QCD. The qualitative interesting feature of Eq. (53) is reflected by the heat evolution equation obeyed by \( G \) :

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial \sigma} \right) G(\xi, \sigma) = 0
\]

with the initial condition given by Eq. (54c) ; here, \( \xi \) is the position, \( \sigma \) the time and \( \frac{1}{x} \text{Im} \Pi(t) \) is the temperature distribution in the region \( 0 \leq \xi \leq \infty \).

The conservation of total heat leads to the duality sum rules :

\[
\int_0^\infty d\xi G(\xi, \tau) = \int_0^\infty d\xi \frac{1}{x} \text{Im} \Pi(t)
\]

which can be generalized to higher moments by using the generating function of Hermite polynomials and their orthogonality properties. The asymptotic behaviour of \( G(\xi, \tau) \) and \( \text{Im} \Pi(t) \) permits us to rewrite Eq. (55) in the FESR form in Eq. (52) .

c) Analytic continuation

Now let us present briefly the method of analytic continuations where different variants have been discussed in the literature \(^\text{32}\) with, however, the unfortunate names of "beyond QCD sum rules" or "alternative to QCD sum rules". In this method the \( \frac{1}{t+Q^2} \)-term of Eq. (46) is replaced by some kernel function.

Ref. (32a) uses for instance, a polynomial in \( t \) as a kernel and then applies the Cauchy theorem to the finite radius \( Q^2 \) contour in the complex-\( Q^2 \) plane. In this way, the author gets the sum rule :
\[ \Pi(Q^2) = \frac{1}{2\pi} \int \frac{C}{t+Q^2} \left( \frac{1}{t+Q^2} - \sum a_n t^n \right) \Pi(t) + \Delta_n \]  

(56a)

where \( \Delta_n \) is the "fit error":

\[ \Delta_n = \frac{1}{\pi} \int_0^\infty \frac{2}{t+Q^2} \left( \frac{1}{t+Q^2} - \sum a_n t^n \right) \text{Im} \, \Pi(t) \]  

(56b)

which should tend to zero and which is controlled by the data. The main difference with previous sum rules is that the role of the data enters only in \( \Delta_n \) whilst the main strength of \( \Pi(Q^2) \) is given by the QCD-expression of \( \Pi(t) \). Such a sum rule might be useful for probing the region of small-\( Q^2 \) behaviour of the Green's function but the fact that the data is only useful for \( \Delta_n \) might decrease the reliability of the predictions of the method.

d) Special sum rules for heavy quarks.

For the heavy quark systems, one can still use the sum rules presented before. However, the fact that \( M_Q^2 \gg \Lambda^2(M_Q \) being the heavy quark mass) allows us to study some other types of sum rules expressed in series of \( 1/M_Q^2 \) in the OPE. SVZ work with the moments:

\[ \left. \frac{1}{(N-1)!} \frac{d^N \Pi}{(Q^2)^N} \right|_{Q^2 = 0} = \int_0^\infty \frac{dt}{t^{N+1}} \frac{1}{\pi} \text{Im} \, \Pi(t) = \frac{C_0}{(M_Q^2)^{N+1}} \left\{ 1 + \sum_{n=1,2} C_{2n} Q_{2n}^2 \right\} \]  

(57)

for the study of the charmonium systems. It is clear that a large number of derivatives increases the role of the ground state but at the same time the effects of high dimension condensates increase like \( N^{2n-1} \). Therefore, a critical value of \( N \) is necessary for optimal information from the sum rule. According to the analysis of Ref. (23a) which includes the effects of dimension 8 condensates into the OPE, it is difficult to find such a value of \( N \) because the high dimension condensate effects are important in Eq. (57) for the charmonium systems. For this reason it becomes useful to evaluate various derivatives at \( Q^2 = Q_0^2 \neq 0 \) as suggested by Ref. (19b). In this way, the convergence of the QCD series can be notably improved. However, numerical estimates of the strength of high dimension condensates are based on the factorization hypothesis which might be very rough for the gluon condensates of high dimensions. We shall see later on that SVZ\(^{11,33}\) and others\(^{2a}\) use the above sum rules for the estimate of the gluon condensate \( \alpha_s < G^2 > \) and the charm quark mass.
What we can conclude from the above catalogues of sum rules is:

various forms of sum rules are complementary to each other in the sense that
their ability in giving the optimal information on physical observables is a
function of the problems to be studied. In the following, we will mainly
concentrate our discussions on the uses of the Laplace-Borel sum rules, which
we shall sometimes compare with the FESR results. The results from the analytic
extrapolation method will only be considered in the estimate of the
subtraction constant.

IV - PHENOMENOLOGY OF THE LIGHT QUARK SYSTEMS

1. Vacuum condensates from $e^+e^- \rightarrow I = 1$ Hadrons.

As a first phenomenological application of our previous discussions,
let us analyze the $\rho$-meson sum rule moment ratios:

$$R(\tau) \equiv -\frac{d}{d\tau} \log \int_0^\infty dt e^{-\tau t} \frac{1}{x} \text{Im} \Pi_\rho(t) ,$$

(58)

where $\Pi_\rho(q^2)$ is the two-point function built from the quark bilinears having
the $\rho$-meson quantum numbers:

$$J^\mu_\rho \equiv \frac{1}{2} (\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$$

(59)

The QCD-expression of the moment is $^{11,15}$:

$$R_\rho^{QCD}(\tau) = \tau^{-1} \{ 1 + C_4 < 0_4 > \tau^2 + C_6 < 0_6 > \tau^3 + \ldots \}$$

(60)

where:

$$C_4 < 0_4 > = \frac{2\pi}{3} \alpha_s < c^2 > (1 + 12\pi \frac{m_u < \bar{u}u> + m_d < \bar{d}d>}{\alpha_s < c^2 >})$$

(61)

$$C_6 < 0_6 > = 6\pi^2 \alpha_s \{ <(\bar{u}\gamma^\mu \gamma^5 \lambda^a u - \bar{d}\gamma^\mu \gamma^5 \lambda^a d)^2 > +$$

$$\frac{2}{9} <(\bar{u}\gamma^\mu \lambda^a u + \bar{d}\gamma^\mu \lambda^a d) \Sigma_{\mu\nu\lambda\phi} \Sigma_{\lambda\nu\mu\phi} \phi_\mu \lambda \phi > \} .$$

(62)

One should notice that contrary to the Laplace $F(\tau)$ sum rule in Eq. (48),
$R(\tau)$ is not sensitive to the leading $\alpha_s$ contribution to the unit perturbative
operator. The spectral function $\text{Im} \Pi_\rho(t)$ is related to the $e^+e^-$ into $I = 1$
Hadrons total cross-sections via the optical theorem:
\[ \frac{1}{\pi} \text{Im} \Pi_\rho (t) \equiv \frac{t}{16 \pi^3 \alpha^2} \sigma_{e^+e^- \to l=1} (t) \]

(63)

The data of \( \sigma_{e^+e^- \to l=1} \) is shown in Fig. 4.1 normalized to \( \sigma_{e^+e^- \to \mu^+\mu^-} \) up to \( \sqrt{t} \approx 2 \text{ GeV} \). Beyond this value of \( t \), we parametrize the continuum using the discontinuity of the QCD

\[ \frac{\sigma (e^+e^- \to l=1)}{\sigma (e^+e^- \to \mu^+\mu^-)} \]

\[ \sqrt{t} [\text{GeV}] \]

**FIG. 4.1**

Feynman diagrams. This phenomenological choice of the continuum threshold will be justified from the stability criterion of the output parameters and for the asymptotic consistency of the QCD and experimental sides of the sum rules. (We confront the phenomenological and QCD sides of the moments \( R(t) \) in the region \( [0, \tau_\text{MAX}] \) by using the MINUIT fitting program with the two parameters \( C_4 < 0_4 > \) and \( C_6 < 0_6 > \). We show in Fig. 4.2 the behaviour of the quality of our fit versus the choice of \( [0, \tau_\text{MAX}] \). At \( \tau_\text{MAX} \approx 1 \text{ GeV}^{-2} \), the constraint is optimal. Below 1 GeV\(^{-2} \), we lose most of the information from low energies while above it the validity of the OPE diminishes.
Our best fit corresponds to the set of values:
\[
\alpha_s < G^2 > \simeq (4 \pm 1)10^{-2} \text{ GeV}^4
\]
\[
C_6 < O_6 > \simeq (0.18 \pm 0.06) \text{ GeV}^6
\] (64)
where we have used the pion PCAC relation in Eq. 22 for the estimate of the $m < \bar{u}u >$ quark condensate.

We give the behaviour of these results versus the changes in $\tau_{\text{MAX}}$. The results are quite stable and indicate that larger values of $C_4 < O_4 >$ require larger values of $C_6 < O_6 >$ because their contributions tend to compensate in the moments $R(\tau)$.

However, the results in Eq. (64) are sensitive to the changes of the continuum threshold $\sqrt{E_c}$. The additional requirement that these values are minimally sensitive to the choice of the unphysical parameter $\sqrt{E_c}$ requires that their derivative with respect to $\sqrt{E_c}$ vanishes. This corresponds to $\sqrt{E_c} \simeq 2 \text{ GeV}$ which corresponds to the phenomenological threshold. This value is also in the range required for a minimal sensitivity of $R_{\text{EXP}}^\rho$ in the choice of $\sqrt{E_c}$ \( \frac{d R_{\text{EXP}}^\rho}{d t_c} = 0 \).

The error bars in Eq. (64) take into account the uncertainties in the choice of $\tau_{\text{MAX}}$ and those from the $e^+e^-$ data. One should, however, notice that in spite of the large uncertainties of the data, the results are quite accurate because the errors tend to compensate for the ratios of the sum rules. The resulting fit of $R(\tau)$ is shown in Fig. 4.5:
One might compare our results with the set of condensates from the FESR
analysis of the same p-channel\textsuperscript{20b,c) :}
\[
\begin{align*}
\langle \alpha_s G^2 \rangle & \approx 0.15(0.33)\text{GeV}^4 \\
C_6 < O_6 > & \approx 0.51(0.75)\text{GeV}^6
\end{align*}
\]
which have been obtained inside the stability region $\sqrt{t_c} \approx 1.7 - 2.4$ GeV.
However, one should notice that the FESR analysis is strongly dependent
on the parametrization of the high-$t$ behaviour of the spectral function
due to the expression of the moments in Eq. (52). Therefore, extracting
the small numbers like the condensate values from the FESR should not be
accurate due to the uncertainties of the $e^+e^-$ data between 1 to 2 GeV.
The set of values in Eq. (65) might only be consistent with the ones from
the moment ratio analysis at the value of $\sqrt{t_c} \approx 2.4$ GeV which is not,
however, the optimal value of $\sqrt{t_c}$ corresponding to our minimum sensitivity
criterion. Then, it is difficult to choose between Eq. (64) and Eq. (65)
from the alone $e^+e^-$ data. Concerning the four-quark condensate, both
results indicate a net deviation from the factorization hypothesis (Eq. (24)) which is expected to be only true in the large $N_c$-limit. Concerning the gluon condensate it is still desirable to estimate it from other sources than the ones above and the charmonium data. The analysis of Ref. 34) based on the two-point functions involving the hybrid $g \bar{d} \gamma_5 G^a_{\nu \mu} \lambda u$ and axial $\bar{d} \gamma^\mu \gamma^5 u$ currents seems to favour the canonical value of the gluon condensate. Due to the importance of the exact value of the gluon condensate for some other phenomenological uses, we are repeating the analysis of Ref. 34) using FESR$^{35}$). Moreover, we need some other consistency tests implied by the uses of a large value of the gluon condensate.

2. Quark masses

We shall be concerned here with the quark mass terms $m \bar{q} q$ of the QCD Lagrangian which should be contrasted with the so-called constituent quark masses involved in the non-relativistic potential models. The light quark mass ratios have been obtained successfully from the comparison of various current algebra Ward identities at zero momentum transfer which permits us to relate the quark masses to the hadron parameters$^{36}$). The success of the current algebra approach is mainly due to the fact that the mass ratio is scale independent or needs not be renormalized. In this way, contrary to the absolute value of the masses, the ratio is well defined$^{36}$):

$$\frac{m_d}{m_u} = 1.8 \pm 0.3 \quad ; \quad \frac{m_s + m_u}{m_d + m_u} = 12.3 \pm 1.7 .$$ (66)

In order to renormalize the masses (like other couplings), we have to choose a renormalization prescription. Within a dimensional regularization and renormalization scheme, the renormalized mass is$^{37}$:

$$m_R = Z_m m_B$$ (67a)

where

$$Z_m = 1 + \gamma_m \frac{1}{\epsilon}$$ (67b)

is the renormalization constant, $d = 4-\epsilon$ is the space-time dimension and $\gamma_m = \gamma(\frac{-\xi}{\epsilon}) + \ldots (\gamma_1 = 2)$ is the mass anomalous dimension. These quantities come from the pole-part of the fermion self-energy diagrams. The running quark mass is a solution of the differential equation: $(t = \frac{1}{2} \log - q^2 / \nu^2)$
with the boundary condition \( \bar{t}(t = 0) = m/v \). It reads to two loops:

\[
\bar{t}(-q^2) = \frac{\bar{t}}{1 + 0.9 \frac{\bar{t}}{s}} \left( 1 + \beta_1 \frac{\bar{t}}{s} + \ldots \right),
\]

where we have introduced the scale- and scheme-independent invariant mass \( \bar{t} \). The running QCD-coupling \( \alpha_s/\pi = \frac{2}{\beta_1 \log q^2/\Lambda^2} \) is well known and is a solution of the differential equation:

\[
\frac{d \bar{t}}{dt} = \alpha_s(\bar{t}), \quad \bar{t}(t = 0) = \alpha_s(\pi),
\]

where \( \alpha_s(\bar{t}) \) is the Gell-Mann-Low \( \beta \)-function. We propose to estimate the invariant mass \( \bar{t} \) using the sum rules. The most sensible quantity is the two-point correlator associated to the divergence of the axial current:

\[
\phi_5(q^2) = i \int d^4x \, e^{iqx} \langle 0| T(\bar{u} A_{1j}^\mu(x)) (\bar{u} A_{1j}^\mu(0))^\dagger |0\rangle,
\]

with

\[
\bar{u}(i\gamma_5)d(x) = (m_u + m_d) \bar{u}(i\gamma_5)d.
\]

The above two-point function has been evaluated to two \(^{38}\) and three \(^{39}\) loops. Power corrections to the perturbative calculations have been included up to the dimension-six condensates \(^{11}\). The low-energy behaviour of the two-point correlator is also controlled from current algebra Ward identities \(^{40}\):

\[
\langle 0| [Q_5, \bar{u} A_{1d}^\mu] |0\rangle = \phi_5(0) = -(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle,
\]

where \( Q_5 \) is the axial-charge. Within the above QCD-expressions, one can immediately write the OPE of the global Laplace sum rule:

\[
F(t) = \tau^- t \int_0^\infty dt \, e^{-t\tau} \frac{1}{\tau} \Omega \phi_5(t).
\]

The spectral function \( \Omega \phi_5(t) \) can be saturated by the two lowest ground states \( \pi \) and \( \pi'(1.3) \). A QCD model has been used for the continuum contribution. Here, one should notice that, contrary to the \( \rho \)-meson channel, one should treat the \( \pi \) and \( \pi'(1.3) \) at the same level due to the Goldstone nature of the \( \pi \) and to the fact that the \( \pi' \)-decay amplitude behaves like \( m_{\pi}^2 \) from arguments based on current algebra analysis. There are different models for the parametrization of the \( \pi' \) effects which satisfy the Goldberger-Treiman and PCAC relations \(^{37,41} \). In this particular quasi-Goldstone channel, corrections due to the finite width and threshold effects will play the role of
stabilisers of the estimate. The sum rule expression of the invariant
mass to three-loops is given in Ref. 10b). We show in Fig. 4.6 the behaviour
of such an invariant mass versus the sum rule variable \( \tau \) and for the values of
the condensates obtained in Eq. (64):

\[ \hat{m}_u + \hat{m}_d [\text{MeV}] \]
\[ t_c [\text{GeV}^2] \]

\[ \Lambda \cong 100 \text{ MeV} \]

\[ \frac{1}{\sqrt{\tau}} \text{(GeV)} \]

**Fig. 4.6.** Behaviour of \( \hat{m}_u + \hat{m}_d \) versus the sum rule scale \( \tau \) for a given
of \( \Lambda \) and different \( t_c \) -
The region A-B represents the sum rule window, i.e. for \( \tau^{-\frac{3}{2}} \) larger than 1.5 GeV, the QCD continuum exceeds the resonance effects whilst below 1 GeV, we might expect that uncontrollable small-size instanton effects break the OPE. One can check that inside the sum rule window the effects of the condensates, even the ones in Eq. (65), are not essential. The stability of the results in the changes of the sum rule scale \( \tau \) is better obtained for a finite width parametrization of the \( \pi^+ \) and for a continuum threshold \( \sqrt{s_c} = 1.7 \) GeV which might coincide with the \( \pi^+ \)-mass fixed by arguments based on the Regge trajectories. Global-duality sum rule gives the optimal estimate for \( 100 < \Lambda_{RS} < 150 \) MeV:

\[
(\hat{m}_u + \hat{m}_d) = (25 \pm 5) \text{MeV} ; \ (\hat{m}_u + \hat{m}_d)(1 \text{GeV}) = (16 \pm 3) \text{MeV}
\] (74)

which is consistent with the optimal lower bound obtained from the positivity of the spectral function\(^{38,42}\). These bounds are also shown in Fig. 4.6. Alternative estimates based on FESR\(^{39,43}\) give values comparable to the ones in Eq. (74). The extension of the above analysis to the strange quark channel is straightforward provided that one includes properly the SU(3)_c-breaking effects due to the strange quark mass, condensates and to the kaon mass. For more details on the derivation of the results from Laplace sum rules we refer to Ref. 10b). Discussions on the extraction of the quark masses from other channels like the divergence of the vector current and the \( \phi \)-meson one are also presented in this work. The weighted average of the quark masses obtained from the sum rules and for \( 100 < \Lambda < 150 \) MeV is:

\[
\begin{align*}
\hat{m}_u & = (8.6 \pm 1.5) \text{MeV} & \hat{m}_u(1 \text{ GeV}) & = (5.1 \pm 0.9) \text{MeV} \\
\hat{m}_d & = (15.2 \pm 2.7) \text{MeV} & \hat{m}_d(1 \text{ GeV}) & = (9.0 \pm 1.6) \text{MeV} \\
\hat{m}_s & = (225 - 319) \text{MeV} & \hat{m}_s(1 \text{ GeV}) & = (133 - 189) \text{MeV} \\
\end{align*}
\] (75)

We have mentioned earlier that these masses are the mass of the QCD Lagrangian. A clean relation between these masses with the so-called "current mass" is not clear because of the scale dependence of the running mass. Only the ratio of the mass value can be compared. The relation of the above masses with the so-called "constituent" mass is not also clear. As first pointed out by Politzer\(^{44}\), the main contribution to the constituent mass comes from the chiral \( < \bar{q}q > \) condensate which arises from the OPE of the fermion self energy:

\[
m_{\text{const}} = \hat{m}(q^2) - < \bar{q}q > (3 + \alpha_s q^2) \frac{4\pi\alpha_s}{q^2}
\] (76)
However, this off-shell definition is unfortunately gauge-dependent ($\alpha_G$ being the covariant gauge parameter).

3. Deviation from kaon PCAC and value of the quark condensates.

You might have noticed that the value of the (pseudo) scalar two-point correlator $\Phi_5(0)$ at zero momentum is controlled by the quark condensate via current algebra Ward identities (Eq. 72). Also, a saturation of the associated spectral function $\text{Im} \psi_5(t)$ by the lowest ground state gives, for instance, the well-known PCAC relation (Eq. 22). In the following, we shall evaluate $\Phi_5(0)$ from the sum rules. This analysis will tell us how good the PCAC relation is for the pion and especially for the kaon. In so doing, we work with the subtracted dispersion relation \(^{45}\):

$$\Phi_5(q^2) - \Phi_5(0)q^2$$

(77)

to which we apply the Laplace operator. In this way, we obtain a global sum rule in terms of the spectral function $\frac{1}{\tau} \text{Im} \psi_5(t)$. Using a straightforward algebraic manipulation, we can combine the above sum rule with the one of $\psi_5(q^2)$ in Eq. (73). Therefore, we obtain \(^{45,46}\):

$$\psi_5(0)^u_s = \int_0^\infty \frac{dt}{\tau} e^{-t\tau}[1-t^2(1+2\frac{\alpha_s}{\pi} + 2m^2_s \tau - \log \left(\frac{m^2_s}{\pi} + \gamma_E\right)] \frac{1}{\tau} \text{Im} \psi_5(t)^u_s,$$

(78)

where $\gamma_E$ is the Euler constant and we have used the fact that $m^2_s >> m^2_u$.

The advantage of the combination of sum rules in Eq. (78) compared to the one obtained directly from the Laplace transform of Eq. (77)\(^{45}\) is the absence of the leading QCD perturbative and non-perturbative terms in Eq. (78). Therefore the sum rule will not be too sensitive to the form of the continuum, which we parametrize using the duality ansatz:

$$-\frac{3}{8\pi^2} \left(\frac{m^2_s}{\pi}\right) e^{-t\tau} \{t_c - 2 - \frac{s}{\pi} s(t_c + \tau^{-1})\},$$

(79)

coming from the discontinuity of the Feynman graphs. Firstly, we test the sum rule in Eq. (78) for the $\bar{u}d$ channel which is well controlled by the pion PCAC. We obtain:

$$\psi_5(0)^d_{ud} = \frac{2m^2_s}{\pi} \{1 + O(m^2_s, m^2_u, \tau)\}$$

(80)

where the optimization scale $\tau^{-1}$ is the order of 1-2 GeV\(^2\). The correction terms to the PCAC relation do not exceed 4% in accordance with the expectations from current algebra \(^{47,40}\). Now, we study the sum rule for the $\bar{u}d$ scalar channel by
assuming that the $\delta$-meson is the lowest meson mass resonance associated to
the divergence of the vector current:

$$<0|\bar{a}_\mu v^\mu(x)_{ud}|\delta>=\sqrt{2} f_\delta M_\delta^2$$

(81a)

with:

$$\bar{a}_\mu v^\mu(x)_{ud} = i(m_u - m_d)\bar{u}d$$

(81b)

The main unknown in the analysis is the $\delta$-decay amplitude, which can be
estimated phenomenologically using its relation with the hadronic kaon
tadpole mass-difference 48) or using 10b) its relation with the $u, d$ mass
difference from the sum rule discussed in Ref. 48). Using the set of
values in Eq. (75), we obtain the correlated set of parameters 10b):

$$f_\delta \approx (1.3 \pm 0.2) \text{MeV} \quad ; \quad \tau_c \approx (2.4 - 3) \text{GeV}^2.$$  

(82)

With such values, we give in Fig. 4.7 the behaviour of $\psi(o)^u_d$ versus the sum
rule scale $\tau$. The "sum rule window" corresponds to $\tau^{1/2} \approx 0.8 - 0.9 \text{ GeV}
to which corresponds the value:

$$\psi(o)^u_d = -(m_u - m_d)<\bar{u}u - \bar{d}d >$$

$$\approx (0.5 \pm 0.2) 10^{-6} \text{ GeV}^4$$

(83)

Solving Eqs 80, 72 and 83, we deduce:

$$\frac{<\bar{d}d>}{<\bar{u}u>} \approx 1 - (1 \pm 0.3) 10^{-2}$$

(84)

which indicates a tiny SU(2)$_F$-breaking in accordance with the analysis in
Ref. 49) and with the one based on the $\rho$-$\omega$ mixing 11) $(1 - 1.5 10^{-2})$. It is
encouraging that the sum rule is able to extract a small deviation like the
one in Eq. (84). Therefore, it can make sense to also study the strange quark
analogue of the above results.
Fig. 4.7 : behaviour of $\Psi(o)_d^u$ versus the sum rule scale $\tau^{-1}$. 

The analysis of $\Psi(o)_s^u$ was first done in Ref. 45). It has been improved and extended since by many other authors (46, 49, 50, 43, 10b,c). The improvements of the analysis are mainly due to the inclusion of the high mass continuum effects. Within the form in Eq. (78) it is easy to check that the leading effects of the quark and gluon condensates of dimension less than or equal to six are not present in the sum rule. This is certainly a significant advantage for a fast convergence of the QCD series compared to the one used in Ref. 50a,b. The behaviour of $\Psi(o)_s^u$ is shown in Fig. 4.8 for different models of the $K'$-parameters and of the continuum. The continuous line is the result of Ref. 46) for the set of values of the $K'$-parameters:

$M_{K'} = 1.45 \quad ; \quad f_{K'} = (0.34 - 0.44) f_K$

and without a QCD continuum. The other curves come from Ref. 50a). The dotted line is the one where the effects of dimension-six condensates have been
Fig. 4.8: Behaviour of $\psi_5(o)^u_s$ versus the sum rule scale $\tau^{-1/2}$ and for various choices of the continuum model.

considered; discontinuous lines correspond to different choices of the QCD continuum thresholds \( a) \tau_c \approx 1.8 \text{ GeV}^2 \); \( b) \tau_c \approx 2.5 \text{ GeV}^2 \); \( c) \tau_c \approx \infty \). In Ref. 50a) the parametrization of the $K'$-effect is done within the framework of a dual-like model which has been tested from some other current algebra low-energy data\(^{51})\). One can notice that within the form of $\psi_5(o)$ given in Ref. 45 and 50a), the role of the QCD continuum is essential for ensuring the stability of the results, whilst in Refs 46,10b,c) with the modified form given in Eq. 78) this is not the case due to the cancellation of the leading effects of the QCD continuum for the combination of the sum rules. For completeness, we also show the value of $\psi_5(o)^u_s$ coming from a single kaon pole plus a QCD continuum\(^{45}\) which starts at the physical threshold $\tau_c \approx 9M_K^2$ (dashed-dotted curve). The results of the above analysis give:

$$\psi_5(o)^u_s \approx (3.5 \sim 4) \times 10^{-3} \text{ GeV}^4$$ \hspace{1cm} (81)

which should be compared with the PCAC estimate:

$$\psi_5(o)^u_s \approx 2M^2_{K'K} \approx 6.3 \times 10^{-3} \text{ GeV}^4$$ \hspace{1cm} (82)

It is clear that SU(3)$_F$ corrections to the kaon PCAC are important. Then, it would be useful to see explicitly such corrections from alternative approaches. Analogous analysis has been done for the divergence of the vector current. For $\psi(o)^u_s$, one has used the $K\pi$ phase shift data for parameterizing the scalar spectral function. This $K\pi$ data can be replaced by a one resonance $^\times$ plus QCD continuum with the parameters:
\[ f_x = 34 \text{ MeV}, \quad M_x = 1.35 \text{ GeV}; \quad \sqrt{t_c} = 1.84 \text{ GeV}, \]  

with which one reproduces the predictions of the strange quark mass value obtained from the Kπ data\(^{48}\). We give in Fig. 4.9 the behaviour of \( \psi(0)_S^u \) for two sets of the \( x \)-parameters and for the value of the strange quark invariant mass \( \hat{m}_S = 250 \text{ MeV} \) and for \( \Lambda = 150 \text{ MeV} \). The arrows indicate

\[ t_c = \infty, \frac{p}{f_K} = 28 \text{ MeV} \]  
\[ \hat{m}_S = 250 \text{ MeV} \]  
\[ (t_c = 3.4 \text{ GeV}^2, \frac{p}{f_K} = 34 \text{ MeV}) \]

Ref 50t  
Ref 46, 10c

**Fig. 4.9**: Behaviour of \( \psi(0)_S^u \) versus the sum rule scale \( t \).
the "sum rule window" at which we obtain the optimal result:  

$$\psi(o)_{u}^{u} \approx - (5 \pm 2) 10^{-4} \text{GeV}^{4},$$  

(84)

where the error has been estimated from the difference between the curves for $\tau_{c} = \infty$ and $\tau_{c} = 3.4 \text{ GeV}$. We also show the result of Ref. 50b) obtained from the uncombined $\psi(q^2)_{u}/q^2$ sum rule similar to the one used originally in Ref. 45). One should notice that unlike our combined sum rule, the analysis in Ref. 50b) is very sensitive both to the strength of the dimension-six condensates (dashed lines) and to the effects of the QCD continuum. Both effects are of the order of factor two at the point $\tau_{c}^{-\frac{1}{2}} \approx 1.25 \text{ GeV}$ where the estimated $\psi(o)_{u}^{u}$ reaches its minimum. A sum rule window from this method does not exist at this level of QCD approximation so that the resulting predictions become misleading. Results similar to the ones in Eq. (84) have been obtained from the isoscalar $I = 0$ scalar current where the authors have assumed that the spectral function is dominated by the $e(1.3)$ meson. Adopting the estimated values of $\psi(o)_{u}^{u}$ and $\psi(o)_{s}^{u}$ from the sum rules and recalling the expressions of the subtraction constants from current algebra Ward identities:

$$\psi(o)_{u}^{u} = -(m_{u} + m_{s}) \langle \bar{u}u + \bar{s}s \rangle, $$

$$\psi(o)_{s}^{u} = -(m_{u} - m_{s}) \langle \bar{u}u - \bar{s}s \rangle,$$

(85)

we obtain the ratio of the condensates:

$$\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} \approx (0.6 \pm 0.2)$$  

(86)

which can indicate a 50% deviation from the expectation of SU(3)$_{F}$ symmetry. Some remarks are in order here. An estimate of $\psi(o)_{s}^{u}$ based on a resonance dominance would imply a positive value of $\psi(o)_{s}^{u}$ which necessarily gives the value of the ratio in Eq. (86) larger than unity. However, we have not found any reason why a one-resonance dominance is justified for the K$\pi$ phase shift data. Actually, a positive value of $\psi(o)_{s}^{u}$ from the sum rule is obtained in the region where the continuum dominates over the resonance effects. There is also an estimate of the $<\bar{s}s>$ condensate coming from baryonic sum rules which has a value quite near the SU(3)$_{F}$-symmetry expectations. However, such results should only be considered as a rough estimate due to some other uncertainties in the baryonic sum rules (choice of the baryon currents, radiative corrections, factorization...). It is still necessary to find some other channels which are sensitive to the ratio in Eq. (86) in order to have some other constraints on this ratio of condensate.

There exist in the literature excellent reviews on the subject which the reader can consult for more details. In general, the nucleon and the \( \Delta \) are described by the lowest dimension interpolating currents:

\[
\Psi_N(x) = \frac{1}{\sqrt{2}} : \psi^T C \gamma_5 \psi + b \phi^T C \gamma^5 \phi :
\]

\[
\phi_\Delta(x) = : \phi^T C \gamma_\rho \gamma_{\mu}^\rho \left( \gamma \gamma_\rho \gamma_{\mu}^\rho \right) \phi :
\]

where \( b \) is an arbitrary mixing parameter. The associated two-point functions have respectively the Lorentz decomposition

\[
S^N_N(q^2) = F_N^N(q^2) + F_N^N(q^2)
\]

\[
S^\Delta_\Delta(q^2) = \{ F_\Delta^1(q^2) + F_\Delta^2(q^2) \} q^\mu q^\nu + ...
\]

Due to their analytic properties \( F_{1,2} \) obeys the standard dispersion relation. Their associated spectral function can be parametrized as:

\[
\text{Im} F_1 = \sum |\lambda^1_1|^2 f(q^2 - M^2_1) + \text{QCD continuum}
\]

\[
\text{Im} F_2 = \sum p_{\gamma} |\lambda^1_1|^2 (q^2 - M^2_1) + \text{QCD continuum},
\]

where \( \lambda^1_1 \) is the coupling of the nucleon to the currents and \( p_{\gamma} \) is the parity of the state. The sum runs over the three first low-lying states. Here, one should notice that unlike the case of mesons, the spacing between the ground state and the excitations is very small, making invalid the naive ansatz: "one resonance" + QCD continuum. Related to this observation, methods like FESR which are sensitive to the high-\( t \) behaviour of the spectral function are not appropriate for the baryon channels. As usual, one studies the OPE of the invariants \( F_i(q^2) \) in the Euclidean region and one takes the Laplace transform \( \tilde{F}_i \) of these expressions. It is straightforward to derive the sum rule for the nucleon mass:

\[
M_N = \tilde{F}_2 \frac{F^N_1}{F^N_2} + \text{"QCD continuum"}
\]

For a reason of dimensionality, \( F_2 \) is controlled to leading order by the chiral \( \langle \bar{u}u \rangle \) condensate. In the naive approximation of one-resonance dominance and for \( b = -1/5 \) as the optimal choice of the mixing \(^{17}\), one obtains the interesting mass relation:

\[
M_N = -24\pi^2 \langle \bar{u}u \rangle Z_N \tau_{on}
\]
where to leading order in the $\alpha_s$-corrections:

$$
Z_N = 1 - \frac{2}{5} M_0^2 \tau_{ON} - \frac{23}{30} \times \alpha_s < G^2 > \tau_{ON}^2 - 64 \pi^4 < \bar{u}u > \frac{2}{5} \tau_{ON}^3 .
$$

(92)

where the factorization assumption of the condensate has been used.

$M_0^2 = (0.2 - 0.5)\text{GeV}^2$ is the scale which parametrizes the mixed condensate (Eq. 29). $\tau_{ON} \approx M_1^2$ is the scale at which the sum rule is optimized.

Eq. (91) has been used in Ref. 2b) for an attempt to relate the nucleon masses from the sum rule and from the Skyrme model. The inclusion of higher order terms in Eq. (90) (excitation + QCD continuum) and in Eq. (92) (radiative corrections) implies a much more complicated form of Eq. (91).

The complete analysis has been done in Ref. 17), by letting $b$ arbitrary.

Ref. 17) uses the sum rule in order to determine the quark condensate as a function of the mixing $b$. We show the result in Fig. 4.10 for different values of $M_0^2$ for the $N$ and $\Delta$ channels.

**Fig. 4.10**: Behaviour of $< \bar{u}u >$ versus the choice of $b$ for different values of $M_0^2$ in $\text{GeV}^2$: (----- : 0 ; - - - 0.2 ; - - - 0.8)
The result suggests that the optimal choice of $b$ is $-1/5$ while the choice of the current made in Ref. 54) ($b = -1$) is outside this range. At the optimal choice, the $\langle \bar{u}u \rangle$ value is not very sensitive to the one of $M_0^2$ but its value is too small, which implies via PCAC a too high value of the quark masses. Therefore Ref. 17) has abandoned the factorization of the four-quark condensate which they have a priori used. In this way, a standard value of the $\langle \bar{u}u \rangle$ condensate is correlated to a violation of the factorization hypothesis which has also been observed independently in the meson sector\textsuperscript{15,16).} What we can conclude from the above analysis is that the baryon sum rules are more peculiar than the meson one. Therefore, the extraction of accurate quantities like the ratio of condensates from this channel is very uncertain. One other point which, however, needs to be clarified is the coefficient of the radiative corrections in the OPE\textsuperscript{55)}, due to the discrepancy between Ref. 17) and Ref. 30). Nevertheless, despite the peculiarity of the baryon sum rules, it is important to notice that the value of the residue $Q_{N}^2$ controlling the strength of the proton decay obtained from the sum rules is quite stable. The sum rules were the first method which noticed before the data on proton decay that the SU(5) minimal model is ruled out.

5. Light exotic mesons.

We mean by exotic mesons those which go beyond the standard quark model. The status of the subject has been reviewed recently in Ref. 56) and we recommend that the reader consult these reviews.

However, one can learn from the above papers that the spectra of the $0^{++}$ scalar mesons are not understood both from the theoretical and experimental sides. Definite tests for the evidence of the exact nature of such mesons are still needed in order to help for deciding the existence of four-quark, $K\bar{K}$ molecules around the 1 GeV region.

The situation of gluonia remains also unclear due to the too many experimental candidates\textsuperscript{1,57)} . However, according to our arguments based on the three-point function analysis of the gluonia decay\textsuperscript{56b)}, we expect that one efficient guide for detecting gluonia is the test of the universality of their couplings to pairs of Goldstone bosons or vector mesons. Analogous argument based on the effective Lagrangian approach has been also advocated in Ref. 58) for the $0^{++}$ channel.
Concerning the mixing of such states with ordinary mesons, it has been found in Ref. 59) that the mixing should be small (less than 10^−6). This result has been confirmed recently from a FESR analysis 60). As a consequence the radiative decay of the O−− gluonium (perhaps the iota) into γγ is predicted to be less than 1.5 keV.

Concerning the nature of the η', it happens that present data like Φ decay or γγ width are not sufficient to decide on the percentage of gluon components of the η' 61). However, a much more theoretical argument based on the divergence of the U(1) current and on the non-zero value of the topological charge already differentiates the η' from the ordinary Goldstone bosons. Therefore, the high value of its mass should be attributed to its gluon component 62). Within this picture, the η' might already be interpreted as the lowest mass gluonium which saturates the U(1)−PCAC relation as does the pion for SU(2)L × SU(2)R, whilst in the absence of the mixing with the q̅q state, the iota, if it is a gluonium could be interpreted as its first excitation.

For the hybrid mesons, the theoretical predictions have been improved. However, due to the expectations of having a large width for such particles, a clean way for identifying them as a true resonance appears to be difficult.

6. K0−K̅0 systems.

The sum rule analysis of the subject has been excellently reviewed in Ref. 63). The authors have estimated the so-called B parameter which controls the deviation of the four-quark operator:

\[ < \bar{K}_0 | \bar{s}_L \gamma_\mu d_L \bar{s}_L \gamma^\mu d_L | K_0 > = \frac{4}{3} F_K M_K^2 B \]

from the vacuum saturation estimate (B = 1) by using FESR-duality sum rules of the two-point correlator built from the ΔS = 2 local four-quark operator. Realizing that the method used is sensitive to the high energy behaviour of the spectral function, the authors have parametrized carefully the "continuum" with the help of an effective Lagrangian consistent with the chiral behaviour of QCD. This parametrization is necessary for having stability of the prediction versus the unphysical choice of the continuum thresholds. In this way, they have obtained the renormalization group invariant quantity:

\[ \hat{B} = (\alpha_s (v^2))^{-2/9} B(v^2) \approx (0.33 \pm 0.09) \]

where v is the MS subtraction scale. Some other results based on the three-point function sum rule range between 0.6 to 1.64). However, it is known that the three-point function analysis should only be considered as a rough estimate.
Actually, one does not have the role of higher excitation and continuum in the analysis under control because of the complexity of the Källen-Lehmann representation in this case. The authors in Ref. 63) have also estimated the H-parameter defined as:

\[
\frac{\text{Im} A_0^o}{\text{Re} A_0^o} = H \sin \delta = s_2 c_3 s_1
\]

which governs the so-called $\varepsilon'$-parameter of the CP-violation; $A_0^o$ is the $I = 0$ amplitude for the $K^0 \to \pi \pi$ decay; $s_1$ and $c_1$ are the Kobayashi-Maskawa mixing angles and $\delta$ the phase of this mixing matrix. Taking the term $\lambda_6 \bar{u}_\mu U^0_\mu U^+$ as responsible for the $\Delta I = \frac{1}{2}$ transition ($U$ being the SU(3) matrix of the Goldstone boson between the vacuum and pseudoscalar states), the authors have obtained:

\[
H \approx 0.09
\]

which is opposite in sign to predictions from various models. As a consequence, the resulting value of the CP-violating parameter $\varepsilon'/\varepsilon$ defined respectively as:

\[
\varepsilon = \frac{A(K_L \to (\pi \pi)_0)}{A(K_S \to (\pi \pi)_0)}
\]

\[
\sqrt{2} \varepsilon' = \frac{A(K_L \to (\pi \pi)_0)}{A(K_S \to (\pi \pi)_0)} - \varepsilon \frac{A(K_S \to (\pi \pi)_2)}{A(K_S \to (\pi \pi)_0)}
\]

is positive. A clean experimental measurement of this ratio should be helpful for a significant test of the QCD-duality approach result or (and) of the standard model.

7. Heavy quark systems.

We have mentioned earlier in section III.4 that sum rules for heavy quark systems are qualitatively different from the light quark ones in the sense that, here, one exploits, in the OPE, the fact that the heavy quark mass is larger than the scale $\Lambda$ of QCD. So it becomes possible to perform the OPE at small momentum transfer. For charmonium systems, SVZ have originally worked at $Q^2 = 0$. The resulting sum rules are typically those in Eq. (57), where the charm quark mass appears explicitly in the QCD series. Moreover, the definition of the quark mass which one should use in this expansion is ambiguous. SVZ have chosen a (gauge-dependent) definition in order to minimize the radiative corrections in the unit operator of the OPE.
Therefore, they worked with the Euclidean mass in the Landau gauge:

\[ m_c(p^2 = m_c^2) = m_c(p^2 = m_c^2)\left\{ 1 - \frac{a_s}{\pi} 2 \log 2 \right\} \]  \hspace{1cm} (98)

where \( m_c(p^2 = m_c^2) \) is gauge invariant but renormalization scheme dependent. SVZ sum rule analysis of the charmonium systems lead to a fit of the Euclidean mass and of the gluon condensate which is the lowest dimension condensate appearing in the OPE. Their results are:

\[ m_c(p^2 = m_c^2) \approx 1.23 \text{ GeV} \]
\[ a_s < G^2 > \approx 0.04 \text{ GeV}^4 \]  \hspace{1cm} (99)

where the latter is considered to be the canonical value of the gluon condensate. SVZ analysis have been improved by Ref. 23a) by the inclusion of dimension 6,8 condensates which destroys the convergence of the series but it is clear that one does not have any good control of the strength of such condensates. From this problem, RRY19b) have worked at \( Q^2 = Q_0^2 \neq 0 \) in order to improve the convergence of the QCD series. A simultaneous fit of the charmonium systems supports the previous result of SVZ. However, according to the Bell-Bertlmann analysis based on the moments ratios in Eq. (49), the value of the gluon condensate has been underestimated 2a)29b).

There is also an ambiguity for translating the value of the sum rule mass into available quark mass definitions 44). Actually, an identification of the sum rule mass with the \( \overline{\text{MS}} \)-running mass or with the so-called constituent mass appearing in the potential models is ambiguous 68). For the analysis involving heavier quarks which are much more static, one needs to know accurately the radiative corrections due to gluon exchange in the OPE because of the importance of Coulomb-like terms. The analysis of the quarkonium built with one light and one heavy quark needs much more care in the OPE series due to the presence of mass singularities in the OPE 13,69). Mass singularities can be subtracted by expressing all heavy quark condensates in terms of the gluon condensate (see e.g Eqs 23 and 30). Interesting information from the inequal mass sum rules is the strength of the D and B mesons decay amplitude \( f_D \) and \( f_B \) analogous to \( f_K \) which are found to be smaller than naively expected 54,67),8b).
8. Three-point functions and form factors.

The success of the two-point function sum rules for explaining the dynamics of hadrons has encouraged many people to extend the analysis to the three-point function\(^{54,67,65,70,71}\). The aim is to evaluate the hadron three-body couplings and hadron form factors. However, the problems are much more complex here than in the two-point function case. There is an arbitrary choice of the vertex configuration but one should be careful on the eventual presence of mass singularities and on the regularity of the QCD series (polynomial in the inverse of the momentum squared). The spectral representation is not completely understood. A postulate of narrow resonances seems to be necessary in order to avoid some eventual anomalous thresholds. Some other authors found that the sum rule variables related to external momenta cannot be treated independently but are correlated to each other\(^{70}\). There is no clean way to treat the effects of higher excited states into the spectral function and so there is no reliable way for controlling the stability of the predictions. The usual attempt to identify the value of \(\tau_0\) of the three and two-point sum rules is very rough. Despite these amounts of unknowns, one should notice that the estimate of the three body couplings is in good agreement with the data. However, one should only consider such results as an estimate (within a factor two or more). Pushing these results within a certain accuracy becomes misleading. For form factors, the merit of the sum rules is the extension of the QCD-applicability to moderate \(q^2 (\approx 1 \text{ GeV}^2)\) where the role of power corrections has been shown to be important. In my opinion, there is also no good control of the accuracy of the results though QCD predictions including power corrections agree nicely with the data and are a real improvement of the previous perturbative asymptotic expression\(^{72}\).

9. QCD at non-zero temperature.

This aspect of the applications of the sum rules has been reviewed recently by Ref. 73) by focusing on the analysis of the \(\rho\)-meson channel. One of the main ingredients of the analysis is the parametrization of the spectral function at high temperature where quasifree gluons and quarks should be considered. It has been noticed that the deconfinement happens at \(T_c \approx 130 \ldots 150 \text{ MeV}\). For \(T > T_c\), there is a sharp modification of the spectrum which can indicate the deconfinement phase transition.
However, there is no unique interpretation of the (existence) disappearance of the resonances and on the (non) vanishing values of the quark and gluon condensates for $T > T_c$.

10. **Composite models of weak interactions.**

Some other applications of the sum rules beyond QCD do also exist in the literature. Among others, the possible implications of the composite assumption for the $W$ and $Z$ bosons are very attractive.

In these models, $W$ and $Z$ are bound states of elementary preon particles which open up the possibility to have a direct $\gamma-Z$ coupling in the much same way as the $\gamma-\rho$ one (in the standard SU(2)$_L \times$ U(1) model, $\gamma-Z$ coupling occurs only via quark loop diagrams). This $\gamma-Z$ coupling is responsible for the SU(2) global symmetry breaking which causes the $W-Z$ splittings. In this framework, preon dynamics is expected to be described around the TeV scale by some hyperstrong gauge group which we take to be SU$(N)_H$ hypercolour. Within this picture, it becomes then possible to use the sum rules in order to explore the dynamical structure of this SU$(N)_H$ theory and to check the consistency of the compositeness assumption.

For spin $\frac{1}{2}$ preons, the sum rule indicates that a continuum preon threshold starting around 1 TeV (this quantity might be interpreted as the compositeness scale) is dual to a value of the hypergluon condensate of 1 TeV. This condition is necessary because of the too small value of $M_W$ compared to the threshold value, which is a situation completely different from that of QCD. For spin 0 preons, one has the same condition of the hypergluon condensate but in addition the scalar $\langle \phi^* \phi \rangle$ condensate is needed to be less than $M_W^2$ for a consistency. We hope that alternative approaches will help to test further the compositeness idea.
V - CONCLUSIONS

We have reviewed various aspects of QCD duality sum rules and some of their phenomenological applications. We have noticed that the results are interesting though there remain some points to be understood. Compared to some other QCD-like approaches, the sum rules are among the few which are able to describe the complex bound state dynamics with the few parameters of the QCD Lagrangian. From this point of view, the sum rules approach is very attractive and powerful!
REFERENCES

1) See e.g. the contribution of F. Couchot at this school.

2) Attempts to relate QCD sum rules with potential models are reviewed by R. Bertlmann : contributions to the Montpellier Workshop on Non-Perturbative Methods (July 1985) edited by S. Narison. An attempt to relate the sum rules with the Skyrme model is in H.G. Dosch and S. Narison, MP 86/10 -HD-THEP 86-6 (to appear in Phys. Lett. B) and Montpellier preprint MP 86/34 (1986). The operator product expansion including vacuum condensates has also been studied in the lattice : for a review see e.g. A. di Giacomo, Montpellier Workshop (1985). Some attempts to relate the sum rules and the lattice approaches are in N.S. Craigie et al, Trieste preprint IC/14/1 (1984) ; E. Bagan et al, Phys. Lett. 152B (1985) 113.

3) J.M. Richard, contribution at this school.

4) E. Brezin, E. Marinari and A. Morel, contributions at this school.


12) For a review see e.g. F. David, contribution at the Montpellier workshop (July 1985) edit. S. Narison.


20) See R.A. Bertlmann in Ref. (2) and R.A. Bertlmann, C.A. Dominguez, M. Loewe, M. Perrottet and E. de Rafael (to appear). For a review on the determinations of the gluon condensate, see e.g. R.A. Bertlmann, Talk given at the XII Intern. Symposium Multiparticle Dynamics, Seewinkel (Austria 1986) Wien preprint UWTh Ph-1986-25.


35) S. Narison (in preparation).


37) For a review on the aspects of dimensional regularization see e.g. S. Narison, Phys. Rep. 84 (1982) 263.


40) For a review on current algebra, see e.g. V. De Alfaro, S. Fubini, G. Furlan and C. Rossetti, North-Holland Pub. (1973) : "Currents in Hadron Physics".


44) H.D. Politzer, Nucl. Phys. B117 (1976) 397. For a review of various attempts to define the quark mass in QCD, see e.g. Ref. 37).


   and references therein ; H.G. Dosch, contribution to the Montpellier
   Workshop (1985) and references therein.
54) B.L. Ioffe, talk given at the Intern. Conf. High Energy Physics,
56) S. Narison, Talk given at the Intern. Conf. High Energy Physics,
   Berkeley (1986), Montpellier preprint PM 86/27 ; Talk given at the 3rd
   Lear Workshop, Tignes, Haute Savoie (1985), Montpellier preprint PM 85/2.
57) B. Jean-Marie, Talk given at the Intern. Conf. on Production and Decay
60) S. Narison (unpublished).
61) Private communication from A. Bramon.
62) For a sum rule estimate of the \( \pi \)-mass and of the U(1) topological
   charge, see e.g : S. Narison, Z. Phys. C26 (1984) 209 and references
   therein.
63) T. Pich and E. de Rafael, Contribution to the Workshop on Non-perturbative
   Methods (Montpellier 1985) ; E. de Rafael, Contribution to the
   European Hadron Facility Conference, Mainz (1986).
64) K.G. Chetyrkin, A.I. Kataev, A.B. Krasulin and A.A. Pivovarov,
65) For a discussion of the spectral representation in the symmetric
   configuration, see e.g. S. Narison and N. Paver, Z. Phys. C22 (1984) 69
   and references therein.
66) For a review, see e.g Refs 11, 2a, and 19b)
67) For a review, see e.g : E. Shuryak, Phys. Rep. 115 (1984) 153 ;
68) S. Narison and R. Petronzio (in preparation)


73) M.E. Shaposhnikov, Lectures given at the CERN-JINF school of Physics, CERN Yellow report 86-03 (1986) 218.

74) H. Fritzsch, MPI/PAE/Pth 76/83 (1983); H. Harari, WIS 85/7 SLAC Summer School (1984); R.D. Peccei, MPI/PAE/Pth 35/84

75) P.Q. Hung and J.J. Sakurai, Nucl. Phys. B143 (1978) 81; For a review, see e.g. D. Schildknecht MPI-PAE/Pth 81/82 (1982) and references therein.

76) For a review, see e.g. S. Narison, Talk given at the EPS-HEP 85 Conference, Bari (1985); Montpellier preprint PM 85/14 (1985); a more recent work is by S.A. Devyanin and R.L. Jaffe, Phys. Rev. D33 (1986) 2615.