STRING VERTEX OPERATORS AND DYNKIN DIAGRAMS FOR HYPERBOLIC KAC-MOODY ALGEBRAS*)

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ABSTRACT

Affine Kac-Moody algebras are obtained from finite Lie algebras by adding a single lightlike direction to the Euclidean root lattice spaces of the latter. The class of hyperbolic Kac-Moody algebras, of which $E_{10}$ is the most celebrated member, are based on a fully Minkowskian root space, which reverts to a finite or affine root system upon the removal of any simple root. In string theory terms, finite Lie, affine Kac-Moody and hyperbolic Kac-Moody algebras are generated by tachyon vertex operators, tachyon plus photon vertex operators and all of the vertex operators for all mass levels respectively. Hyperbolic algebras are closely related to the exceptional Lie algebras, which is basically the reason they do not extend beyond rank 10. The 136 possible hyperbolic Dynkin diagrams between the ranks 3 and 10 are classified and exhibited.

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1. INTRODUCTION AND MOTIVATION

The algebra E_{10}, often mentioned in the context of string theory, belongs to the class of hyperbolic Kac-Moody algebras whose mathematical theory is not at a stage of development comparable to that of their finite Lie or affine Kac-Moody counterparts. What is currently available in the mathematical literature [1-3] on the subject consists of general conditions on the Cartan matrix, the certainty that higher (m^2 > 0) string vertex operators will become part of the generators (resulting in an exponential growth in the number of generators, in contrast to the polynomial growth in the affine case) and upper bounds on the degeneracies of root spaces. Our aim in this note is to take a simple (but one hopes, useful) step towards the characterization of such algebras by enumerating their Dynkin diagrams (of rank > 3). A hint of the procedure and partial results are already given in Ref. [1]; its detailed implementation, however, leads to a non-trivial amount of work and a surprising proliferation of cases. One can say that these are all "exceptional", both in the sense of not belonging to infinite algebra series, and also in the intimate ties they have with the diagrams of the ordinary exceptional Lie and affine algebras. Hence, although finite in total number, they have to be treated and displayed individually. In fact, the reason behind the finite number and the absence of hyperbolic algebras of rank>10 is essentially the exceptional-ness and finite number of their familiar ancestors. That the highest allowed dimension for a hyperbolic root space of signature (++...++) is the same as that for the superstring stands out as an intriguing fact.

The interest of physicists and mathematicians in Lorentzian Kac-Moody algebras (of which the hyperbolic ones form a subclass) stems largely from the most promising method to realize them, which is through an algebra of vertex operators with momenta in a Minkowski space. Versions of vertex operators have proved their utility [4] in integrable systems, in boson-fermion conversions, in endowing the heterotic string with internal symmetry, in dual resonance models and in the construction of affine Kac-Moody algebras. It is especially the last two contexts that suggest the potential usefulness of Lorentzian Kac-Moody algebras the most strongly. Let us examine the mathematical one first. As shown by Goddard and Olive, one passes from a finite Lie algebra with Euclidean roots (momenta) to an affine one by adding a single new lightlike direction to the root space. (The original Frenkel-Kac-Segal construction [5] uses momenta of the Euclidean vertex operator; this is essentially a "unitary gauge" approach.) The natural space in which to imbed the resulting singular root space with one zero on the diagonal of its metric is a Minkowskian one, with a second lightlike root independent of the first. We will see in Section 2 that this gives a Lorentzian algebra of exponential growth. (To have an hyperbolic algebra, this last simple root has to be added to the Dynkin diagram in such a way that the removal of any simple root from the resulting diagram leads to a known finite or affine Lie algebra.) Apart
from being of interest as natural extensions of affine algebras, Lorentzian algebras are also known to be related to exceptional structures such as the 26-dimensional Leech lattice and the "Monster", the largest sporadic finite group.

As for the physical or string theory motivation, one can argue that string theorists have been "speaking in the prose of Lorentzian algebras" all along, because of the correspondence between the basic three-string vertex and a commutator of two vertex operators, giving a third one. Hence, the "structure constants" of such an algebra should determine the form and strength of the three-string vertex, in conformity with Witten's expectation [6] that $E_{10}$ must be related to string field theory, presumably in 10 dimensions. One may then further speculate that the lower rank, especially the $r = 4$ hyperbolic algebras, may be related to string interactions in our four-dimensional space-time.

In this context, an especially curious but not commonly emphasized fact is that Lorentzian algebras work even without latticizing the momenta; all one needs is an integral value for the dot product of the two momenta entering the commutator [see Eq. (4)]. But this is automatically ensured if the three momenta are on the mass-shell values allowed by the string spectrum! A different application of hyperbolic algebras to string theory has also been given recently by Kostelecky and Lechtenfeld [7], who use bosonized ghosts in place of Fubini-Veneziano fields in vertex operators.

The paper is organized as follows: in Section 2, we discuss the changes brought about in the vertex operator algebra, such as the emergence of vertex operators beyond the photon, when the root space becomes Minkowskian. In Section 3A, we give definitions and describe our method for the classification of hyperbolic Dynkin diagrams. Strictly hyperbolic cases are classified in Section 3B, followed by the hyperbolic ones in 3C. The latter subsection also contains a detailed proof of the impossibility of hyperbolic diagrams beyond rank 10. The results are displayed in Tables 1 and 2. Section 4 ends the paper with a discussion mostly concerning Lorentzian algebras that are not of the hyperbolic type.

2. - ALGEBRA OF VERTEX OPERATORS IN MINKOWSKI SPACE

Before plunging into a list of Dynkin diagrams, it may be useful to demonstrate with a simple example that a dramatic change in the vertex operator algebra ensues when a new Minkowskian root is added to a familiar diagram. Take the (1) diagram for $A_2^2$, represented by

```
1 2 3
```

and extend it to

\begin{align*}
\beta_1 &= \sqrt{2} \left( 1, 0, 0, 0 \right), \\
\beta_2 &= \sqrt{2} \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, 0 \right), \\
\beta_3 &= \sqrt{2} \left( -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right), \\
\beta_4 &= \sqrt{2} \left( 0, 0, -1, 0 \right).
\end{align*}

Note that two independent lightlike vectors $k_+$ and $k_-$ can be defined through

\begin{align*}
k_+ &= \beta_1 + \beta_2 + \beta_3 = \left( 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\
k_- &= \beta_1 + \beta_2 + \beta_3 + \beta_4 = \left( 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).
\end{align*}

The infinite (but of polynomial growth) root system of the affine Kac-Moody algebra $A_2^{(1)}$ consists of integral superpositions of $\beta_1, \beta_2$ and $k_+$ such that the resulting lengths are $\sqrt{2}$ or 0. Let us now review the first Goddard and Olive article in Ref. [2] for the construction of the corresponding generators. The root $\rho^\mu$ ($\rho^2 = 2$) now appears as the momentum in the integrated tachyon vertex operator

\begin{equation}
A(\rho) \equiv \oint \frac{dz}{2\pi i} : \exp \left( i \rho \cdot Q(z) \right) :,
\end{equation}

where $Q^\mu(z)$ is a standard Fubini-Veneziano field, and all conventions are as in the mentioned paper. We have to multiply (3) with a cocycle factor $c(\rho)$ to correct signs in some commutators; we thus arrive at the step operators $e(\rho) = c(\rho)A(\rho)$.

The centre-of-mass momentum operator $p^\mu$, where $\mu$ represents directions spanned by $\beta_1, \beta_2$ and $k_-$ only, serves as the three-dimensional Cartan subalgebra and completes the algebra of the $e(\rho)$ to $A_2^{(1)}$. The basic commutator $[e(\rho), e(\zeta)]$ gives a result proportional to

\begin{equation}
\frac{1}{(2\pi i)^2} \oint dz' \oint dz \left( z - z' \right)^{\rho \cdot \zeta} : \exp \left( i \rho \cdot Q(z) + i \zeta \cdot Q(z') \right) :.
\end{equation}
where the $z$ integration encircles $z'$, excluding the origin and the $z'$ integration is about the origin of the $z'$ plane. Given the restriction of $\rho$ and $\zeta$ to the span of $\beta_1, \beta_2$ and $\beta_3$, we have $-2 < \rho \cdot \zeta < 2$; or, equivalently, $0 < (\rho \cdot \zeta)^2 < 8$. Clearly, the result is zero if $\rho \cdot \zeta > 0$ [as $(\rho \cdot \zeta)^2 > 4$ shows $\rho \cdot \zeta$ is not a root]; the vertex operator corresponding to $(\rho \cdot \zeta)$ if $\rho \cdot \zeta = -1$, and a "photon" vertex operator when $\rho \cdot \zeta = -2$, through differentiation of the integrand once with respect to $z$. When $\rho = (\hat{\tau}_n, n k_\perp) = -\zeta$ (here $\hat{\tau}$ is the transverse part of length $\sqrt{2}$), we have in particular the result $\hat{\tau} \cdot p + n k_\perp \cdot p_\perp$, revealing that the central term may be thought of as one of the two lightlike components of the centre-of-mass momentum operator. This completes our sketchy review of the Goddard-Olive treatment of the affine case; the finite Lie algebra is, of course, given by the restriction $n = 0$ everywhere.

Let us now append $k_\perp$ to the former root space. Most of the new features to be discussed also hold for Lorentzian algebras which are not hyperbolic. If we take $\rho = (\hat{\tau}_n, n k_\perp)$ and $\zeta = (s, n k_\perp)$ with $\hat{\tau}^2 = \hat{s}^2 = \rho^2 = \zeta^2 = 2$; $\rho \cdot \zeta$, which determines the outcome of the commutator (4) can now take the values $\hat{\tau} \cdot \hat{s} = -\rho \cdot \zeta$, unbounded from above or below. Still, (4) shows that $\rho \cdot \zeta > 0$ trivially gives zero. Goddard and Olive show that even more general multiple commutators vanish if the total squared momentum exceeds 2. In terms of string theory, this simply reflects the fact that there are no particles with squared mass below the tachyon. On the other hand, $\rho \cdot \zeta = N < -3$ now results in $(1/(N-1!))d^{N-1}/dz^{N-1}$ being applied in (4) prior to the $z$ integration. This clearly brings in vertex operators beyond that of the photon; thus the number of generators, or root degeneracy, increases (modulo subtleties involving null states which need not concern us at this point) with $(\rho \cdot \zeta)^2$ essentially as the number of states in the dual resonance model in the same space-time dimension does, i.e., exponentially with the mass.

A more detailed characterization of such algebras, say, in the form of a closed expression for all possible commutators is beyond the scope of the present paper. For such questions, we refer the interested reader to the last paper of Ref. [3] and Ref. [8], where recent progress is reported. Let us, however, note a number of general features.

(i) Extending the affine root space with one null direction to a full Minkowski space lattice reveals that the central term $c$ and the derivation operator $d$ may be viewed as two additional components of the extended Cartan subalgebra, given by $k_\perp \cdot p_\perp$ and $k_\perp \cdot p_\perp$ respectively. In the usual treatments of affine algebras this natural symmetry between $c$ and $d$ is not apparent, as $d$ is treated as a gradation operator outside the algebra.

(ii) The algebra respects a "Regge trajectory" type relation between $\mu^2 \equiv -\xi (\rho \cdot \zeta)^2$ and the rank of the most highly symmetric space-time tensor at that $\mu^2$. 


For brevity, let us refer to this rank as "spin", which is what it becomes in four dimensions, and denote it by J. To see the relation, note that for $\rho \cdot \zeta = -J+1$, the derivative $d^{J}/dz^{J}$ will act inside (4), resulting in, among other things, a term of the highest symmetry given by

$$\rho_{\mu_{1}} \cdots \rho_{\mu_{J}} \oint \frac{dz}{2\pi i} : \frac{dQ^{{\mu_{1}}}}{dz} \cdots \frac{dQ^{{\mu_{J}}}}{dz} \exp(i(\rho + \zeta) \cdot Q(z)) :$$

(5)

Now, a correspondence established [9,10] between the derivatives of the vertex operator in the form

$$\left( \frac{d^{n_{1}}}{dz^{n_{1}}} Q^{{\mu_{1}}} \right) \cdots \left( \frac{d^{n_{k}}}{dz^{n_{k}}} Q^{{\mu_{k}}} \right) \exp(i \rho \cdot Q)$$

and the states created by the mode operators $\alpha^{{\mu}}_{n}$ in the combinations $\alpha^{{\mu}}_{-n_{1}} \cdots \alpha^{{\mu}}_{-n_{k}}$ shows that (5) represents the "leading spin" J. On the other hand, we have $\mu^{2} = -\frac{1}{4}(\rho^{2} + \zeta^{2} + 2\rho \cdot \zeta) = -2J+1$, which is exactly the old Regge relation between the square of the mass and angular momentum. This also shows that "unit intercept" is built into Lorentzian algebras.

A generalization of the above result to the commutator of two higher leading vertex operators of spins and momenta $(J_{1}, \rho_{1})$ and $(J_{2}, \rho_{2})$ (where $\rho_{1}^{2}$ and $\rho_{2}^{2}$ are no longer equal to 2 but related to $J_{1}$ and $J_{2}$ through the Regge relation) can be obtained by a somewhat more complicated version of the steps leading to (4) and (5), with the result

$$J_{\text{Maximum (resultant)}} = J_{1} + J_{2} - \rho_{1} \cdot \rho_{2} - 1.$$  

(6)

(iii) As the tachyon operators commute with the Virasoro operators $L_{n}$, all higher vertices obtained from (4) will also commute with them. This follows from the Jacobi identity. One can see a consequence of this in its simplest form by putting $\rho \cdot \zeta = -2$ in (4). After an integration by parts, Goddard and Olive find

$$\epsilon_{\mu} \psi^{{\mu}} = \frac{1}{2} (\rho - \zeta)_{\mu} \oint \frac{dz}{2\pi i} : \frac{dQ^{{\mu}}}{dz} \exp(i(\rho + \zeta) \cdot Q(z)) :$$

(7)

Here, as expected, the photon vertex operator $\psi^{{\mu}}$ is contracted with a polarization vector $\epsilon_{\mu} = \frac{1}{2} (\rho - \zeta)_{\mu}$, with $\epsilon \cdot k = \epsilon(\rho + \zeta) = 0$. Furthermore, the longitudinal component $k \cdot \psi$, whose Fock space counterpart would correspond to a state of a zero norm created by the $L_{-1}$ operator, is a total derivative and thus identically vanishes! For higher vertex operators, the raw result of (4) requires considerable manipulation before the correspondences with Fock space states and the structure of the appropriate polarization tensors become apparent. We will content ourselves with
treating the case \( \rho \xi = -3 \), \((\rho \xi)^2 = -2\) in some detail. The basic method is to
look for a suitable combination of the three different but equivalent expressions
obtained from (4), namely:

\[
\frac{1}{2!} \oint \frac{dz}{2\pi i} \left\{ \xi_{\mu} \frac{d^2}{dz^2} Q^{\mu} + \xi_{\nu} \frac{d^2}{dz^2} Q^{\nu} \right\} \exp \left( i(\rho + \xi) \cdot Q \right) = \left( \rho \leftrightarrow \xi \right) = -\frac{1}{2!} \rho_{\mu} \xi_{\nu} \oint \frac{dz}{2\pi i} \left\{ \frac{d^2}{dz^2} Q^{\mu} \frac{d^2}{dz^2} Q^{\nu} \right\} \exp \left( i(\rho + \xi) \cdot Q \right).
\]

(8)

The result can be put in the form (the \( z \)-integral, normal ordering and the exponen-
tial are understood; \( \frac{d}{dz} \) represents \( d/\text{d}z \))

\[
\epsilon_{\mu \nu} \frac{d^2}{dz^2} Q^{\mu} \frac{d^2}{dz^2} Q^{\nu} + \left\{ \epsilon_{\mu \nu} \frac{d^2}{dz^2} Q^{\mu} + 2 \epsilon_{\mu \nu} \frac{d^2}{dz^2} Q^{\nu} \right\}
\]

\[
+ \frac{1}{2} \left\{ \frac{d^2}{dz^2} Q^{\mu} + \frac{1}{2(d-1)} \left[ 10 g_{\mu \nu} + (d+4) k_{\mu} k_{\nu} \right] \frac{d^2}{dz^2} Q^{\mu} \right\} = (d-1) \left\{ \frac{1}{2} \left[ \rho_{\mu} \rho_{\nu} + \xi_{\mu} \xi_{\nu} \right] - \frac{1}{2(d-1)} + (d+4) \left[ \rho_{\mu} \xi_{\nu} + \rho_{\nu} \xi_{\mu} \right] - 10 g_{\mu \nu} \right\} = \frac{1}{4(d-1)} \left\{ \left( d-6 \right) \left[ \rho_{\mu} \rho_{\nu} + \xi_{\mu} \xi_{\nu} \right] - \left( d+4 \right) \left[ \rho_{\mu} \xi_{\nu} + \rho_{\nu} \xi_{\mu} \right] - 10 g_{\mu \nu} \right\}.
\]

(9)

where \( d \) = dimension of space-time (or rank of the algebra in the hyperbolic case),
\( k_{\mu} = \rho_{\mu} - \xi_{\mu} \), \( k^2 = -2 \), \( \epsilon_{\mu \nu} = \rho_{\mu} - \rho_{\nu} \) and

\[
\epsilon_{\mu \nu} = \frac{1}{4(d-1)} \left\{ \left( d-6 \right) \left[ \rho_{\mu} \rho_{\nu} + \xi_{\mu} \xi_{\nu} \right] - \left( d+4 \right) \left[ \rho_{\mu} \xi_{\nu} + \rho_{\nu} \xi_{\mu} \right] - 10 g_{\mu \nu} \right\}.
\]

(10)

Note that \( \epsilon_{\mu \nu} \) satisfies the physical "spin-2" conditions

\[
\epsilon_{\mu \nu} = \epsilon_{\nu \mu} \quad ; \quad k^\mu \epsilon_{\mu \nu} = \epsilon_{\mu \nu} = 0.
\]

(11)

The form of the other two terms in (9) agrees with their physical null state
counterparts given in Ref. [9]. More precisely, the counterpart of the last term
in (9) is the famous \( k^2 = -2 \) state which is of positive norm for \( d < 26 \), null for \( d = 26 \) and physical but of negative norm for \( d > 26 \). Needless to say, the effort
involved in bringing \( k^2 < -4 \) into a "manifestly physical" form is much greater.

This concludes our brief survey of some novel aspects of vertex operator
algebras that arise from Minkowskian roots. In the remaining sections we will
largely limit the discussion to Dynkin diagrams of hyperbolic algebras.
3. - DYNKIN DIAGRAMS OF HYPERBOLIC ALGEBRAS

A. Definitions and method

Let us begin by defining a hyperbolic Dynkin diagram more precisely than we have in the informal discussion of the previous section. A Dynkin diagram is said to be of the hyperbolic type if it is not one of those for finite Lie algebras or (twisted or untwisted) affine Kac-Moody algebras and if it reduces to such diagrams (or some disconnected combination of them) upon the deletion of any of its points. If all such deletions result only in finite Lie algebra diagrams, the original diagram is said to be strictly hyperbolic. We shall see in detail that there are no strictly hyperbolic algebras of rank >4, no hyperbolic algebras of rank >10 and that the total number of such algebras is finite (albeit over a hundred!), confirming results stated in Ref. [1].

It is clear that the rank-2 case is special, from both the diagrammatic and the vertex operator point of view. Indeed, it has been shown [11,12] that there are infinitely many such algebras. At the other end of the spectrum, ranks 7, 8, 9 and 10 are given in Kac [1] and are 18 in number. The 118 cases between the ranks 3 and 6 constitute the chief content of the present study.

The general strategy in searching for hyperbolic Dynkin diagrams of rank r+1 is as follows: (i) draw all possible Lie and/or affine (including semi-simple) diagrams of rank r; (ii) add an extra root, trying all possible lengths; (iii) try connecting the new root to the old ones in all the ways consistent with the meaning of Dynkin diagrams; for example,

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  1 —> 2
     |     |
     3 —> 4
```

is not allowed since it inconsistently assigns both of the roots 3 and 4 two different lengths (we adhere to the convention that the arrow points from the long to the short root); (iv) test the resulting diagram by removing any point to see whether it reduces to (perhaps a disconnected combination of) known finite or affine algebras, the twisted ones being included among the latter. A diagram that survives the test is of the hyperbolic type.

If such a case-by-case approach seems unsatisfactory to the reader, he should recall that the Dynkin diagrams (or equivalently, the Cartan matrices) of the five exceptional Lie algebras are found in a similar way, as the standard treatment in Jacobson [13] or Cahn [14] shows. It is no coincidence, but a reflection of the fact that the new algebras are also exceptional, that one has to resort to the same method here. In fact, the rank >7 algebras are essentially built around the E-series, while G2 and F4 play a prominent role for lower rank. We now apply the
method and present the results in order of increasing rank, starting with the strictly hyperbolic cases.

B. Strictly hyperbolic algebras

It is not difficult to see that these stop at rank 4. Consider the possible diagrams with five points. Those with triangles or squares are ruled out, as the deletion of a point outside the closed subdiagram would give us a closed diagram which does not correspond to any finite Lie algebra. The next closed figure, a pentagon, represents $A_4^{(1)}$ if the roots are of the same length. The diagram

![Pentagon Diagram]

although leading to an "acceptable" looking Cartan matrix if decoded naively, must be ruled out as it implies the roots 1, 2, 3 and 4 are long and short at the same time! (The matrix corresponding to this diagram could be the rank-5 strictly hyperbolic Cartan matrix whose existence is left as a possibility in Kac [15].) A pentagon of the form

![Pentagon Diagram]

or its dual yields an affine rather than a finite algebra when root 4 is removed. This leaves tree diagrams of three sorts: (a) a linear one, (b) one in a $D_5$-type configuration, (c) one with four dots connected to a central point. Note that a $G_2$ triple line is forbidden as it would lead to $G_2^{(1)}$ or other non-Lie subdiagrams. Thus, only single and/or double lines may be used in (a), (b) and (c). The all-single-line cases are $A_5$, $D_5$ and $B_4^{(1)}$. In (c), we may use at most one double line to find

![Tree Diagrams]

but removal of 1, 2 or 3 produces the affine algebra $B_3^{(1)}$. The dual version fails similarly. In (b) again only one double line is allowed. Of three such diagrams, the failures of

![Tree Diagrams]

and

![Tree Diagrams]

are evident upon dropping 5, while

![Tree Diagrams]

is $B_4^{(1)}$. In the linear option (a), one may use two double lines only in the form (modulo duals)

![Tree Diagrams]
but these are the affine algebras $A_8^{(2)}$ and $G_4^{(1)}$. Other attempts only produce $F_4^{(1)}$, $E_6^{(2)}$, $B_5$ or $G_5$. The arguments establishing the absence of strictly hyperbolic algebras of higher rank proceed similarly, with certain simplifying restrictions such as the use of at most one double line per diagram and at most three branches per vertex. See also the proof of the absence of hyperbolic algebras for $r > 10$ in Section 2.6, where more detail is provided.

The allowed strictly hyperbolic algebras of ranks 3 and 4 are displayed in Table 1. The reader can easily check that these are the only possibilities by using the general method in Section 2A and the restrictions mentioned subsequently. Let us briefly note a few interesting points. (a) All of the diagrams of rank 3, except the first one, are based on the exceptional group $G_2$. The unique rank-4 diagram can be thought of as an $F_4$ diagram closed upon itself, suggesting that the absence of strictly hyperbolic algebras of higher rank is related to the absence of suitable higher rank exceptional algebras. $E_6, 7, 8$ are too large for the strictly hyperbolic case; however, they perform a similar function in the hyperbolic one as we shall see later. (b) The diagrams 2, 4 and 5 for rank 3 indicate three different simple root lengths (from now on we will frequently omit the qualifier "simple", since these will be the only roots we shall be concerned with), a situation not encountered in finite Lie algebras, although found in the twisted affine series $A_{2l}^{(2)}$.

C. Hyperbolic algebras

The proof that there are no hyperbolic algebras of rank $>10$ proceeds along lines similar to the proof for strictly hyperbolic ones, except for the fact that we now allow affine (untwisted or twisted) algebra diagrams as possible subdiagrams. To show the impossibility of rank $>11$, we need to establish a number of intermediate results. As we shall see, some of these rules will not apply for lower ranks, leading to a proliferation of diagrams for $r = 3, 4, 5$ and 6.

C.1) All closed subdiagrams up to $(r-2)$-gons are forbidden, as the removal of one of the two outer points leaves us with a closed diagram (or diagrams) from which at least one root "dangles". There are no such finite or affine algebras.

C.2) An $(r-1)$-gon with a dangling root is also forbidden for $r > 10$. To see this, remove the sixth root from the junction; one is left with at best another hyperbolic algebra.

C.3) $G_2$, $A_2^{(1)}$ and $A_2^{(2)}$ are obviously ruled out, as a subdiagram of at least four points containing them can always be isolated; such a diagram is not finite or affine.
C.4) We come to r-gons. With only one type of root, we have the affine algebra \( A_{r-1}^{(1)} \). We must next rule out r-gons with double lines to be left with tree diagrams only.

C.5) Double lines may not be introduced into an r-gon \( (r \geq 11) \) since it is always possible to delete a root so as to leave a double line in the middle rather than at the ends of a subdiagram. Again, there are no such finite or affine algebras.

We are now left with tree diagrams subject to the following restrictions which again hold for \( r \geq 11 \).

C.6) At most three branches may be connected to a vertex. Otherwise we could isolate a subdiagram of more than five points containing such a vertex. (Obviously this may be circumvented for \( r = 6 \) since \( A_4^{(1)} \) is of rank 5.)

C.7) There cannot be more than two vertices per diagram. Otherwise we could find and remove a point so as to end up with a connected diagram with two vertices and a branch of two more points beyond one of them.

C.8) A double line may not be directly connected to a vertex as no such configuration is found among finite or affine diagrams.

C.9) Double lines may only be used at the ends of diagrams. This is again obvious, recalling the structure of Lie and affine diagrams of rank \( > 5 \).

C.10) Four double lines are obviously ruled out, as are three double lines, albeit less obviously. A diagram with three double lines which are placed at three branch ends by C.9 must have a junction where these three branches meet. None of the double lines may be directly attached to the vertex by C.8. Hence, removal of the point at the very end of one of the double lines results in an unacceptable diagram with two double lines and one "protrusion" or dangling line left after the removal.

C.11) The above-mentioned subdiagram, with two double lines at the two ends of a branch and protrusions dangling from the same branch is also unacceptable as a hyperbolic diagram for \( r \geq 11 \). To see this, just remove the double line farthest away from the protrusion.

C.12) A linear diagram of equal length roots with double lines added at both ends belongs to one of the finite or affine series \( B_r, C_r, G_\text{r-1}^{(1)}, \text{"} 2 \text{r} \text{"} \) or \( D_r^{(2)} \). Hence a hyperbolic diagram subject to all the previous restrictions can admit at most one double line at one end.
C.13) By C.7, a diagram ending in a double line can have at most two vertices; but in fact for \( r > 0 \) it cannot have any. To rule out the two-vertex case, remove the root farthest from the double line. To rule out the single-vertex case, note that the vertex must be situated as far away from the double line as possible and must be either of the \( D^r_6 \) or \( E_{6,7,8}^{(1)} \) (shorter side) type. But the former diagram is just \( B^{(1)}_{r-1} \), while removing the outermost point on the double line in the latter case gives something beyond \( E_8^{(1)} \). This brings us back to C.12; thus double lines are out altogether!

C.14) We are now restricted to single-line tree diagrams with at most three vertices. These can be only of two kinds: (i) one with two \( D^r_6 \) type vertices, (ii) another one with one \( D^r_6 \) and one \( E_{6,7,8}^{(1)} \) type (shorter side) vertex. Other possibilities such as lengthening the E-vertex or using two E-vertices can be ruled out by removing the point beyond the E-vertex or shortening one of the two E-vertices.
Now, (i) is already \( D^r_6 \), while deleting an extreme point from the E-vertex leaves the case (ii) with something bigger than the \( E_8^{(1)} \) diagram.

C.15) We are left with a single type of root, forming a Y-shaped diagram. Following Kac, we denote this by \( T_{n,m,\lambda} \) \((n \geq m \geq \lambda)\), where \( n,m,\lambda \) refer to the lengths of the branches including the vertex; thus \( n+m+\lambda = r+2 \geq 13 \). The combination \((n,2,2)\) is just \( D^r_3 \); all others of the form \((n,m,\lambda)\) \((n,m \geq 3, \lambda \geq 2)\) can easily be checked to give diagrams beyond \( E_8^{(1)} \) upon root deletions; hence this last class of candidates also cannot be hyperbolic Q.E.D.

A striking aspect of the above proof is again the special role played by the E-series. It is clear that hyperbolic algebras stop at rank 10 essentially because the finite and affine E-series stop at \( E_8^{(1)} \); hyperbolic algebras of ranks 7, 8, 9, 10 exist because \( E_{6,7,8}^{(1)} \) and \( E_{6,7,8}^{(1)} \) exist. A glance at the high rank end of Table 2 will make this assertion clearer.

We can now refer the reader to Table 2 where hyperbolic Dynkin diagrams of ranks between 3 and 10 are displayed in order of increasing rank. Within a given rank, the order roughly corresponds to increasing number of double, triple or quadruple lines and/or loops. Algebras with more than one root length sometimes have distinct dual partners obtained by interchanging root lengths, which is accomplished by changing the directions of the arrows; sometimes the algebra is self-dual. A very interesting point is the appearance of twisted and untwisted affine algebras as subalgebras of the same hyperbolic algebra. This is based on the fact that the duals of \( B^r_1 \), \( C^r_1 \), \( F_4 \) and \( G_2 \) are \( A^{(2)}_{2r-1} \), \( D^{(2)}_{r+1} \), \( E_8^{(2)} \) and \( D_8^{(3)} \). The twisted affine series \( A^{(2)}_{2r} \) and the twisted algebra \( A^{(2)}_{2} \) also make a number of
hyperbolic algebras possible. The lesson is that the twisted Kac-Moody algebras are as indispensable as the finite or untwisted affine ones (and indeed, appear on an equal footing) in the construction of hyperbolic algebras.

\[ r = 3: \]

The large number of rank-3 diagrams is due to the unique rank-2 diagrams representing \( A_1^{(1)} \), \( A_2^{(2)} \) and \( G_2 \). These, in all possible combinations with an additional point, generate the rank-3 hyperbolic cases. The organization of the table is based on the mentioned subdiagrams. Note the occurrence of the algebras (11-14, 17 and 18) with three different root lengths.

\[ r = 4: \]

In the light of the arguments linking string theory in \( r \) dimensions to a hyperbolic algebra of the same rank, this class of diagrams (including especially the single strictly hyperbolic one) is of special interest. We note that \( A_1^{(1)} \) and \( A_2^{(2)} \), which are already affine, cannot appear as subdiagrams. \( G_2 \), in contrast, still survives through its affinized forms \( G_2^{(1)} \) and \( D_4^{(3)} \). The first diagram is the lowest member of what is left of the A-series in the hyperbolic case; this series extends to \( r = 9 \) as we shall see. The determinant of the Cartan matrix of a rank-\( r \) member of this "A-series" is \( -(r-1) \); in this sense these may be thought of as the groups "SU(-n)", \( 8 \geq n \geq 3 \). Diagram 3 possesses an unusually high degree of symmetry, being invariant under the 24 operations of the permutation group \( P_4 \). There are some diagrams of three root lengths after 11; in 12 we encounter for the first time a diagram of four different root lengths! The Cartan matrix for the algebra 16 has determinant \(-1\), a rare property it shares with \( E_{10} \).

\[ r = 5: \]

The algebra \( F_4 \) is what makes the cases 6, 8 and 9 possible. 9 and 10 have three different root lengths.

\[ r = 6: \]

Diagram 2 has appeared previously in Ref. [7]. Diagram 3 is strictly peculiar to rank 6, being based directly on \( D_4^{(1)} \). The algebras 4, 5, 8, 9, 11 and 12 owe their existences to \( F_4^{(1)} \) and its twisted dual \( E_6^{(2)} \). Diagrams 12 and 13 involve three different root lengths.
\[ r = 7: \]

All the diagrams for \( r > 7 \) have already been given by Kac [1]. Case 2 can be thought of as a contribution of the rank 6-diagram 2 or a cross between the E- and the D-series. Similarly, 3 is a cross between the E- and B(C)-series that started with 4 from rank 5 and 6 from rank 6. All algebras are based on \( E_6 \).

\[ r = 8: \]

Diagram 2 is denoted by \( T_{4,3,3} \) in Ref. [1]. It is the first example of its kind, while 1, 3 and 4 are higher rank members of series we have previously seen. All diagrams are built upon the E-series in its finite and affine forms. This will also hold for the remaining \( r = 9 \) and 10 cases.

\[ r = 9: \]

Diagram 2 is \( T_{5,4,2} \); otherwise everything is as for \( r = 8 \).

\[ r = 10: \]

The "A-series" has disappeared as predicted in 2.C.2, there being no affine algebra beyond \( E_6^{(1)} \) to support it. The diagram 1 is \( T_{7,3,2} \), alias the famous \( E_{10} \), deserving the latter name both because of its rank and the fact that its Cartan matrix \( A_{ij} \) has determinant \(-1\), in conformity with the general result \( \text{det}(A_{ij}[E_n]) = 9-n \), where \( E_n = E_6^{(1)} \). Thus with \( E_{10} \) and the algebras represented by diagrams 2 and 3 the hyperbolic algebras come to an end.

4. DISCUSSION

Or do they? While in the strict sense they do, Lorentzian extensions with roots still in a latticized Minkowski space are possible. These are distinct from the more restricted hyperbolic class in at least two important respects: (i) the Dynkin diagram no longer has only finite or affine proper subdiagrams. (ii) The number of points on the diagram may, in general, be greater than the rank of the algebra. The second possibility, for which we will give an example, arises from the fact that a number of simple roots greater than the dimensionality of space-time may be needed to reach all of the lattice points. Interesting examples of such diagrams have been discussed by Goddard and Olive [2] in the context of even self-dual Minkowskian lattices denoted by \( \Gamma_{8n+1,1} \). Here \( (8n+1,1) \) indicates the signature and the allowed dimensions of spaces in which such lattices can occur. The unique lattice for \( n = 1 \) is known to correspond to the root lattice of \( E_{10} \).
That the Cartan matrix has determinant $-1$ and the simple roots are of equal length are essential to this result. One of the two possible diagrams given by Goddard and Olive for $n = 2$ illustrates both novelties. This is the diagram

```
   o---o---o---o---o---o---o---o---o---o---o---o
```

which can be thought of as two $E_8^{(1)}$ diagrams tied together by an additional point in the middle. This means 19 simple roots furnish a basis for this 18-dimensional lattice. Note that the Cartan matrix determinant for this diagram vanishes, reflecting the linear dependence of the 19 simple roots. The situation becomes extreme for $\mathbb{H}^{25,1}$; the Dynkin diagram is infinite [2]! Thus once again a 26-dimensional Minkowski space-time emerges as a limiting case, just as it does for the unitarity and covariance of the bosonic string. We find it very intriguing that the numbers 26, 10 and 4 appear as upper bounds on the ranks of a special class of Lorentzian algebras, hyperbolic algebras and strictly hyperbolic algebras, respectively. Our conclusion is that a deeper understanding of such algebras will provide new insights into string theory and vice versa. Recent history supports this assertion: vertex operators, invented for dual model calculations, have proved to be ideal tools for the realization of Kac-Moody and Lie algebras, whereas the vertex operator representation of Lie algebras with roots in self-dual lattices made the heterotic string possible.

Regarding the actual classification of Lorentzian diagrams, we are not aware of a systematic attempt along the direction of the present paper. However, a number of papers by Vinberg [16] examining the discrete groups in Lobachevski spaces may be relevant. Vinberg's basic approach * is to translate Coxeter's work [17] on the classification of discrete reflection groups for spheres $S^n$ or Euclidean spaces $E^n$ to hyperboloidal (Lobachevski) spaces $H^n$. As Coxeter's method leads to a classification of Lie algebra root configurations, this translation may very well provide a method for the description of Lorentzian root structures, although Ref. [16] does not directly address this problem.

* The figures at the end of the first article of Ref. [16] are not to be interpreted as conventional Dynkin diagrams. For example, a list that appears superficially to resemble finite Lie algebra diagrams also includes the diagram for the affine algebra $G_2^{(1)}$, our hyperbolic algebra 16 of rank 4 and a $G_2$-type diagram with $m$ lines, with $m > 5$. All in all, the overlap (at the level of superficial similarity of diagrams) between our work and Vinberg's consists of 11 diagrams plus the "$E_{18}$" case discussed in the last section.
ACKNOWLEDGEMENTS

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REFERENCES


**TABLE 1**

**STRICTLY HYPERBOLIC ALGEBRAS**

<table>
<thead>
<tr>
<th>Rank 3:</th>
<th>Subgroups (dual subgroups)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1. <a href="#">Diagram</a></td>
<td>$B_2, A_2, B_2$</td>
</tr>
<tr>
<td>2. 2. <a href="#">Diagram</a></td>
<td>$G_2, A_1 \times A_1, G_2$</td>
</tr>
<tr>
<td>3. 3. <a href="#">Diagram</a></td>
<td>$G_2, A_2, G_2$</td>
</tr>
<tr>
<td>4. 4. <a href="#">Diagram</a></td>
<td>$G_2, A_1 \times A_1, B_2$</td>
</tr>
<tr>
<td>5. 5. <a href="#">Diagram</a></td>
<td>$B_2, A_1 \times A_1, G_2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rank 4:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1. <a href="#">Diagram</a></td>
</tr>
</tbody>
</table>

The subgroup in the $n$-th position from the left is obtained when the root numbered $n$ is removed.
TABLE 2
HYPERBOLIC ALGEBRAS

Rank 3:

<table>
<thead>
<tr>
<th></th>
<th>G</th>
<th>G (dual)</th>
<th>Subgroups (dual subgroups)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td>$A_2$, $A_1 \times A_1$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>2</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
<td>$B_2$, $A_1 \times A_1$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>3</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td>$(1)$, $A_1 \times A_1$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>4</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td>$A_1^{(1)}$, $A_2$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>5</td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
<td>$A_2$, $A_1^{(1)}$, $A_2$</td>
</tr>
<tr>
<td>6</td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
<td>$B_2$, $A_1^{(1)}$, $B_2$</td>
</tr>
<tr>
<td>7</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td>$(1)$, $(1)$, $A_1^{(1)}$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>8</td>
<td><img src="image15" alt="Diagram" /></td>
<td><img src="image16" alt="Diagram" /></td>
<td>$G_2$, $A_1 \times A_1$, $A_1^{(1)}$</td>
</tr>
<tr>
<td>9</td>
<td><img src="image17" alt="Diagram" /></td>
<td><img src="image18" alt="Diagram" /></td>
<td>$G_2$, $A_1^{(1)}$, $G_2$</td>
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</tbody>
</table>

The groups in parentheses in the n-th position from the left are those obtained when the point number n is removed from the dual diagram.
Subgroups (dual subgroups)

10. \(A_2, A_1 \times A_1, A_2^{(2)}\)

11. \(B_2, A_1 \times A_1, A_2^{(2)}\)

12. \(B_2, A_1 \times A_1, A_2^{(2)}\)

13. \(G_2, A_1 \times A_1, A_2^{(2)}\)

14. \(A_2^{(2)}, A_1 \times A_1, A_2\)

15. \(A_2^{(2)}, A_1 \times A_1, A_2\)

16. \(A_2^{(2)}, A_2, A_2^{(2)}\)

17. \(A_2^{(2)}, B_2, B_2\)

18. \(A_1^{(1)}, A_1 \times A_1, A_2^{(2)}\)

19. \(A_2^{(2)}, A_1^{(1)}, A_2^{(2)}\)
Rank 4:

<table>
<thead>
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<th>Subgroups (dual subgroups)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td><img src="image1" alt="Graph 1" /></td>
<td>$A_3, A_2 \times A_1, A_3, A_2^{(1)}$</td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Graph 2" /></td>
<td>$A_2^{(2)}, A_2^{(1)}, A_3, A_3$</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Graph 3" /></td>
<td>$A_2^{(1)} A_2^{(1)} A_2^{(1)} A_2^{(1)}$</td>
</tr>
<tr>
<td>4</td>
<td><img src="image4" alt="Graph 4" /></td>
<td>$C_2^{(1)} (D_3^{(2)}), A_1 \times B_2, A_2 \times A_1, B_3(C_3)$</td>
</tr>
<tr>
<td>5</td>
<td><img src="image5" alt="Graph 5" /></td>
<td>$B_3(C_3), B_3(C_3), A_2 \times A_1, A_2^{(1)}$</td>
</tr>
<tr>
<td>6</td>
<td><img src="image6" alt="Graph 6" /></td>
<td>$C_2^{(1)} (D_3^{(2)}), B_3(C_3), A_3, B_3(C_3)$</td>
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<tr>
<td>7</td>
<td><img src="image7" alt="Graph 7" /></td>
<td>$C_2^{(1)} (D_3^{(2)}), C_2^{(1)} (D_3^{(2)}), C_2^{(1)} (D_3^{(2)}),$ $A_1 \times A_1 \times A_1$</td>
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<tr>
<td>8</td>
<td><img src="image8" alt="Graph 8" /></td>
<td>$D_3^{(2)}, A_1 \times B_2, B_2 \times A_1, C_2^{(1)}$</td>
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<tr>
<td>9</td>
<td><img src="image9" alt="Graph 9" /></td>
<td>$C_2^{(1)}, D_3^{(2)}, C_2^{(1)} (D_3^{(2)}),$ $C_2^{(1)}, D_3^{(2)}$</td>
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</tbody>
</table>
10. \(C_3(B_3), A_1 \times A_1 \times A_1, C_3(B_3), C_2^{(1)}(D_3^{(2)})\)

11. \(A_4^{(2)}, A_1 \times B_2, A_1 \times A_2, B_3(C_3)\)

12. \(A_4^{(2)}, A_1 \times B_2, A_1 \times B_2, A_4^{(2)}\)

13. \(A_4^{(2)}, A_1 \times B_2, B_2 \times A_1, D_3(C_2)\)

14. \(B_3, A_1 \times A_1 \times A_1, A_4^{(2)}, C_3\)

15. \(A_4^{(2)}, A_1 \times A_1 \times A_1, C_2^{(1)}(D_3^{(2)}), A_4^{(2)}\)

16. \(G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, A_2 \times A_1, A_3\)

17. \(D_4^{(3)}, A_1 \times A_2, A_2 \times A_1, G_3^{(1)}\)

18. \(G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, B_2 \times A_1, B_3(C_3)\)

19. \(C_3(B_3), A_1 \times B_2, G_2 \times A_1, G_2^{(1)}(D_4^{(3)})\)

20. \(G_2^{(1)}(D_4^{(3)}), G_2^{(1)}(D_4^{(3)}), A_1 \times A_1 \times A_1, A_3\)

21. \(G_2^{(1)}(D_4^{(3)}), G_2^{(1)}(D_4^{(3)}), A_2 \times A_1, A_2^{(1)}\)

22. \(G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, A_1 \times G_2, G_2^{(1)}(D_4^{(3)})\)

23. \(D_4, G_2^{(3)}(D_4^{(3)}), A_1 \times G_2, A_2^{(1)}(G_2^{(1)})\)

24. \(D_4, D_4^{(3)}(D_4^{(3)}), G_2, G_2^{(1)}\)
Rank 5:

G

G (dual)

Subgroups (dual subgroups)

1. $D_4$, $A_4$.

2. $A_3 \times A_1$, $A_3^{(1)}$.

3. $B_3^{(1)}(A_5^{(2)})$, $B_4(C_4)$, $B_4(C_4)$, $A_3 \times A_1$, $A_3^{(1)}$.

4. $B_3^{(1)}(A_5^{(2)})$, $B_4(C_4)$, $A_4$.

5. $B_4^{(1)}(A_5^{(2)})$, $B_3^{(1)}(A_5^{(2)})$, $B_3^{(1)}(A_5^{(2)})$.

6. $D_4$, $A_1 \times A_1 \times A_1$.

7. $F_4$, $F_4$, $A_4 \times A_2$, $A_3 \times A_1$, $B_3^{(1)}(A_5^{(2)})$.

8. $B_3^{(1)}(A_5^{(2)})$, $A_1 \times B_3(C_3)$, $B_2 \times A_1 \times A_1$.

9. $B_4(C_4)$, $D_4(2)^{(1)}$.

10. $F_4$, $C_3^{(1)}(D_4^{(2)})$, $F_4$, $B_4(C_4)$, $B_4(C_4)$.
9. \[ A_6^{(2)}, A_1 \times B_3(C_3), A_2 \times B_2, A_1 \times B_3(C_3), F_4 \]

10. \[ B_4(C_4), A_1 \times A_1 \times B_2, A_1 \times C_3(B_3), \]
\[ A_5(3), A_6 \]

Rank 6:

1. \[ D_6, D_5, A_5, A_5, A_4 \times A_1, A_5^{(1)} \]

2. \[ D_5, D_5, D_5, A_1 \times A_1 \times A_1 \times A_2, D_4 \times A_1, D_4^{(1)} \]

3. \[ A_1 \times A_1 \times A_1 \times A_1 \times A_1 \times A_1 \]

4. \[ E_6, A_1 \times C_4, A_2 \times A_3, A_3 \times A_2, B_4 \times A_1, F_4^{(1)} \]

5. \[ F_4^{(1)}(E_6^{(2)}), A_1 \times F_4, A_2 \times C_3(B_3), A_3 \times A_2, A_4 \times A_1, B_5(C_5) \]

6. \[ B_4(A_7^{(2)}), A_1 \times B_4(C_4), A_2 \times A_1 \times B_2, A_4 \times A_1, B_5(C_5) \]

7. \[ B_4(A_7^{(2)}), B_4(A_7^{(2)}), B_4(A_7^{(2)}), A_1 \times A_1 \times B_2, D_4 \times A_1, D_4^{(1)} \]

8. \[ F_4(E_6), F_4(E_6^{(2)}), A_1 \times A_1 \times C_3(B_3), A_3 \times A_2, D_4 \times A_1, B_4^{(1)}(A_5^{(2)}) \]
9. \( C_4 \times D_5 \), \( A_1 \times C_4 \times B_4 \), \( A_2 \times C_3 \times B_3 \), \( B_2 \times B_3 \times C_3 \), \( A_1 \times F_4 \), \( E_6 \times F_4 \)

10. \( B_4 \times A_7 \), \( A_1 \times B_4 \times C_4 \), \( A_1 \times B_2 \times B_2 \), \( A_1 \times B_4 \times C_4 \), \( B_4 \times A_7 \), \( D_5 \times C_6 \)

11. \( E_6 \), \( C_4 \), \( E_6 \), \( F_4 \), \( D_5 \), \( F_4 \)

12. \( F_4 \times A_7 \), \( A_1 \times B_4 \times C_3 \times B_3 \), \( C_3 \times B_3 \times A_2 \), \( C_4 \times B_4 \times A_1 \), \( A_6 \)

13. \( B_4 \), \( A_1 \times B_4 \), \( B_2 \times A_1 \times B_2 \), \( C_4 \times A_1 \)

Rank 7:

1. \( E_6 \), \( D_6 \), \( D_6 \), \( A_7 \), \( A_7 \), \( A_5 \times A_1 \), \( A_5 \)

2. \( D_5 \), \( A_1 \times D_5 \), \( A_1 \times A_2 \times A_3 \), \( A_4 \times A_1 \times A_1 \), \( E_6 \), \( E_6 \)

3. \( B_5 \times A_9 \), \( A_1 \times B_5 \times C_5 \), \( A_2 \times A_1 \times B_3 \times C_3 \), \( A_4 \times B_2 \), \( D_5 \times A_1 \), \( E_6 \), \( B_6 \times C_6 \)

Rank 8:

1. \( E_7 \), \( E_7 \), \( D_7 \), \( D_7 \), \( A_7 \), \( A_7 \), \( A_6 \times A_1 \), \( A_6 \)

2. \( E_7 \), \( E_7 \), \( A_1 \times A_6 \), \( A_1 \times A_6 \), \( A_1 \times A_2 \times A_3 \), \( A_5 \times A_2 \), \( E_6 \times A_1 \), \( E_6 \)