LATTICE CONSTRUCTIONS
OF FERMIONIC STRINGS*

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ABSTRACT

A brief review is given of the covariant lattice construction of fermionic strings. Recent developments that are discussed include a simple proof of multi-loop modular invariance, the construction of new supercurrents, the partition functions of $D_n$ theories and their Atkin-Lehner symmetries, and an explicit representation of the Leech lattice in terms of $D_1$ lattices.

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This paper consists of three parts, all dealing in some way with string constructions using lattices, or strings compactified on lattices. The first, and most important part is an explanation of the covariant lattice construction of fermionic strings in dimensions ten and below. In the second part we discuss some work with unfortunately inconclusive results, namely an attempt to find non-supersymmetric lattice theories with a vanishing one-loop cosmological constant. Using Atkin-Lehner transformations, we classify all partition functions that have that property, but have not succeeded in finding any examples. In the third part we give an explicit representation of the Leech lattice in terms of $D_1$ lattices. This also provides an explicit fermionization of string theories compactified on the corresponding torus.

1. The Covariant Lattice Construction

The covariant lattice construction is one of several ways of obtaining string theories in four dimensions [1-7]. Our purpose here is to explain this construction and some of its properties in a systematic way, putting ideas that have been presented in several papers [5], [8-12] in their logical order. One will not find here a discussion of the consequences of these models for string theory and its relevance for physics. We have commented on this previously, and the reader is undoubtedly able to draw his own conclusions about this. Instead, we will concentrate on explaining how the construction works. For a more detailed review see [13].
1.1 History and Heuristics

Before going into details, it might be useful to explain the basic idea of the construction, as well as its history. The covariant lattice construction is the happy marriage of two applications of lattices to string theory: the use of lattices in “compactification” and the bosonic lattice construction of the NSR model and its ghost system. Although the use of lattices for the description of torus compactification of bosonic or type-II strings is part of string pre-history, the for our purpose more relevant lattices on which the momenta of left- and rightmovers are treated asymmetrically first appeared in the construction of the heterotic string. The historical development started with a 16-dimensional lattice $\Gamma_{16}$ on which the momenta of 16 left-moving bosons were quantized [14], and this lattice was subsequently enlarged in two ways, each of which expanded the set of bosons whose momenta lie on the lattice. The first step was the class of theories constructed by Narain [3], who realized that in string theories in $d = 10 - 2n$ dimensions one can enlarge the lattice from $\Gamma_{16}$ to $\Gamma_{16+2n,2n}$ (the semicolon separates left and rightmovers) by including on it the momenta of all bosons not linked to flat space-time coordinates.

The second way of enlarging the lattice is to include the momenta of the bosonized NSR fermions and the corresponding ghosts. This bosonic representation was first considered in [15] for an entirely different purpose, namely to get a convenient framework for describing fermionic string interactions and in particular fermion emission. The fermionic sector of ten-dimensional superstrings or heterotic strings is then described by a “covariant” lattice $\Gamma_{5,1}$, with vectors whose first five entries are $D_5$ weights, while the last one is the ghost charge. It will be discussed in more detail below. The use of this idea for the construction of new string theories was initiated in [8], and developed further in [9] and [5]. To describe ten-dimensional heterotic strings one combines the lattices for the left- and rightmovers to $\Gamma_{16,5,1}$. The last logical step is then to combine this enlargement of the lattice with the one considered by Narain. The result is a set of string theories described by lattices of the form $\Gamma_{16+2n,2n+5,1}$, which include the momenta of every world-sheet degree of freedom other than scalars related to flat space-time coordinates. Hence the
sequence of studying larger and larger lattices reaches its end here.

In any of these steps of enlargement one has as trivial special cases the lattices which are simple direct products of their constituents, e.g. Narain lattices of the form $\Gamma_{16} \otimes \Gamma_{2n;2n}$, covariant lattices for ten-dimensional strings of the form $\Gamma_{16} \otimes \Gamma_{5;1}$, or covariant lattices for d-dimensional strings of the form $\Gamma_{16+2n;2n} \otimes \Gamma_{5;1}$. The crucial point is however that this is too restrictive: if one directly formulates the necessary consistency conditions in terms of the enlarged lattices one finds more general solutions. In the three cases mentioned above these new solutions are respectively the class of theories constructed in [3] (a generalization of left-right symmetric torus compactifications), the non-supersymmetric ten-dimensional heterotic strings [16], [17], (a generalization of the supersymmetric ones) and a class of chiral four-dimensional heterotic strings [5], which from this point of view can be regarded as a generalization of Narain compactifications.

The non-trivial consistency conditions that still have to be satisfied by imposing requirements on the lattices are modular invariance and world-sheet supersymmetry. Other conditions, such as reparametrization and conformal invariance are already implicit in this set-up; for example conformal invariance essentially determines (after subtraction of ghost contributions) the dimension of the lattice. The condition that modular invariance imposes is well-known, at least as long as the NSR-sector is not part of the lattice. In order to satisfy it, the left-right momentum lattice must be even and self-dual with respect to a metric with opposite signature for left- and right-movers. The inclusion of the NSR-sector does not alter this very severely: again the combined lattice must be self-dual, but this time some specified vectors must have odd rather than even length, with respect to a lorentzian metric.

The precise conditions that the lattice has to satisfy will be listed below, where also the conditions for world-sheet supersymmetry will be added, but the essential point is that any such lattice gives rise to a consistent string theory. The rather complicated rules for modular invariance which one encounters in other constructions are simply encoded in the self-duality of the lattice. Here “modular invariance” means of course modular invariance at arbitrary order in string perturbation the-
ory (at least if one makes the usual assumptions about the multi-loop partition functions).

All of this can be formulated in an even simpler way by switching to a different lattice formulation, which we will call the even formulation. Here one replaces the odd self-dual covariant lattice by and even self-dual one, from which the spectrum can be derived more easily. These even lattices play also a useful role in proving modular invariance.

1.2 Covariant Lattices

As the above suggests, we want to construct new string theories starting from an entirely bosonic formulation, in which one never really has the 10 right-moving Majorana-Weyl fermions of the NSR model. Nevertheless to gain insight in this construction it is useful to start by bosonizing the NSR system in 10 dimensions. The bosonization of the Majorana-Weyl fermions is straightforward. The fermions $\psi^k$ are replaced by real linear combinations of exponentials of the form

$$\psi^k = e^{iX_k} c_k, \quad \psi^{-k} = e^{-iX_k} c_{-k} \quad (k = 1, \ldots, 5),$$  \hspace{1cm} (1.1)

where $X_k$ is a set of five new bosons, and $c_k$ are cocycle factors. These operators act on states with momenta quantized on the $D_5$ weight lattice, which form representations of the (Wick-rotated) Lorentz-group $SO(10)$.

If one would try to arrive at a lattice formulation involving only these five bosons, one would not succeed because the boundary conditions of the $\beta, \gamma$ ghost system are correlated with those of the fermions. In bosonic language this correlation implies that the ghost charge lattice is not simply tensored with the $D_5$ lattice, but that they combine in a less trivial way. To make this explicit one has to bosonize the ghosts, which can be done using the construction of [15]:

$$\beta = e^{-i\phi} \partial \xi, \quad \gamma \sim e^{i\phi} \eta.$$  \hspace{1cm} (1.2)

Here $\xi$ and $\eta$ form an auxiliary fermionic system needed to make the ghosts commute rather than anticommute. It will not play a role in the following because its
boundary conditions are not linked non-trivially with those of the other fields. The non-trivial boundary conditions are bosonically represented by the $\phi$ ghost charge quantization condition.

To ensure that the ghosts always have the same boundary conditions as the fermions, one must require that this ghost charge is integral for tensor-representations of $D_5$ (i.e. for representations in the conjugacy classes $(0)$ and $(v)$) and half-integral for spinors (i.e. the conjugacy classes $(s)$ and $(c)$). This can be summarized by extending the lattice with one extra dimension representing the ghost, and requiring that all vectors on the extended lattice must belong to one of the four conjugacy classes $(0), (v), (s)$ or $(c)$ of $D_5$. We will denote vectors on this lattice as $(\Lambda | q)$, where $\Lambda$ is a $D_5$ weight and $q$ is the ghost charge.

Operator products of vertex operators related to this lattice have the general form

$$e^{i\lambda \cdot X(z)} e^{i\phi(z)} e^{i\lambda' \cdot X(w)} e^{i\phi(w)} = (z - w)^{\lambda - \lambda'} q^{\lambda' - \lambda} e^{i(\lambda + \lambda') X(w)} e^{i(q + q') \phi(w)} + \ldots, \quad (1.3)$$

from which we learn that the natural inner product on it has signature $(+++++--)$. Therefore we will denote the lattice with this inner product as $D_{5,1}$ rather than $D_5$.

From the ghost energy momentum tensor one finds that the mass formula for states corresponding to this lattice is

$$\frac{1}{8} m^2 = \frac{1}{2} (\lambda^2 - q^2) - q + \mathcal{N} - 1, \quad (1.4)$$

where $\mathcal{N}$ is the number operator for the oscillator contributions. The linear term is related to the ghost number current anomaly. Excitations of the auxiliary $\xi, \eta$ system have been ignored here.

There is by no means a one-to-one correspondence between lattice states and physical states. This was already not the case for the $D_5$ lattice, and the additional ghost charge leads to even more states. The light-cone spectrum is represented an infinite number of times on the lattice, and one can recover it by fixing a value for
the ghost charge. Choosing different ghost charge sectors can be achieved by acting
with the picture changing operator. Although all vacuum choices are physically
equivalent, some are more convenient than others. In particular it would be useful
to have an easy way to recover the light-cone count of physical states, and the usual
form of the mass-formula. This is achieved by choosing the canonical ghost vacua,
with charge \( q = -1 \) for bosons and \( q = -\frac{1}{2} \) for fermions. The light-cone states are
then obtained by considering only states with lattice momenta of the form

\[
\left( \lambda_{\text{lightcone}}, 0 \mid -1 \right) \\
\left( \lambda_{\text{lightcone}}, -\frac{1}{2} \mid -\frac{1}{2} \right)
\]  

(1.5)

and oscillators in the Cartan subalgebra of \( D_4^{\text{lightcone}} \) acting on these states (as well
as the eight transverse bosonic oscillators). It is not difficult to see that this is in
deed precisely the spectrum obtained with transverse Neveu-Schwarz and Ramond
oscillators, acting on the usual vacua. Furthermore the masses of these states are
correctly given by (1.4). In other pictures one cannot read off the Lorentz represen-
tations of the states from the lattice in such a straightforward way, although all
states are certainly there.

The theory we have obtained so far has many diseases, which are all cured by the
GSO-projection. In bosonic language this projection corresponds to keeping only
the conjugacy classes (0) and (s) on the lattice. On such a lattice all lorentzian
inner product are integral, so that all operator products are local. Furthermore one
notices that the lattice is now self-dual with respect to the lorentzian metric, but it
is not even: the spinors have odd norm (= length-squared).

One suspects that this self-duality must have a direct relation with modular
invariance, which is a known consequence of the GSO-projection. At one loop this
was indeed shown to be the case by Lerche and Lüst [8], who also explained why the
fact that the lattice is odd rather than even is no problem, but is in fact precisely
what is needed. The argument uses the partition function that one can associate
with the lattice, which has the form

\[
\mathcal{L}_{5,1} = \sum_{\lambda, q} e^{ix\tau(\lambda^2 - q^2 - 2q)} e^{2\pi i \eta},
\]

(1.6)
and which is multiplied with the usual \( \eta \)-functions for the oscillator contributions.\(^*\)

The second factor is put in to provide a spin-statistics minus sign in the partition function for the contribution of the space-time fermions. It is trivial to check that this function is invariant under \( \tau \rightarrow \tau + 1 \) precisely because the odd spinor norm is compensated by the linear ghost charge anomaly term. It is harder to check that it also transforms properly under \( \tau \rightarrow -1/\tau \). Proving this involves nothing more than the usual techniques, but one has to close ones eyes for the fact that one is dealing with ill-defined quantities (partition functions for lorentzian lattices) in this calculation.

Indeed, the partition function (1.6) is neither well-defined, nor is it the “physical” partition function we ought to consider. In [8] it is shown that the partition function of the light-cone states can be obtained by formally dividing (1.6) by the partition function of an even lorentzian self-dual lattice \( D_{1,1} \), which is also (formally) modular invariant, and is equally ill-defined. Although it is somewhat unsatisfactory to have to deal with such ratios of ill-defined quantities, and although we will present a well-defined alternative below, we have mentioned it here because it demonstrates quite clearly how the odd-self-duality, the ghost number current anomaly and the spin-statistics sign conspire to make the theory modular invariant.

There is nothing surprising about the fact that the superstring turns out to be modular invariant, because this is already known. The point is however that the entire argument carries over to a much larger class of theories. If we stay in ten dimensions for the moment, and combine the right-moving \( NSR \)-sector with the left-moving lattice partition function of the heterotic string (\( E_8 \otimes E_8 \) here for definiteness) or with a left-moving \( NSR \)-sector of a type-II string, then we can describe these theories by lattices of the form

\[
(E_8 \otimes E_8)_L \otimes (D_{5,1}^{0s})_R \quad (\text{heterotic})
\]
\[
(D_{5,1}^{0s})_L \otimes (D_{5,1}^{0s})_R \quad (\text{type \( \text{II}B \)})
\]
\[
(D_{5,1}^{0s})_L \otimes (D_{5,1}^{0c})_R \quad (\text{type \( \text{II}A \)})
\]

\(^*\) In the following we disregard phases and weights in the transformation of the lattice partition function which are compensated by these \( \eta \)-functions.
Here the superscripts on $D_{5,1}$ indicates the conjugacy classes that are present. These are the cases where the left and right sectors are separately modular invariant. Nothing in the above argument requires this however. It goes through as long as the \textit{combined} left-right lattice is self-dual and odd, with space-time spinors having odd length. In order to respect Lorentz-invariance the components that belong to $D_{5,1}$ conjugacy classes above will have to belong to such conjugacy classes in general, but these conjugacy classes may correlate non-trivially with other parts of the lattice, instead of the direct product structure above. The conjugacy classes (v) and (c) of $D_{5,1}$ may then also appear, as long as they do so in such a way that the complete lattice is self-dual.

Exactly the same argument applies also in $d = 10 - 2n$ dimensions. In this case one never needs to consider the $2n$ Majorana-Weyl fermions that belong to the “internal” or “compactified” dimensions. One uses instead $n$ bosons, which are on equal footing with the usual bosonic degrees of freedom. Only the $d$ Majorana-Weyl fermions with space-time indices remain somewhat special, since the lattice momenta of the corresponding $d/2$ bosons are constrained by Lorentz-invariance to be weights of $D_{d/2}$. Combining them with the bosonized ghosts as above, and with all other bosonic degrees of freedom other than space-time coordinates, we get theories described by lattices of the form

$$\Gamma_{16+2n;5+2n,1} = \Gamma_{16+2n;3n+d/2,1},$$

where the last $d/2 + 1$ components are required to belong to one of the four conjugacy classes of $D_{d/2,1}$. The generalization to type-II theories is completely straightforward.

It is easy to check that the formal proof of one-loop modular invariance given above goes through under exactly the same conditions, \textit{i.e.} the lattice must be odd self-dual, with odd lengths for space-time spinors, and with inner product $(-)^{16+2n}(+)^{2n+5}(-)$.

One may worry that all this relies rather heavily on formal manipulations with ill-defined partition functions. One may also worry that modular invariance may
not hold at higher loops. We will discuss this below, after introducing the even lattice formulation. The conclusion will be that the conditions we gave here do indeed guarantee modular invariance. It should be pointed out here that below ten dimensions there are additional conditions on the lattice arising from world-sheet supersymmetry, to be discussed later.

1.3 The Even Lattice Map

Although the covariant lattices introduced in the previous section specify the fermionic string theory completely, they are not very pleasant to deal with, because of their rather cumbersome metric, and because some of their vectors have odd norm. As explained above, for heterotic string theories in $d = 10 - 2n$ dimensions we need odd self-dual lattices of the form $\Gamma_{16+2n;5+2n,1}$, where the last $6 - n$ right-moving components belong to one of the four conjugacy classes of $D_{5-n,1}$. Furthermore the space-time spinors have to correspond to vectors of odd length.

Now we will introduce a map of any such lattice to an even self-dual lattice. This map will have the useful property that it preserves modular invariance at all loop orders. Essentially, what we are going to do is to associate with any heterotic string theory (or type-II theory) a bosonic string theory, by mapping the right (or left) moving superconformal system to a bosonic one. This map works in general only in this direction.

This idea originates from [18], where it was used to argue that the fermionic strings are “vacua” of the bosonic string. Although this remains an attractive idea, this is not our interest here. For us the lattice map will merely be a useful tool for proving certain properties of fermionic strings. The form of the lattice map that we will use appeared first in [19], and in [20] it was shown that it preserves one-loop modular invariance. This was extended to higher loops in [11].

Basically, the idea is very simple. If we decompose one of the lattices of interest with respect to the $D_{5-n,1}$ conjugacy classes, we see that it consists of four sets of vectors

$$(\Delta_0, 0) + (\Delta_v, v) + (\Delta_s, s) + (\Delta_c, c),$$

(1.8)
where the second entry denotes the $D_{5-n,1}$ conjugacy classes, and the $\Delta$'s are simply the vectors associated with them on the rest of the lattice. Now define a new lattice $\Gamma_{16+2n,8+2n}$ by writing down a decomposition identical to (1.8), but with the second entry interpreted as a conjugacy class of $D_{8-n}$ instead of $D_{5-n,1}$. That is, we map

$$D_{5-n,1} = D_{4-n}^{\text{lightcone}} \quad D_{1,1}^{\text{ghost}} \quad D_{4-n}^{\text{lightcone}} \quad D_{4}^{\text{ghost}} = D_{8-n}.$$  \hspace{1cm} (1.9)

Here $D_{1,1}^{\text{ghost}}$ represents the longitudinal and ghost components of vectors on the covariant lattice.

Observe that the old and the new lattice have the same number of conjugacy classes, so that if the old lattice was unimodular, so is the new one. If one now computes the cos and inner products of the conjugacy classes of $D_{5-n}$ in comparison with those of $D_{5-n,1}$, one finds that modulo two the only change is in the norm of the spinors, which change from odd into even length. But the spinors of $D_{5-n,1}$ were precisely the odd vectors on $\Gamma_{16+2n,5+2n,1}$. Hence the map preserves unimodularity and all mutual inner products, but it maps all the odd vectors into even ones. The result is a lattice $\Gamma_{16+2n,8+2n}$ which is even self-dual with respect to the metric $\left[(-)^{16}; (+)^8\right]$. Thus, by use of this one-to-one map, we are able to find an alternative even lattice formulation of heterotic string theories. Again the generalization to type-II is completely straightforward.

Notice that we are considering a map between lattices (even with different dimensions) and not a map between individual vectors. One should regard this as only a bookkeeping trick, which enables us to describe the lattices in a more convenient way. If we add an $E_8$ factor to the right lattice, we get an even self-dual lorentzian lattice $\Gamma_{16+2n,16+2n}$ on which the bosonic string can be compactified. The physical light-cone spectrum is obtained by applying a set of truncation rules to the bosonic string spectrum, which are designed to mimic exactly the corresponding procedure for the covariant, odd lattice. To get these rules, consider the physical state selection rule (1.5). It tell us to keep only states with special components in the $D_{1,1}^{\text{ghost}}$ sublattice, belonging to conjugacy classes $(v)$ and $(s)$ of that lattice. Since $D_{1,1}^{\text{ghost}}$ is mapped to $D_{4}^{\text{ghost}}$ by the lattice map, we take the identical conjugacy classes of
$D_4^{\text{ghost}}$ to identify the physical states. Thus one physical state is represented by an entire conjugacy class of $D_4^{\text{ghost}}$. One can choose a representative of the $D_4^{\text{ghost}}$ conjugacy class that makes all this even simpler. Namely, if one chooses an arbitrary weight of $8_s$ and $8_v$ to represent the spinor and vector conjugacy class, then the mass of the light-cone states is correctly given by the compactified bosonic string mass formula in which one should include the contribution from the $D_4^{\text{ghost}}$ components. Fixing these $D_4^{\text{ghost}}$ components arbitrarily, we find thus that the physical states must have $\Gamma_{16+2m;16+2n}$ lattice vectors whose last 12 (i.e. $D_4^{\text{ghost}} \otimes E_8$) components must be

$$
(1,0,0,0,(0)^8) \quad \text{for bosons}
$$

$$
\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},(0)^8\right) \quad \text{for fermions}
$$

(1.10)

These lattice states may be excited by all bosonic oscillators, except the twelve in the Cartan subalgebra of $D_4^{\text{ghost}} \otimes E_8$.

Although these rules may look peculiar at first sight, they do give exactly the desired result, namely the spectrum of the covariant lattice theory. They were first formulated in [18] and [19], and were a completely ad hoc prescription for getting the fermionic string from the bosonic one. We see here that they do in fact follow from the physical state projection rule explained in sect 1.2, which originates from conformal field theory.

To some extent, we can drop the covariant lattice altogether, and work with even self-dual lattices. This is unfortunately not possible if one wants to study scattering amplitudes. For this purpose one needs the complete machinery of conformal field theory, and we have not succeeded to cast that into even lattice form. However, the even lattice formulation is extremely useful for discussing the spectrum.

### 1.4 Multi-loop Modular Invariance

The even lattice formulation is also quite useful for proving multi-loop modular invariance (with the highest degree of rigor that can be achieved at present.) To discuss multi-loop modular invariance properly, one would have to write down the multiloop partition function for the fermionic string, and its dependence on the
supermoduli, which has not been done yet. However, one usually assumes that as far as its modular transformation properties are concerned, the contribution of the NSR-fermions and the superghost system has the form

\[ (\text{Det} D_{1/2})^5 \alpha^{-n} (\text{Det} D_{3/2})^{-1}, \]

where \( D_{1/2} \) denotes the Dirac determinant, and represents the contribution of two Majorana Weyl fermions and \( D_{3/2} \) is the Rarita-Schwinger determinant (which only has ghost contributions). The subscript \( \alpha \) denotes the spin-structure. In general, this combination of determinants is multiplied with some \( \alpha \) dependent factor representing the contribution from all other world-sheet degrees of freedom.

The point is now that the modular transformation properties of these determinants are known explicitly [21]. Using them, we can show that the following two spin-structure dependent quantities have identical modular transformations at any loop order

\[ Y_\alpha = \left( \frac{\text{Det}_{1/2}}{\text{Det}_{3/2}} \right)_\alpha, \]

\[ X_\alpha = (\text{Det}_{1/2})^4 \sum_\beta (\text{Det}_{1/2})^8_\beta \]  

(1.11)

Thus if one replaces in a fermionic string partition function \( Y_\alpha \) by \( X_\alpha \), the new one will be modular invariant if and only if the old one was.

This replacement is precisely what the lattice map achieves. To see that one has to make a simple basis transformation from spin-structure basis to conjugacy class basis. This is trivial, and is moreover explained in detail in [11]. Details aside, comparing \( Y_\alpha \) and \( X_\alpha \) the reader will undoubtedly recognize that the first represents the \( D_{1,1} \) ghost and longitudinal part of the covariant lattice while the second represents \( D_4 \) conjugacy classes, multiplied with the \( E_8 \) partition function.

Using the above we conclude then that the following statements are equivalent:

- The covariant lattice \( \Gamma_{16+2n;5+2n,1} \) is odd self-dual (with odd norms for the spinors, as discussed above).
The lattice $\Gamma_{16+2n;3+2n} \times E_8$ obtained by means of the lattice map is even self-dual.

The bosonic string, compactified on the latter lattice is multi-loop modular invariant.

The fermionic string defined by the odd self-dual covariant lattice is multi-loop modular invariant.

The beauty of the above reasoning lies in the fact that the ghost system is determined by the $N = 1$ conformal system, i.e. by a world-sheet property, that is the same for any fermionic string. Hence the replacement of $Y$ by $X$, or the lattice map, is equally useful for any fermionic string theory, and in any space-time dimension.

1.5 World-sheet Supersymmetry

Up to now we have freely replaced world-sheet fermion by bosons. One has to be a little bit more careful about this, because one may endanger world-sheet supersymmetry. In the NSR sector of heterotic strings world-sheet supersymmetry is a manifest (local) $N = 1$ supersymmetry, which transforms the fermions $\psi^\mu$ into the bosons $X^\mu$ and vice-versa. Replacing $\psi^\mu$ by bosonic fields does not in itself destroy world-sheet supersymmetry. In constructing four-dimensional strings we do however something more drastic than just bosonization. We allow the bosons related to the internal world-sheet fermions to mix freely with the "compactified" bosons. Since our philosophy is to construct new string theories directly in four dimensions, we start always with a completely bosonic formulation, i.e. in practice we never bosonize anything. Therefore it is not automatically guaranteed that the result respects world-sheet supersymmetry.

World-sheet supersymmetry is an essential ingredient of fermionic strings. One thing that goes wrong if one does not have it is Lorentz-invariance. World-sheet supersymmetry is used to remove unphysical degrees of freedom from the covariant action of fermionic strings, so as to write them entirely in terms of the physical,
light-cone degrees of freedom. Although one can always write down such a light-cone action, it is important that it came from a covariant action. This makes it possible to write down a Lorentz-algebra, and in particular the generators $L^i_-$ that are needed to extend the light-cone algebra to a covariant one. In supersymmetric strings in ten dimensions one of the terms appearing in the expression for $L^i_-$ is $\int \bar{\psi}^i T_F$, where $\psi^i$ are the transverse components of the fermions, and $T_F$ is the world-sheet supercurrent.

One of the things that can go wrong with Lorentz invariance is that $[L^i_-, L^j_-]$ simply does not yield the answer required by the algebra of the Lorentz group. This situation occurs in theories with a conformal anomaly, and can be avoided by choosing the field content so that the conformal anomaly cancels. It might appear that nothing prevents us from simply using the Lorentz algebra of ten-dimensional strings, regarding 6 of the eight 8 transverse Lorentz indices as internal. This algebra closes, and it is indeed a valid choice in torus compactifications. However, closure of the algebra is not enough. In addition, one has to require that $L^i_-$ takes physical states into physical states. Otherwise one faces a problem which is even worse than the conformal anomaly, namely the states of the theory won’t even fit into representations of the Lorentz group. (In the presence of a conformal anomaly, the Lorentz algebra does not close, but for the bosonic string and the NSR formulation of the superstring it is still possible to assemble the massive states into representations of $SO(d - 1)$, the little group of massive particles)

This is indeed precisely what happens, and one sees this most easily in the chiral sector of chiral string theories. At first sight, chirality and string theory are almost contradictory, because one would think that chiral states can always be excited into massive states. Massive chiral states would however violate Lorentz invariance, so a really clever mechanism is needed to ensure that all massive states combine into non-chiral pairs, while the massless ones do not. This is achieved by world-sheet supersymmetry. Indeed, this restriction on massive chiral states turned out to be powerful enough to determine all the conditions of world-sheet supersymmetry for the class of theories considered in [5].
A second way in which world-sheet supersymmetry enters is through the picture changing operator. This operator has the form $\mathcal{P}_{+1} = e^{i\phi} T_F$, where $\phi$ is the bosonized ghost of section (6.1). It is essential for the calculation of scattering amplitudes, and it too should take physical states into physical states, and have a well-defined action.

The problem of finding a bosonic realization of world-sheet supersymmetry consists thus of two parts

1. Find a purely bosonic expression for $T_F$ that satisfies the supersymmetry algebra.

2. Make sure that it acts properly on all states.

The algebra that the supercurrent $T_F$ must satisfy can be encoded in operator products, and reads (with finite terms omitted)

$$T(z) T_F(w) = \frac{3/2}{(z-w)^2} T_F(w) + \frac{1}{(z-w)^2} \partial_w T_F(w)$$
$$T_F(z) T_F(w) = \frac{10}{(z-w)^3} + \frac{2}{(z-w)} T(w)$$ (1.12)

The first of these equations states that it must have conformal weight $\frac{3}{2}$. The form of the supercurrent is further restricted because we want it to be Lorentz invariant in $d$ dimensions. This means that it must have the form

$$T_F = \psi^\mu \partial X_\mu + T^{\text{int}}_F ,$$ (1.13)

where the second term is some expression of conformal weight $\frac{3}{2}$, constructed out of $3n$ bosons. Here $n$ is defined by $d = 10 - 2n$. The $3n$ bosons consist of 2n "compactified" bosons and $n$ bosons representing the "compactified" world-sheet fermions. Subtracting the space-time part of the supercurrent in (1.12), we see that $T^{\text{int}}_F$ has to satisfy it with "10" replaced by $10 - d$, and $T$ by $T^{\text{int}}$, the energy momentum tensor of $3n$ bosons.
The first equation is quite easy to satisfy. The general solution has the form

\[ T_F^{\text{int}} = \sum_i \tilde{B}(i) \cdot \partial \tilde{X} e^{i\tilde{r} \cdot \tilde{X}} + \sum_i A(i) e^{i\tilde{r} \cdot \tilde{X}}, \]

(1.14)

where \( i^2 = 1, i^2 = 3, \) and \( \tilde{r} \cdot \tilde{B} = 0. \) For chiral theories we will have to require that \( \tilde{B} = 0, \) for reasons that will become clear when we discuss the spectrum.

The second equation in (1.12) imposes more complicated conditions on \( A \) and \( \tilde{B}, \) and we do not know its general solution. However, as we will discuss in the next section, we do know many special solutions. First however we want to address the second point mentioned above, the action on \( T_F \) on the states in the theory.

We have identified two operators whose action should take physical states into physical states, namely \( L^i- \) and \( \mathcal{P}_{+1}. \) These operators consist usually of several terms, each of which contains an exponential of the form \( \exp(iw \cdot X), \) where \( w \) is a vector on the lattice. We will write such vectors as \( (w_L; u_R, v_R \mid q), \) where the four entries denote respectively the \( 16 + 2n \) dimensional left part, the \( 3n \) dimensional right internal part, the \( \frac{d}{2} \) dimensional components in the space-time lattice, and the ghost charge. Using this notation, we see that for a term in (1.14) one gets vectors \( w \) equal to

\[ w^{L^i} = (0; \tilde{r}, \tilde{V} \mid 0) \text{ or } (0; \tilde{r}, \tilde{V} \mid 0) \quad \text{for } L^i- \]
\[ w^{PC} = (0; \tilde{r}, 0 \mid -1) \text{ or } (0; \tilde{r}, 0 \mid -1) \quad \text{for } \mathcal{P}_{+1}, \]

where \( \tilde{V} \) denotes a vector weight of \( D_{d/2}. \) Adding such a vector to a lattice vector should give another lattice vector, since otherwise one would leave the physical sector. For the picture changing operator one should furthermore require that its action on all states is local. This implies that \( w \) must have integral dot product with all lattice vectors, so that it must lie on the dual of the lattice. Fortunately our lattices are self-dual, so that these requirements on \( w \) are compatible.

The requirements one gets from \( L^i- \) and \( \mathcal{P}_{+1} \) are also compatible with each other, because in covariant lattices the \( D_{d/2} \) components and the ghost charge belong to the same conjugacy class of \( D_{d/2,1}. \) What we conclude from this is that
for every vector $\vec{l}$ and $\vec{i}$ appearing in $T_{\vec{p}}^{\mu\nu}$ the lattice must contain vectors of the form $w = (0; \vec{l}, (v))$ and $(0; \vec{i}, (v))$, where $(v)$ denotes generically the vector conjugacy class of $D_{d/2,1}$. Of course the condition is the same in the even formulation, with $(v)$ denoting a conjugacy class of $D_{8-n}$.

There is yet another way to arrive at this conclusion, which we mention here because it uses more directly the requirement of supersymmetry, and can also be used to identify other world-sheet supersymmetries on the lattice. What these conditions are imposing on the theory is that the supersymmetry transformation not only respect the local properties of the fields, but also, on Riemann surfaces with non-trivial topology, their global properties, that is, the boundary conditions of all fields. In uncompactified fermionic strings this can always be achieved by choosing appropriate periodicities for the supersymmetry parameter $\epsilon$. If the supercurrent consists of more than one term, as in (1.14), this is however not always possible, since different terms may require different periodicities for $\epsilon$. In bosonic language these boundary conditions depend on the soliton sector, i.e. the lattice momentum $\vec{w}$ of the state on which the supersymmetry generator acts. States in this sector which are created by vertex operators of the form $\exp(i\vec{w}' \cdot \vec{X})$ pick up a phase $\exp(2\pi i \vec{w}' \cdot \vec{w})$ if one takes the string coordinate $\sigma$ to $\sigma + \pi$. When one considers one term of the supercurrent acting on it, then such a state is mapped into a state which picks up a different phase when moved around a closed loop. This phase change is $\exp(2\pi i \vec{p} \cdot \vec{w} + 2\pi i \phi(\vec{w}))$, where we have denoted the phase of $\epsilon$ in the sector $\vec{w}$ as $\phi(\vec{w})$. The vector $\vec{p}$ appearing here represents just one term in the supercurrent (i.e. it is one of the vectors $\vec{l}$ or $\vec{i}$ used above). If there were just one term, one can always adapt $\phi(\vec{w})$ to cancel the phase, so that supersymmetry respects all boundary conditions. If the supercurrent has a second term with a vector $\vec{p}'$, then this cancellation of the phase by $\phi(\vec{w})$ can work if and only if $\vec{p} - \vec{p}'$ has integer dot-product with any $\vec{w}$ on the lattice. But then it must lie on the lattice itself.

This argument tells us thus that the difference of any two vectors appearing in the supercurrent must lie on the lattice. This includes also the space-time part of the supercurrent. For that part the vector $\vec{p}$ belongs to the vector conjugacy class of $D_{d/2}$, so that we arrive at the exactly the constraints mentioned above.
For the space-time part of the supercurrent, these conditions imply simply that the roots of $D_{d/2}$ must be on the lattice. This is however always the case, and it explains why no extra world-sheet supersymmetry conditions are needed in ten dimensions.

1.6 Construction of Supercurrents

There are two simple ways of obtaining supercurrents that satisfy the second equation of (1.12). The first is to construct one out of free fermions, and bosonizing it. Such a free fermion supercurrent has the form

$$ T_F(z) = \frac{1}{6} f_{abc} \psi^a \psi^b \psi^c, \quad (1.15) $$

where the $\psi^i$'s are $6n$ fermions related to the $3n$ internal bosons by bosonization. If $f_{abc}$ are the structure constants of a semi-simple Lie algebra, then one can check that (1.12) is satisfied [22]. The generic choice for this Lie algebra (which has to be of dimension $6n$) is $SU(2)^{2n}$. In dimensions below five there are additional solutions, which are of some interest for type-II strings, but not for chiral heterotic strings. To obtain a supercurrent of the form (1.14) one simply bosonizes the $6n$ fermions into $3n$ bosons. There is some freedom in doing that, because on may associate the fermions in different ways with the $6n$ $SU(2)$ generators, but there is one especially simple way which yields only norm 3 vectors in the supercurrent. This is to take

$$ T_F = N \sum_{i=1}^{n} [\psi_i^3 \psi_i^2 \psi_i^3 + \psi_i^{-1} \psi_i^{-2} \psi_i^{-3}] $$

where $\psi^i$ and $\psi^{-i}$ denote the two real fermions one can construct out of a single boson. The normalization constant is easily determined from (1.12). The vectors $\vec{i}$ one gets from this supercurrent are, for $n = 1$,

$$ \vec{i} = (\pm 1, \pm 1, \pm 1), \quad (1.16) $$

For larger $n$ one just adds sets of 8 such vectors in orthogonal three-dimensional subspaces, so that altogether one has $8n$ vectors. This supercurrent was first used by Kawai et. al., and led to their “triplet constraint”.
Although one might think that all supercurrents that satisfy (1.12) can be written as a trilinear expression of free fermions, this is not the case. A counterexample, found in [10] and [23] has vectors \( \vec{t} \) of the form (for \( n = 1 \))

\[
\begin{align*}
\vec{t}_1 &= (\sqrt{3},0,0) \\
\vec{t}_2 &= (0,\sqrt{3},0) \\
\vec{t}_3 &= (0,0,\sqrt{3})
\end{align*}
\]

(1.17)

This pattern can be repeated \( n \) times if \( n > 1 \). By bosonizing the free fermionic supercurrent, one can only get vectors of the form \( (\pm1, \pm1, \pm1, 0, \ldots, 0) \), i.e., there must exist a basis in which three entries are \( \pm1 \), and the others zero. For one norm 3 vector one can always achieve that, but not necessarily simultaneously for several vectors. In particular it is not possible for (1.17). There is an analogous problem in the construction of Frenkel-Kac generators of \( E_8 \), where it is known that the \( D_8 \) spinor generators cannot be written as fermion bilinears, but require a "transcendental" expression [24].

The second method for generating supercurrents owes its inspiration to asymmetric orbifolds, and one automatically arrives at it if one attempts to represent asymmetric orbifolds by covariant lattices [7], [23], [12], [25]. In this case one starts by taking the ten-dimensional form also in the internal space, i.e.

\[
T^\text{int}_F = \sum_{i=1}^{2n} \psi^i \partial Y_i,
\]

where we denote the internal bosons as \( Y_i \). The fermions can be bosonized trivially. Although the rest of the supercurrent is already bosonic, it is of no use for us since in covariant lattice constructions such a supercurrent (with \( \vec{B} \neq 0 \) in (1.14)) cannot lead to chiral theories. We can try to re-express the factors \( \partial Y \) as sums of exponentials of the form \( \exp(i\vec{r} \cdot \vec{X}) \), where \( \vec{r}^2 = 2 \), i.e. by rebosonizing them. These expressions are constrained by the fact that they must satisfy the same operator products as the \( \partial Y \)'s. An equivalent way of saying this is that they must be the generators of a \( U(1)^{2n} \) algebra. One can find such subalgebras by rotating the \( \partial Y \)'s
into the root system of a Lie algebra. One way of finding such solutions is to use the Weyl rotations of the Lie algebra $G$ belonging to the lattice, and apply the conjugation that maps twists into shifts.

A Weyl twist $\sigma$ can always be realized as an element $\tilde{\sigma}$ of $G$, the group of which $G$ is the Lie algebra. In the Lie algebra one can define a new basis such that $\tilde{\sigma}$ lies in a maximal torus. This allows us to define a new set of bosons $\partial X^i(z)$ that are linear combinations of the $\partial Y^i$ and vertex operators of the form $e^{i\tilde{\sigma} \cdot \tilde{X}}$, in such a way that

$$\tilde{\sigma} = \exp(2\pi i \oint \frac{dz}{2\pi i z} \tilde{\sigma} \cdot \tilde{X}(z)).$$

The basis transformation that maps the $X^i$s to the $Y^i$s is generated by conjugation on the Lie algebra, and preserves all operator products. Clearly the inverse of this transformation of the Lie algebra yields an expression for the $\partial Y^i$ in terms of the $\partial X^i$ and vertex operators of the form $e^{i\tilde{\sigma} \cdot \tilde{X}}$. Manifestly from (1.18), $\tilde{\sigma}$ leaves $\partial X^i$ fixed and performs an orthogonal rotation on the $\partial Y^i$s.

For chiral theories, we need only consider elements of the Weyl group that have no eigenvalues equal to 1 when acting upon the Cartan subalgebra. Such elements will be called non-degenerate. For non-degenerate elements the expression for $\partial Y^i$ in terms of $X^i$ cannot contain any of the currents $\partial X^i$.

We choose a complex eigen-basis $\{\partial Y^A, \partial Y^\bar{A}\}$ ($A=1,2,3$) for the $\partial Y^i$, such that $\tilde{\sigma}$ has eigenvalue $e^{2\pi i \theta_A}$ on $\partial Y^A$. In this basis $\partial Y^\bar{A}$ may be written as

$$\partial Y^\bar{A} = \sum_{\tilde{\alpha}} a_{\tilde{\alpha}}^A e^{i\tilde{\alpha} \cdot \tilde{X}}$$

(1.19)

where $\tilde{\alpha} \cdot \tilde{\sigma} = \theta_A \mod 1$ when $a_{\tilde{\alpha}}^A \neq 0$.

The supercurrent, which is necessarily real, has the form

$$T_F(z) = \lambda^A \partial Y^A \bar{\lambda}^1 \partial Y^1$$

(1.20)

and trivially bosonizing the fermions yields a supercurrent of the form (1.14) with $\bar{B}(I) = 0$. By construction $T_F$ satisfies the appropriate operator products.
If one manages to construct such a bosonic supercurrent in some way it is not \textit{a priori} obvious that a covariant lattice exists that can accommodate it. In the present construction such a lattice can be found, however. One starts with a Narain lattice $\Gamma_{16+2n;2n}$, which has the Lie-algebra $G$ in a Frenkel-Kac realization on the right lattice. The \textit{covariant} lattice corresponding to this Narain lattice is $\Gamma_{16+2n;2n} \otimes D_{6,1}^{0,8}$, which can be mapped to the even lattice $\Gamma_{16+2n;2n} \otimes E_8$.

This lattice already allows the new supercurrent, but the theory is not chiral. This is due to the presence of roots extending $D_8$ to $D_8 \subset E_8$, which always destroy chirality. If we could find a shift vector that projects out all such roots while preserving the constraint vectors $(i, (v))$, then we have a chance of constructing a chiral theory. Such a vector is given to us by the foregoing construction, and consists of the vector $\vec{v}$ defined in (1.18), with additional components in the $E_8$ factor to represent the twisting of the world-sheet fermions which in orbifold constructions accompanies the twists of the bosons. Furthermore one may add components in the left-moving part of the lattice, so that the shift-vector is by no means unique.

Using this shift vector one may modify the Narain theory to get a genuine covariant lattice which does not have a separate $E_8$ factor. This proves that non-trivial lattices exists for any of these supercurrents. In fact previous experience suggests that a large number of covariant lattices exists for each supercurrent.

More details regarding this construction may be found in [12], where all supercurrents of this kind are enumerated.

1.7 The Spectrum: Generalities

To discuss the spectrum of covariant lattice theories we will use the even lattice formulation, and we begin by summarizing the previous sections. To specify a covariant lattice theory in $d = 10 - 2n$ dimensions one must provide

- A lattice $\Gamma_{16+2n;8+2n}$ with left- and right dimensions as indicated
- A supercurrent of the form (1.13), (1.14), that satisfies the $N = 1$ supersymmetry algebra.
The lattice must satisfy the following requirements.

1. It must be even and self-dual with respect to a Lorentzian metric of the form \( \text{diag}((-)^{16+2n};(+)^{8+2n}) \)

2. The last \( 8 - n \) components of any vector must belong to one of the four conjugacy classes \((0),(v),(s)\) or \((c)\) of the Lie algebra \( D_{8-n}^{\text{space-time}} \).

3. It must contain all vectors of the form \( \vec{w}_L = 0, \vec{w}_R = (\bar{l}, (v)) \) and \( (\bar{l}, (v)) \) for all \( \bar{l} \) and \( \bar{l}' \) appearing in the supercurrent. This condition applies obviously only to dimensions less than 10.

The physical states are obtained by making a decomposition of \( D_{8-n} \) to its subalgebra \( D_{4-n} \times D_4 \). One keeps only states of which the last four components are fundamental vector or spinor weights of \( D_4 \), and counts them with multiplicity one. One finds then that the following general rule holds for the content of the \( D_{8-n} \) conjugacy classes

\[
\begin{align*}
(0) & \rightarrow \text{ odd rank tensors} \\
(v) & \rightarrow \text{ even rank tensors} \\
(s) & \rightarrow \text{ spinors of chirality} + \\
(c) & \rightarrow \text{ spinors of chirality} -
\end{align*}
\]  

(1.21)

This holds only for the pure lattice states. Excitations by bosonic oscillators with a space-time index may of course change the spin and the chirality of states.

The last fact one has to know about the spectrum is the mass formula. This is extremely simple: It is just the mass formula of the compactified bosonic string:

\[
\begin{align*}
\frac{1}{8} m_L^2 &= \frac{1}{2} w_L^2 + \mathcal{N} - 1 \\
\frac{1}{8} m_R^2 &= \frac{1}{2} w_R^2 + \bar{\mathcal{N}} - 1 
\end{align*}
\]  

(1.22)

Here \((\vec{w}_L; \vec{w}_R)\) denotes a vector on \( \Gamma_{16+20;8+120} \). Notice that the entire vector contributes to the mass formula, including the \( D_4 \) components. Since every physical state necessarily is a fundamental vector or spinor of this \( D_4 \), the lowest possible value of \( \frac{1}{8} m_R^2 \) is \(-\frac{1}{2}\). It follows that if we are only interested in massless states and
tachyons, then we can ignore the right-moving oscillators, since they would increase
the mass by too much.

1.8 Roots and Local Symmetries

We define a root of the lorentzian lattice to be any vector of the form \( (r; 0) \) or
\((0, r) \), where \( r \) is some vector of norm 2. The presence of such vectors tell us exactly
all local symmetries of the theory, because all gauge bosons (including gravitons and
gravitino’s) originate from such vectors. The nature of the corresponding physical
state depends of course on the conjugacy class of the space-time lattice \( D_{8-n} \) to
which it belongs. We have the following possibilities:

Conjugacy class (0)

This includes first of all the roots of \( D_{8-n} \) itself, which are always present.
Combined with a left oscillator excitation they yield the graviton, the dilaton and
the antisymmetric tensor \( B_{\mu\nu} \). If the oscillator belongs to the Cartan-subalgebra
of the Frenkel-Kac group of the left lattice, one gets a gauge-boson. Since there
are always \( 16 + 2n \) such oscillators, one always gets a gauge-group of rank \( 16 + 2n \).
Furthermore any root of the left lattice belongs in this class. Although such a root
itself is not a physical state, one can combine it with a \( D_{8-n} \) root on the right to
get the remaining gauge bosons of the Frenkel-Kac gauge group of the left lattice.

If there are any roots belonging within \( D_{8-n} \) to conjugacy classes other than (0),
than we can conclude that \( D_{8-n} \) is regularly embedded in some larger Lie-algebra.
For \( d \geq 4 \) this can only be \( D_{k} \) or \( E_{6}, E_{7}, E_{8} \).

Conjugacy class (\( v \))

If a root belongs to (\( v \)) of \( D_{8-n} \) then it is part of some larger \( D_{k} \) root-system
on the right lattice. This can obviously only happen below ten dimensions. If such
roots are present, they imply the absence of chiral fermions, since the \( D_{8-n} \) spinors
come automatically in chiral pairs, a fact familiar from torus compactification of
chiral theories. The roots themselves are scalars on the right lattice, and can be
combined with a bosonic oscillator excitation from the left to obtain a space-time
vector. These vector bosons are gauge bosons of a gauge-group associated with the right lattice, which in fact is a (non-regular) subgroup of the Frenkel-Kac group of that lattice [10]. Such gauge groups are of no interest in heterotic strings, since their existence would not allow chirality, but in type-II strings one may still get chiral fermions from the left part of the lattice.

Conjugacy classes (s) or (c)

In this case $D_{8-n}$ extends to an exceptional group. Such roots can be excited on the right by a bosonic oscillator, or they may be combined with roots of the left lattice. Such space-time states are gravitino's or gaugino's, and this indicates that one has a theory with a certain number of space-time supersymmetries. There is a lot more that can be said about this.

1.9 Space-time Supersymmetry

It is not difficult to work out how many supersymmetries one gets in 4, 6, 8 and 10 dimensions, if the space-time factor $D_{8-n}$ extends to one of the exceptional groups. The result is:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$D_n$</th>
<th>$\subset$</th>
<th>$E_m$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$D_5$</td>
<td>$\subset$</td>
<td>$E_6$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$E_7$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$E_8$</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>$D_6$</td>
<td>$\subset$</td>
<td>$E_7$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$E_8$</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$D_7$</td>
<td>$\subset$</td>
<td>$E_8$</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>$D_8$</td>
<td>$\subset$</td>
<td>$E_8$</td>
<td>1</td>
</tr>
</tbody>
</table>

For even more intriguing results in two dimensions we refer to [5].

* This relation between space-time supersymmetry and exceptional groups is in fact completely general, and applies for instance also to Calabi-Yau compactified string theories. The reason is that space-time supersymmetry constrains the vertex operator algebra so that it can always be mapped to an exceptional algebra.
It seems that everything that's known about supersymmetry and supermultiplets can be found back in the representation theory of the exceptional groups. For example, consider $E_6 \supset D_5$ in four dimensions. The smallest representations of $E_6$ are the singlet $(1)$, the $(27)$ and the $(78)$, which is the adjoint. Their $D_5$ content is

$$(1) \rightarrow (1)$$

$$(27) \rightarrow (10) \oplus (16) \oplus (1)$$

$$(78) \rightarrow (45) \oplus (16) \oplus (\overline{16}) \oplus (1)$$

According to our rules, the $D_5$ singlets contain no physical states, while the $(10)$ yields a scalar and the $(16)$ a fermion. Thus the $(27)$ contains the scalar multiplet. In a similar way one can see that $(78)$ contains the vector multiplet. Note that to count the spinors we should consider only the $(16)$, because the $(\overline{16})$ is just a CPT conjugate.

In fact, one can push this further, as it is possible to relate the "unphysical" left-over singlet components to the auxiliary fields of $N = 1$ supergravity. Vertex operators for such auxiliary fields have recently been investigated [26]. It is conceivable that the above procedure works also for extended supergravities and/or higher dimensions, as well as for massive multiplets with arbitrary high spins, by using the representation theory of the appropriate exceptional group. More details may be found in [27].

We have not said much about massive states, but something needs to be said about them in supersymmetric theories. One would expect that they too belong to supermultiplets. This can in fact be proved by using the triality properties in a somewhat unexpected way. These properties are known to be related to supersymmetry in ten dimensions, and to clarify the relation between the Neveu-Schwarz-Ramond and the Green-Schwarz formalism. In those arguments, one uses triality rotations in the transverse Lorentz group. Clearly that is of no use below ten dimensions, since one has a different Lorentz group. Instead we will show that one can use the $D_4$ that in our formulation describes the ghosts and longitudinal modes for this purpose. This part of the lattice is dimension independent, so that what follows holds in arbitrary dimensions.
The $D_4$ factor plays a role in distinguishing fermions from bosons, because the former have as their $D_4$ components weights of the spinor representation ($8_s$), whereas the latter have weights of the vector representation ($8_v$). These vectors are related by triality rotations. It is a special property of the embedding of $D_4$ in the exceptional groups that triality becomes an inner automorphism of the exceptional group. This means in particular that for every ($8_v$) of $D_4$ contained in some representation of an exceptional group, there is also an ($8_s$) and ($8_c$). For the spectrum that statement translates into having a fermion for every boson, and hence into supersymmetry at all levels (at least as far as multiplicities are concerned).

1.10 Tachyons

Tachyons can not appear in the conjugacy class (0) of $D_{8-n}$, because the physical state projection rule ensures that the minimal mass of a physical state in that sector vanishes. So there are no vectorial tachyons. One might also expect absence of fermionic tachyons, but this is not as easy to see directly from the lattice. Below ten dimensions the spinor weights of $D_{8-n}$ have norm less than 2, so that they would be tachyonic unless there are components of the lattice vector in the right-moving internal dimensions.

World-sheet supersymmetry guarantees that this is indeed the case. If there were no internal components, the spinorial lattice vector would have half-integer inner product with the constraint vectors of world-sheet supersymmetry. This argument does not tell us the length of the internal components, although in individual cases it can be verified that they must at least have the length required to make the spinor massless. General arguments tell us that the minimal mass in the Ramond sector is always zero, and this principle should ensure that the above is true for all choices of the supercurrent.

This leaves tachyons in the conjugacy class ($v$) of $D_{8-n}$. They are always scalars, and they may indeed occur in non-supersymmetric theories. Because of the possibility of having internal components, they need not be of minimal mass (i.e. the mass of the tachyon of the Neveu-Schwarz model). There many examples of non-supersymmetric theories which are tachyonfree.
1.11 Massless fermions and scalars

There is no general principle to decide what the massless fermion and scalar spectrum for a given lattice is. One simply has to calculate the physical state content of all the conjugacy classes. Such states are of course given by lattice vectors of the form \((w_L; w_R)\) with \(w_L^2 = w_R^2 = 2\), and where \(w_R\) has components in the spinor or vector conjugacy class of \(D_{5-n}\).

The representation of these fields is always one with a weight of total norm 2, which may be composed of shorter weights of simple factors of the gauge group (this concept is less useful if the group has \(U(1)\) factors). If \(w_L\) is in fact a root, then we know that the lattice must also contain the vector \((0; w_R)\), which can be excited with a right-moving oscillator into either a vector-boson or a gravitino. Hence scalars in the adjoint representation can only exist in theories with an extra gauge-group \(G_R\) associated with the right lattice (such theories are never chiral), and fermions in the adjoint representation can only exist in locally supersymmetric theories.

1.12 Type-II strings

Type-II strings can be discussed in a way very similar to heterotic strings. The lattices one has to consider now are of the form \(\Gamma_{2n+5,1;2n+5,1}\), which must be self-dual with respect to a metric \((-)^{2n+5}(+)^{2n+5}(-)\). The last \(6 - n\) components of vectors on the left and right lattice must belong to conjugacy classes of \(D_{5-n,1}\), and the lattice must be odd in such a way that space-time fermions correspond to vectors of odd length. Any such lattice can be mapped to an even self-dual lattice \(\Gamma_{8+2n,8+2n}\) by using the lattice map discussed above in both sectors.

To satisfy the conditions for world-sheet supersymmetry one has to choose supercurrents in both the left and the right-moving sector, and require that the corresponding constraint vectors are on the lattice. Of course the supercurrents used in the left and the right sectors may be different, and if one wants to get chiral fermions, it is sufficient to get them from one sector, and use the other sector to generate the gauge group. (It is necessary to do it like this because, as explained above, in our class of theories chiral fermions and gauge symmetries in the same
sector are mutually exclusive). In four dimensions, one can by suitable choices of the left and right sector obtain \( N = 1, 2, 3, 4, 5, 6 \) and \( N = 8 \) supergravity [10].

Theories of this kind have been constructed in [28], [10], and [29]. The existence of large numbers of such theories briefly renewed hopes of getting the standard model from type-II theories. In the class we studied in [10] that turned out to be impossible. This was then proved under more general conditions in [29].

1.13 Lattice Construction

In ten dimensions the lattices that lead to consistent heterotic and type-II strings can be classified completely [9]. Because of ten-dimensional Lorentz-invariance, there is only one possible supercurrent, and its constraint vectors are automatically present, as explained above. The lattices we are looking for are even self-dual ones of the form \( \Gamma_{16,8} \) for heterotic strings, and of the form \( \Gamma_{8,8} \) for type-II strings, where all the eight-dimensional subspaces must be \( D_8 \) lattices. Classification of such lattices is easy, because they can be mapped to Euclidean lattices by simply changing the signature of the metric. This preserves even self-duality because one is dealing with \( D_8 \) factors. To classify ten-dimensional heterotic strings, one looks for regular embeddings of \( D_8 \) in the Frenkel-Kac groups belonging to the 24-dimensional even self-dual Niemeier lattices. To classify the type-II strings one does the same with the two 16-dimensional even self-dual lattices. In this way one obtains quite easily the known ten-dimensional heterotic strings with rank 16 gauge groups [16], [17], and the usual type-IIA and type-IIB theories, as well as the type-II theories found in [30].

Constructing lower-dimensional strings has not yet been done so systematically, because the number of possibilities is much larger. One way of finding examples, although not necessarily the most efficient one, is to start with some root lattice of a semisimple, simply laced group, plus the conjugacy classes needed for the constraint vectors. Then one adds conjugacy classes with integral inner products and even norm, until no new ones can be found. The resulting lattice is even self-dual and satisfies the world-sheet supersymmetry constraints. For a more detailed discussion, examples and other constructions see e.g. [5], [31], [13].
2. Partition Functions and the Cosmological Constant

In this section we summarize some results on calculations of the partition function and the cosmological constant for a subclass of the theories described above, namely for heterotic strings based on the "canonical" supercurrent (1.16). Our motivation for studying these partition function was to investigate the possibility of finding non-supersymmetric theories with vanishing one-loop cosmological constant, due to Atkin-Lehner symmetry or perhaps because of other reasons. The results of this study are inconclusive: although such partition functions do indeed exist, we have not succeeded in finding any lattices which realize them, nor can their existence be ruled out. We simply summarize the results on the partition functions here, as they might perhaps be useful for further work in this direction. A similar study for $Z_3$-orbifolds (with similar conclusions) has been presented recently in [32].

If the supercurrent is of the form (1.16), then the right lattice is necessarily of the form $D_{8-n} \times (D_1)^{3n}$, where $D_1$ has a root lattice consisting of the even integers, and a weight lattice consisting of all integers and half-integers. In general, if we write a lattice in this way, we mean that the root lattice is always present, while in addition some sublattice of the weight lattices (i.e. some set of conjugacy classes) appears. This does not rule out the possibility that there are additional roots, formed out of combinations of $D_{8-n}$ and $D_1$ conjugacy classes.

We will assume that the left lattice can also be written as a product of $D_1$ factors. This is not as restrictive as it might appear to be, since many larger lattices can be decomposed in this way. For example, all the Niemeier lattices can be broken into $D_1$ factors (see next section), and we have some reasons to believe that the same is true for the class of lattices which admits (1.16).

The one-loop partition function for such a theory can be written as a sum of products of $\theta$-functions with (half)-integer characteristics, usually denoted as $\theta_2$, $\theta_3$ and $\theta_4$. To write down the partition function, one must sum the contributions of each of the conjugacy-classes of the lattice. Each such conjugacy class is a combination of conjugacy classes of the $D_1$ factors with a $D_{8-n}$ conjugacy class, and the partition function is a product of the contributions of each simple factor.
Within each $D_1$ factor, the conjugacy class (0) contributes $\frac{1}{2}(\theta_3 + \theta_4)$, (v) contributes $\frac{1}{2}(\theta_3 - \theta_4)$, and (s) and (c) yield $\frac{1}{2}\theta_2$ (for our purposes $\theta_1$ has a vanishing first argument, and does not contribute). The contribution of the $D_{8-n}$ factor is slightly different due to the physical state projection rule and the spin-statistics factor: the effect is that the contribution of (0) and (v) interchanges, and that the spinor conjugacy classes have an extra $-$ sign. In $d = 10 - 2n$ dimensions the theta functions from the space-time factor of the lattice have furthermore a power $4 - n$.

The expression for the one-loop graph without external lines has thus the form

$$A_{\text{1-loop}} = \int \frac{d\tau}{(Im\tau)^2} \frac{1}{\eta^{12}\eta^{24}} \sum C_{r_2, r_3, r_4} \theta^{n_2} \theta^{n_3} \theta^{n_4} \theta_2 \theta_3 \theta_4,$$

where $\theta_i$ and $\bar{\theta}_i$ denote $\theta_i(0|\tau)$ and $\theta_i(0|\bar{\tau})$, and where the coefficients $C$ are determined by the procedure sketched above. We use bar's to denote left-movers here (a change of convention in comparison with earlier papers). The sum is over all combinations of integers $n_i$ and $\bar{n}_i$ subject to the constraints $n_2 + n_3 + n_4 = 2n + 4$ and $\bar{n}_2 + \bar{n}_3 + \bar{n}_4 = 16 + 2n$.

Although there are many terms that can appear in this expression, it can be greatly simplified by making use of modular invariance, of the "triality" or "supersymmetry" identity $\theta_3^4 + \theta_4^4 = 0$, and by using the fact that the lattice is constrained by world-sheet supersymmetry.

One can easily see that the latter implies that the number of spinor entries on the right lattice must be even, so that $n_2$ is even. Because the lattice is lorentzian even, $\bar{n}_2$ must then be even as well. Since modular transformations permute all the theta-functions, it follows that all $n_i$ and $\bar{n}_i$ must be even.

Modular invariance can be used to write the partition function in terms of modular orbits, which we define as

$$\mathcal{M}(n_2, n_3, n_4; \bar{n}_2, \bar{n}_3, \bar{n}_4) = \frac{1}{\eta^{4+2n} \bar{\eta}^{16+2n}} \theta^{n_2} \theta^{n_3} \theta^{n_4} \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4 + 5 \text{ modular transforms},$$

that is, one adds to the first terms with argument $\tau$ the same terms with arguments $\tau + 1, 1 - \frac{1}{\tau}, \frac{\tau}{\tau+1}, \frac{1}{1-\tau}$, and $-\frac{1}{\tau}$. This sequence is generated by successive applications
of the generators $\tau + 1$ and $-\frac{1}{\tau}$ of the modular group at genus one. If one more step in the sequence gives us back the original term without a phase-change (which is true in this case), then the result is modular invariant.

Since these six terms yield the six permutations of the three theta functions (with some phases), we can classify the different functions $M$ up to simultaneous permutations of the barred and unbarred indices. Using the theta-function identity we can then reduce this set even more, namely to 4,6,8 and 11 functions in 10,8,6 and 4 dimensions respectively.

In 10 dimensions the right lattice is always a $D_8$ factor, and by arguments similar to the ones used above we can then conclude that the integers $n_i, \bar{n}_i$ must in fact be multiples of four rather than two. This reduces the number of orbits to just two, for which one may take $M(4,0,0,8,8,0)$ and $M(4,0,0,16,0,0)$. The second of these has a tachyon pole $-128(q\bar{q})^{-1}$ (where $q = e^{i\pi \tau}$), whereas the first is tachyon free. The first must thus be the one appearing in the partition function of the $O(16) \times O(16)$ string, whereas all ten-dimensional tachyonic strings have partition functions which are linear combinations of the two.

In the other dimensions one can have two kinds of tachyons, with poles $(q\bar{q})^{-1}$ and $(q\bar{q})^{-(1/2)}$ in the partition function. Hence there are 4,6 and 9 independent tachyon-free linear combinations of partition functions in 8,6 and 4 dimensions respectively. In general, their integrals over the fundamental domain are unrelated real numbers, and one would not expect that they can cancel for simple integer linear combinations. This expectation was proved to be false in [33], where it was pointed out that some integrals may vanish due to Atkin-Lehner symmetry. This symmetry is not manifest in the partition function, and becomes evident only if one writes the $\tau$-integral as an integral of a function which is invariant under a subgroup of the modular group, integrated over an extended modular domain. This extended domain must be chosen in such a way that the integral remains the same.

In such extended domains one has transformations in addition to those in the modular group, which take the extended domain into itself. The simplest example of
such a transformation is $\tau \to -\frac{1}{2}\tau$, and this is in fact the only one which we will need. Given such an Atkin-Lehner transformation one can construct vanishing integrals by simply taking any (not modular invariant) partition function, subtracting its Atkin-Lehner transform, and summing the difference over a modular orbit in the way described above. It is not always possible to re-express the Atkin-Lehner transform in terms of $\theta$ and $\eta$ functions of argument $\tau$ rather than $2\tau$ (the transformations for doing this may be found in [33]). Furthermore the result may not have only even powers of $\theta$ (none of the examples in [33] does). Finally, the resulting partition function may be tachyonic, or it may vanish. In all these cases the result is of no interest to us. Using only the transformation $\tau \to -\frac{1}{2}\tau$ and eliminating all such cases, we are finally left with 1 Atkin-Lehner orbit in six dimensions, and 3 in four dimensions. Using other AL-transformations did not yield anything new. To check whether anything is missed in this analysis, we have calculated the integrals over the basic modular orbits (of course combined with the necessary $Im\tau$ and $\eta$ factors as in (2.1)), and checked for cancellations among integral linear combinations. Within the numerical accuracy, we did not find any (for coefficients smaller than 100).

The results for 8,6 and 4 dimensions are listed in the following tables. The last two entries give respectively the coefficients of the the two tachyon poles $(q\bar{q})^{-1}$ and $(q\bar{q})^{-1/2}$.

Eight Dimensions

<table>
<thead>
<tr>
<th>label</th>
<th>$M$ arguments</th>
<th>Leading pole</th>
<th>Subleading pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2,2,2,14,2,2)</td>
<td>0</td>
<td>-1536</td>
</tr>
<tr>
<td>2</td>
<td>(4,2,0,0,18,0)</td>
<td>72</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(4,2,0,4,14,0)</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>(4,2,0,8,10,0)</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(4,2,0,12,6,0)</td>
<td>-24</td>
<td>-512</td>
</tr>
<tr>
<td>6</td>
<td>(4,2,0,16,2,0)</td>
<td>-56</td>
<td>-1024</td>
</tr>
</tbody>
</table>

There are no Atkin-Lehner combinations.
Six Dimensions

<table>
<thead>
<tr>
<th>label</th>
<th>$\mathcal{M}$ arguments</th>
<th>Leading pole</th>
<th>Subleading pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4,2,2,0,18,2)</td>
<td>0</td>
<td>1152</td>
</tr>
<tr>
<td>2</td>
<td>(4,2,2,8,10,2)</td>
<td>0</td>
<td>128</td>
</tr>
<tr>
<td>3</td>
<td>(4,2,2,16,2,2)</td>
<td>0</td>
<td>-1792</td>
</tr>
<tr>
<td>4</td>
<td>(4,4,0,12,8,0)</td>
<td>-16</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(4,4,0,16,4,0)</td>
<td>-48</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>(4,4,0,20,0,0)</td>
<td>-80</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>(6,2,0,14,6,0)</td>
<td>-32</td>
<td>-512</td>
</tr>
<tr>
<td>8</td>
<td>(6,2,0,18,2,0)</td>
<td>-64</td>
<td>-1152</td>
</tr>
</tbody>
</table>

The Atkin-Lehner combination is $2(1) - 4(2) + (3) - 3(4) + (5)$.

Four Dimensions

<table>
<thead>
<tr>
<th>label</th>
<th>$\mathcal{M}$ arguments</th>
<th>Leading pole</th>
<th>Subleading pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4,4,2,12,8,2)</td>
<td>0</td>
<td>-256</td>
</tr>
<tr>
<td>2</td>
<td>(4,4,2,16,4,2)</td>
<td>0</td>
<td>-768</td>
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<td>3</td>
<td>(4,4,2,20,0,2)</td>
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<td>-1280</td>
</tr>
<tr>
<td>4</td>
<td>(6,2,2,10,10,2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(6,2,2,18,2,2)</td>
<td>0</td>
<td>-2048</td>
</tr>
<tr>
<td>6</td>
<td>(6,4,0,2,20,0)</td>
<td>72</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>(6,4,0,6,16,0)</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>(6,4,0,10,12,0)</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>(6,4,0,14,8,0)</td>
<td>-24</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>(6,4,0,18,4,0)</td>
<td>-56</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>(6,4,0,22,0,0)</td>
<td>-88</td>
<td>0</td>
</tr>
</tbody>
</table>

The Atkin-Lehner combinations are

\begin{align*}
(A) & \quad 10(1) - 10(2) + 4(3) - 13(8) - 2(9) - (10) \\
(B) & \quad -7(1) + 9(2) - 4(3) + (8) - 2(9) + (10) \\
(C) & \quad -3(1) + (2) + (7) - 2(8) + (9)
\end{align*}

The third partition function has a rather remarkable property: it is purely imagi-
nary, and hence it contains no terms with equal powers of $q$ and $\bar{q}$. If one could find a string theory with this partition function it would have at any level an equal number of physical (i.e. satisfying left-right mass equality) bosons and fermions, but it would not have supersymmetry in the “unphysical”, unmatched states. Clearly, this theory does not have local supersymmetry, since that would imply that the partition function vanishes identically (the arguments in section 1.9 are valid for states with unequal left and right mass), but in a sense it would have global supersymmetry.

Despite its potential existence, we have not found any example of a four-dimensional non-supersymmetric theory with vanishing cosmological constant. The cosmological constant is in fact easy to calculate using the above. One has to read off the lowest states in the spectrum from the lattice, and match the leading behavior of the partition function with the basic modular functions listed above. If one then calculates the contribution of each basic modular function to the one-loop path-integral numerically, the calculation of the cosmological constant is reduced to an algebraic problem. We have done this for many examples, and found that the cosmological constant appears with both signs. A slightly worrisome pattern is the fact that the coefficient of modular orbit (11) in four dimensions turned out to be positive in all cases, whereas it should vanish for AL-orbits. If it could be proved that this is always the case, then one would have proved a no-go theorem for Atkin-Lehner symmetry in this class of models.

3. THE LEECH LATTICE

The Leech lattice [34] is somewhat special among the 24 Niemeier [35] lattices since it has no vectors of norm 2. Nevertheless it is not as different from the other lattices as one might think. The other lattices can be written as a product of Lie-algebra lattices with a list of conjugacy classes (see e.g [36]). This looks manifestly impossible for the Leech lattice, and indeed no such explicit representation was given in [36]. However, only a slight generalization of the definition of a Lie-algebra lattice is needed to make this possible. One simply has to allow factors $D_1$, which is the obvious extrapolation of $D_n$ to $n = 1$. 
In some examples of four-dimensional strings we constructed in [5] we made use of Niemeier lattices written in terms of $D_n$ factors. If one allows $D_1$ factors this turned out to be possible even if they contained $A_n$ or $E_n$ factors, and the result could be derived directly from the tables in [36]. This led to the speculation that it should be possible to write even the Leech lattice in terms of 24 $D_1$ factors. To prove this one has to find the list of conjugacy classes that defines the lattice, which because of self-duality contains $2^{24}$ classes. After some experimentation it was indeed possible to determine this list, and we give here its generators, from which the entire list can be constructed by addition.

$$(c, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, s, 0, v)$$

$$(v, 0, 0, v, 0, 0, c, s, c, s, 0, 0, v, 0, 0, v, 0, c, s, s, v, 0)$$

$$(v, 0, s, 0, v, 0, 0, 0, s, c, s, s, s, c, s, s, c, s, s, s, s, s, v, 0)$$

$$(0, v, v, 0, v, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, v, 0, c, s, c, s)$$

$$(0, v, v, 0, s, 0, s, 0, 0, 0, s, s, c, s, s, s, c, s, s, s, s, s, s)$$

$$(c, c, v, 0, v, 0, 0, 0, c, s, c, s, 0, 0, v, 0, 0, v, 0, c, s)$$

$$(v, 0, c, s, v, 0, s, 0, 0, 0, s, s, s, s, c, s, s, s, c, s, s, s)$$

$$(c, c, s, s, v, 0, v, 0, 0, 0, c, s, c, s, 0, v, 0, 0, v, 0)$$

$$(0, v, c, s, c, s, v, 0, s, 0, 0, 0, s, s, s, s, c, s, s, s, c, s)$$

$$(v, 0, s, s, s, s, v, 0, s, 0, 0, 0, c, s, c, s, 0, v, 0, 0, 0)$$

$$(c, c, v, 0, c, s, c, s, v, 0, s, 0, 0, 0, c, s, s, c, s, s, s)$$

$$(s, s, c, s, s, s, s, s, v, 0, v, 0, 0, 0, c, c, s, c, s, 0, v, 0)$$

It is not important how this list was obtained, as long as it can be shown that it does generate the Leech lattice. To show this, one can check that the 12 generators have even norm and integer inner products, that they are independent (i.e. that one gets $2^{24}$ different conjugacy classes by adding these vectors in all possible ways), and that none of the $2^{24}$ classes contains vectors of norm 2. As a further (though unnecessary) check we have verified that the number of norm 4 vectors is also correct.
This is probably only of academic interest, but it does remove any doubts, expressed by some, about the possibility of fermionizing a bosonic string on the Leech lattice. Lattices written in terms of $D_n$ factors can always be represented by free fermions with periodic and antiperiodic boundary conditions and a combination of $GSO$ projections.

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