1. Introduction.

Recently it has been shown \cite{1,2,3,4} that the metric of the moduli space of superstring compactified on Calabi-Yau vacua \cite{5} has a restricted Kahler structure as implied by $N = 2$ space-time supersymmetry \cite{6,7} when these manifolds are thought of as vacua of type II superstrings. \cite{8,9} These restrictions have also been recently derived \cite{2} for superstrings compactified on arbitrary internal $(2,2)$ superconformal field theories \cite{10} by exploiting the Ward identities of $N = 2$ world-sheet supersymmetry on the four-point function of the moduli fields which is related to the curvature tensor of the moduli space.

$N = 2$ space-time supersymmetry requires the moduli space, related to deformations of the Kahler class $(1,1)$ harmonic form and of the complex structure $(2,1)$ form to be a product space \cite{3,1,9}

$$N = H_A \times H_B$$

with restricted Kahler potentials of the form \cite{1}

$$K_{A(B)} = - \log V_{A(B)} \quad (A = (1,1) \text{ moduli})$$

$$B = (2,1) \text{ moduli}$$

with $V_{A(B)}$ given, in a certain choice of coordinates for the moduli, by \cite{6,7}

$$V_{A(B)} = - e^{- f_A(B) + f_B(A)} - \frac{1}{2} \left[ \frac{\partial^2 A(B)}{\partial \phi_A(B) \partial \phi_B(B)} - \frac{\partial^2 A(B)}{\partial \phi_B(B) \partial \phi_A(B)} \right]$$

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In this paper we will (somewhat improperly) use the word Calabi-Yau and their moduli space for generic $(2,2)$ internal superconformal field theories.
\( f_A(B), \phi_A(B) \) are two holomorphic functions of the complex moduli fields \( \mathbf{A}(B) \) which determine all low-energy couplings of heterotic\(^\dagger\) strings as well as type II strings\(^\dagger\) compactified on an arbitrary (2,2) system.

In the field theory limit the functions \( \gamma_A(B) \) are given by\(^\dagger,3,4\)

\[
\gamma_A = V \int J \wedge J \wedge J, \quad \gamma_B = -i \int \mathbf{A} \wedge \mathbf{A},
\]

where \( V \) is the volume of the Calabi-Yau manifold, \( J \) is the Kahler form as a function of the (1,1) moduli and \( \mathbf{A} \) is the holomorphic (3,0) form as a function of the (2,1) moduli.

In type II theories, due to the occurrence of extra massless scalar fields (Ramond-Ramond scalars), the Kahler spaces of the moduli are enlarged to quaternionic spaces,\(^\dagger,4,7\) as required by \( N = 2 \) space-time supersymmetry.\(^6,7,13\)

This phenomenon occurs when the moduli fields are members of \( N = 2 \) hypermultiplets, is the case for the (2,1) moduli in type IIA strings and for the (1,1) moduli in type IIB strings.

If one also includes in the sector of massless excitations the dilaton and the space-time axion (four-dimensional antisymmetric tensor), the complete manifolds are\(^3,1\)

\[
\mathbf{N} = \frac{SU(1,1)}{U(1)} \times \mathbf{N}_A \times \mathbf{N}_B
\]

in heterotic strings,

\[
\mathbf{N}' = \mathbf{N}_A \times \mathbf{N}_B
\]

in type IIA strings,

\[
\mathbf{N}'' = Q_A \times \mathbf{N}_B
\]

in type IIB strings, where \( Q_A \) and \( \mathbf{N}_B \) are restricted Kahler manifolds of complex dimension \( (1,1) \), \( (2,1) \) respectively and \( Q_A \), \( \mathbf{N}_A \) are quaternionic manifolds of (real) dimension \( 4(N+1)-4 \) which contain, as submanifolds, the Kahler manifolds \( \frac{SU(1,1)}{U(1)} \times \mathbf{N}_B \) and \( SU(1,1) \times \mathbf{N}_A \) respectively. The \( SU(1,1) \) part refers to the dilaton, axion sector. The relation between these manifolds are called \( c \) and \( s \)-map in Ref. 1. These quaternionic manifolds were called dual quaternionic manifolds in Ref. 1 where it was pointed out that their metric should be completely determined in terms of the same holomorphic functions \( f_A(B) \) which determine their Kahler submanifolds \( \mathbf{N}_A(B) \) (s-map). Particular cases of such quaternionic manifolds which correspond to symmetric or homogeneous Kahler manifolds have been discussed in the literature.\(^14,15\)

Although the existence of such manifolds was established, no explicit construction of these spaces has been carried out yet. In Ref. 1 the simpler problem of determining the \( s \)-map for rigid supersymmetry was solved. In that case the dual hyper-Kahler manifolds of dimension \( 4n \), from complex (rigid) Kahler manifolds of dimension \( n \) were obtained.

In the present paper we solve the problem of the \( s \)-map in local supersymmetry. The result is not only of physical but also of mathematical interest in view of the fact that it allows the construction of continuous families of quaternionic manifolds, allowed by \( N = 2 \) supergravity (Einstein spaces of negative curvature with a specific value), whose geometry in a certain coordinate system, depends entirely on the very same holomorphic function of the restricted Kahler manifolds of the moduli space of (2,2) superconformal theories.\(^1-4\)
We anticipate some of the properties shared by all these dual manifolds. Let us call the preferred coordinate systems for these manifolds \((E^8, C_i, S)\) where \(E^8, (a = 1, \ldots, n)\) are the original coordinates of the restricted Kahler manifold, (moduli fields) \(C_i, (i = 1, \ldots, n+1)\) are the Ramond-Ramond scalars and \(S\) is a complex field related to the dilaton and the (space-time) axion.

(a) At each point of the moduli space \(E^8 = \mathbb{R}^8(0, \mathbb{R}^8 = 0)\), the Ramond-Ramond scalar parametrizes a \(\text{SU}(1, n+1)\) manifold.

(b) At each point \((\text{Re} C_i = 0, \text{Im} C_i = \text{Im} C_i)\) \(E^8(0, \mathbb{R}^8 = 0)\) the \((E^8, S)\) fields parametrize a \(\text{SU}(1, n+1)\) manifold when \(n\) is the original Kahler manifold. This is the Kahler manifold of the heterotic string when the charged matter fields are set to zero.

(c) The quaternionic manifold \(Q_{n+1}(n+1)\) has at least \(2n+4\) isometries acting on all coordinates but the \(E^8\) coordinates.

(d) The dual quaternionic manifolds are Einstein spaces with negative curvature \(R = -8(n+3)(n+1)\).

(e) Those moduli which have vanishing Yukawa couplings\(^\text{16}\) together with their Ramond \(N = 2\) partners from a Kahler quaternionic submanifold \(\text{SU}(2, n) / \text{SU}(2) \times \text{SU}(n) \times \text{U}(1)\) of the original (non-Kahler) quaternionic manifold.

2. Properties of Dual Quaternionic Spaces.

Let us anticipate a particular result stated in Ref. 1. In the case of symmetric quaternionic spaces, all of them are 3-maps of symmetric Kahler spaces with the exception of the \(Sp(n) / Sp(1) \times Sp(n)\) series. All symmetric\(^\text{15}\) restricted (or homogeneous)\(^\text{15-21}\) Kahler manifolds correspond to a holomorphic function \(f\) which is a polynomial of degree two or degree three in the \(z\).\(^\text{15}\) These particular polynomials lead to a vanishing (for quadratic functions) or constant (for cubic functions) Yukawa coupling\(^\text{16}\) in heterotic strings.\(^\text{1,2,9}\) In particular, vanishing Yukawa coupling means that the \(f\) function is quadratic and in that case the restricted Kahler manifold is always \(\text{SU}(1, n) / \text{SU}(1) \times \text{SU}(n)\) and the dual quaternionic manifold \(\text{SU}(2, n+1) / \text{SU}(2) \times \text{SU}(n+1) \times \text{U}(1)\) which is also Kahler.\(^\text{1}\) These results have been confirmed in special cases in Refs. 17, 1, and 7, where the moduli metric was computed for orbifolds and for Calabi-Yau spaces obtained by tensoring several copies of the \(N = 2\) minimal series.\(^\text{2}\)

Now let us consider the general case in which, for arbitrary choice of the holomorphic function \(f(z)\) the restricted Kahler manifold is not symmetric nor homogeneous. This is of course the case of interest for generic Calabi-Yau spaces.

The metric of the dual quaternionic manifold is derived\(^\text{1,16}\) by performing a dimensional reduction from \(D = 4\) to \(D = 3\) dimensions of \(N = 2\) supergravity coupled to \(n\)-vector multiplets with holomorphic function \(f(z)\) \((n = 1, \ldots, n)\).

The \(N = 2\) Lagrangian for vector multiplets is\(^\text{6,7}\) (bosonic part)
\[
\mathcal{L} = -\frac{1}{2} R - \frac{i g_4}{2} \text{tr} F \wedge \star F + \frac{1}{4} \text{tr} \ast F \wedge F + \frac{1}{4} \text{tr} F \wedge F + \frac{1}{4} \text{tr} \ast F \wedge F + \frac{1}{4} \text{tr} \ast F \wedge \ast F
\] (2.1)
with
\[ h_{ij} = \frac{1}{6} \left( F_{ij} + F_{ij}^* \right) \]
\[ R_{ij} = \frac{1}{4} \left( F_{ij} + F_{ij}^* \right) \]
\[ \mathbf{X} = -\ln 2 \sqrt{h} = -\ln \left[ 1 + \frac{1}{2} \left( h_{\alpha \beta} h_{\gamma \delta} (F_{\alpha \beta} F_{\gamma \delta}^*) \right) \right]. \]

\[ F(x) = \lambda^2 F(x) \text{ and } \mathbf{F}(x) = x^{\alpha \beta} \mathbf{F}(x). \]

Dimensional reduction from D = 4 (N = 2) to D = 3 (N = 4) is obtained using a triangular gauge for the vierbein
\[ e^\mu_\nu = \begin{pmatrix} e^{\mu_1}_1 & 0 \\ e^{\mu_2}_1 & e^{\mu_2}_2 \\ e^{\mu_3}_1 & e^{\mu_3}_2 \end{pmatrix}, \quad \nu = 1, \ldots, 4; \quad \mu = 1, \ldots, 3 \]
and for four-vectors we have
\[ e^\mu_\nu = (\pi_1, e^\mu_2, \zeta^1, \zeta^4), \quad (\zeta^1 = \pi_4^1). \]

The Lagrangian (2.1) reduced to three dimensions, after a Weyl rescaling
\[ e^{\mu_1}_1 = \frac{1}{2} \phi - \frac{1}{4\phi} (\partial_\mu \phi)^2 + \frac{1}{4} \phi^2 \psi^2, \]
the Lagrangian describing the scalar manifold is
\[ e^{-1}X = -\frac{1}{2} \pi_1 \pi_1^* \phi^2 - \phi_2 \frac{1}{2} \frac{1}{2} (C\tilde{C})_1^1 (\pi^{-1})^1_1 (C\tilde{C})_1^1. \]
\[ x \left[ \frac{1}{2} \partial_\mu S + (C \bar{C}) R^{-1} \partial_\mu C \right] - \frac{1}{4} \left( C \bar{C} \right) R^{-1} \partial_\mu C \bar{C} R^{-1} \left( C \bar{C} \right) \right] \]

\[ + \left( \frac{1}{2} \partial_\mu S + (C \bar{C}) R^{-1} \left( C \bar{C} \right) \right) \left[ \frac{1}{2} \partial_\mu S - \frac{1}{2} \partial_\mu \bar{S} R^{-1} \left( C \bar{C} \right) \right] \]

\[ \times R^{-1} \left[ \partial_\mu \bar{C} - \frac{1}{2} \partial_\mu \bar{S} R^{-1} \left( C \bar{C} \right) \right] . \]  

(2.12)

Positivity of the kinetic energy (in Eq. (2.10)) required \( f_{AD} \) and \(-h_{ij}\) to be positive definite metrics.

Eq. (2.12) defines a manifold for the \( (n+1) \) complex scalar fields \( S, S^a \), \( C \), that according to Ref. 1, should be the dual quaternionic manifold \( Q \) obtained by the \( s \) map from the restricted Kahler manifold \( K \) (specified by the holomorphic function \( f(x) \)). More specifically, as mentioned in the Introduction, the Lagrangian given by Eq. (2.12) (in \( D = 4 \) dimensions) describes the (low-energy) interactions of moduli fields \( (S^a) \), dilaton and axion \( S \) and the Ramond scalars \( C \) present in type II superstrings. For a generic compactification on a \( (2,2) \) system the \( S^a \) moduli refer to the moduli in Calabi-Yau manifolds) to deformation of the Kahler class in type IIB strings and to deformation of the complex structure in type IIA strings.

The dual quaternionic manifolds have therefore (quaternionic) dimensions \( h_{(2,1)} \) in type IIA and \( h_{(1,1)} \) in type IIB, where \( h_{(1,1)} \) and \( h_{(2,2)} \) are the Hodge numbers of the manifold.

It is our aim to prove that the metric given by Eq. (2.12) is quaternionic and discuss its properties.

Let us first discuss properties (a), (b), (c), and (e). An alternative form for the metric given by Eq. (2.12) is

\[ e^{2K} = -k_{AB} \partial_\mu S^A \partial_\mu S^B - k_{CD} \partial_\mu C^D \partial_\mu C^D - k_{CE} \partial_\mu C^E \partial_\mu C^E - k_{C} \partial_\mu C \partial_\mu C \]  

(2.13)

where

\[ k_{CD} = \partial_\mu S^C \partial_\mu \bar{S}^D - \frac{1}{2} \partial_\mu \bar{S} R^{-1} \left( C \bar{C} \right) \]

\[ k_{CE} = \partial_\mu S^C \partial_\mu \bar{C} - \frac{1}{2} \partial_\mu \bar{C} R^{-1} \left( C \bar{C} \right) \]  

and

\[ k = -\ln \left( S + \bar{S} + \frac{1}{2} \left( C \bar{C} \right) R^{-1} \left( C \bar{C} \right) \right) . \]  

(2.14)

The above equations show that for fixed \( C \) \( \left( \partial_\mu C = 0 \right) \) and \( \bar{C} = 0 \) the manifold \( Q \) contains the Kahler submanifold \( S^U / U(1) \) with coordinates \( (S, S^a) \), while for fixed \( S \) it contains the Kahler submanifold \( \bar{S}/U(1) \times SU(n+2) / U(1) \times SU(n+2) \) with coordinates \( (S, C) \). The first manifold is the Kahler manifold for heterotic strings which contains the dilaton, the axion and the moduli fields, the second manifold is the manifold of Ramond-Ramond scalars for fixed values of the moduli.

We can also remark that if the matrix \( F(S, C) \) is holomorphic, i.e., does not depend on \( S \) then the manifold \( Q \) is Kahler with Kahler potential \( K+S+K+S \). In view of Eq. (2.2) this is the case if the function \( f(S) \) is a quadratic polynomial and the Kahler metric proportional to \( R_{AB} \), \( R_{CD} \) and their hermitian conjugates.

\*The standard metric for the manifold is best seen by making the following (holomorphic for fixed \( S \) field redefinition: \( S + \bar{S} + \frac{1}{2} \left( C \bar{C} \right) R^{-1} \left( C \bar{C} \right) \).

\*Note that the covariant derivative in Eq. (2.13) just reproduces the terms in the Kahler metric proportional to \( R_{AB} \), \( R_{CD} \) and their hermitian conjugates.
This corresponds to the statement (n) in the Introduction.

Let us now consider another important property of the $Q$ manifold, i.e. its isometries. We may note from Eq. (2.10) that among the $4(n+1)$ scalar fields only $2n+1$ may appear with non-polynomial interactions. These are the true moduli $(\xi^a, \phi, S)$ which correspond to the deformation of the manifolds and the dilaton. These are actually at least $2n+3$ isometries associated with the axion and the Ramond fields

$$ S + S + 42 = 2C \gamma - \gamma \gamma $$

$$ C = C + 4 \beta + \gamma, \tag{2.15} $$

where $\alpha, \beta, \gamma$ are $2n+3$ real parameters. An additional isometry is the scale-transformation $S \rightarrow S, C \rightarrow C^{-1/2}$. Therefore the $Q$ manifold has at least $2n+4$ isometries.

We may compare these results with the simpler case of rigid supersymmetry described in Appendix B of Ref. 1. In rigid supersymmetry, the $N = 2$ gravity sector (graviton and graviphoton) was missing (this sector contains 4 degrees of freedom) so we had a correspondence between a (rigid) restricted Kahler manifold of complex dimension $n$ with a hyper-Kahler manifold of complex dimension $n$ and a hyper-Kahler manifold of quaternionic dimension $n$. Since a hyper-Kahler manifold can be regarded as a limiting case (for vanishing $Sp(1)$ connection) of a quaternionic manifold (as in local supersymmetry) the Einstein space structure of the latter is replaced by the Ricci flatness of the former.\textsuperscript{15,1}

3. The Geometry of the $Q$-Manifolds

In this section we will prove that the manifold defined by Eq. (2.10) is indeed a quaternionic manifold.

Let us recall that for quaternionic (4d-dimensional) manifolds\textsuperscript{13,19-21} there are three locally defined (1,1) tensors $(\omega^a, \mathcal{B}, \eta)$ which satisfy the quaternionic algebra

$$ \omega^a \times \omega^b = -\eta_{a b} + \epsilon^{a b c} \omega^c. \tag{3.1} $$

Moreover, the three two-forms

$$ \omega^a = \frac{1}{2} \epsilon_{a b c} \frac{d x^b}{d y^c}, \quad \eta = \eta_{a b} \omega^a \omega^b \tag{3.2} $$

are covariantly constant with respect to a $Sp(1)$ connection $\omega$

$$ d \omega + \omega \wedge \omega = 0, \quad J = J^{a b} \omega^a \omega^b. \tag{3.3} $$

The $Sp(1)$ curvature is proportional to the $J$ two-forms

$$ d \omega + \omega \wedge \omega = i \lambda J \tag{3.4} $$

for some constant $\lambda$.

The holonomy group of a quaternionic manifold is a subgroup of $Sp(1) \times Sp(4)$. In addition quaternionic manifolds are Einstein spaces with $R_{\mu \nu} = \ldots$

\textsuperscript{*}Wedge product of forms $dx \wedge dy$ will be denoted by $dxdy$.\textsuperscript{13}
Consistent coupling to supergravity requires $\lambda$ to be negative and fixed to $-1$. So only quaternionic manifolds with negative curvature can be coupled to supergravity. We will see later that this property is automatically satisfied for the $Q$-manifolds irrespective of the holomorphic function $f(z)$.

Let us consider the original Kähler manifold $K_n$ with Kähler (closed) two-form given by

$$J = i e^A d\bar{A}, \quad \langle e^A = e^A d\bar{A} \rangle.$$  \hspace{1cm} (3.4)

The Kähler metric is

$$g_{\alpha\beta} = e^A (e^A)^{\alpha\beta}.$$  \hspace{1cm} (3.5)

It is very convenient to define a $n \times (n+1)$ matrix $P^A$ as follows:

$$P^A = e^A, \quad P_A = -e^A e^A.$$  \hspace{1cm} (3.6)

Note that $P$ satisfies

$$P^2 = 0 \quad (S^2 = 1)$$  \hspace{1cm} (3.7)

$$P^2 P = -\frac{1}{SN} \left( N - \frac{\text{NORM}}{SN} \right)$$  \hspace{1cm} (3.8)

(3.9)

The vierbein one forms are then given by

$$e = P d\bar{z}$$

$$\mathbf{E} = e^{(n+1)/2} P^{-1} \left[ d\mathbb{C} - \frac{1}{2} \mathbb{D} - 1 \right]$$

$$w = 2e^{(n+1)/2} \left[ d\mathbb{C} - \frac{1}{2} \mathbb{D} - 1 \right]$$

$$v = e^{n+1} \left[ d\mathbb{C} + \mathbb{D} - 1 \right]$$

The Lagrangian for the quaternionic manifold takes the form

$$-e^{-i} \mathbf{E} = e^{-i} \mathbf{E} + \mathbf{E} \mathbf{E} + v \mathbf{E} + v \mathbf{E}$$

$$-\frac{1}{2} \sum_{\alpha=1,2; i=1, \ldots, n+1} e^A (e^A)^{\alpha\beta}$$

(3.11)

in terms of the $2(n+1)$ component vierbeins

$$e^A = \left( e^+, e^- \right)$$

$$e^I = \left[ \begin{array}{c} u \\ e^A \end{array} \right], \quad e^{-I} = \left[ \begin{array}{c} v \\ e^A \end{array} \right].$$

(3.12)

*We will use capital letters for flat indices, small letters for curved indices, initial letters of the alphabet $A, A$ run from 1 up to $n$ while middle letters $i, j$ run from 1 up to $n+1$.

*The $\otimes$ symbol denotes the sum of the product of components of two one-forms.
To find the connections we compute the exterior derivatives of the vierbein one-forms

$$\text{d}e = -\nabla e$$

$$\text{d}v = \nabla v + uu + EE$$

$$\text{d}u = \left[ -\frac{1}{2} (v v) + \frac{\text{Eng}_v}{2 \text{Eng}} \right] u - EE$$

$$\text{d}e = \left[ -\omega - \frac{1}{2} (v v) + \frac{\text{Eng}_v}{2 \text{Eng}} \right] E$$

$$-\omega = \frac{\text{Eng}}{2} \text{Eng} - d(N-1) \text{Eng} E$$.

(3.13)

Here $\omega$ is the connection for the original Kahler manifold $K_{\omega}$

$$\omega = \frac{\text{Eng}_v}{2 \text{Eng}} + \frac{\text{Eng}}{2} \left( d(N-1) + \text{Eng}^{-1} \rho + \text{Eng}^{-1} \text{Eng} + \text{Eng}^{-1} \text{Eng}^{-1} \rho \right)$$

and

$$W_j + i Y_j = -\frac{1}{2} F_{ij}.$$

(3.15)

The curvature two-form for $K_{\omega}$ is

$$\text{d}\omega - \frac{\text{Eng}_v}{2 \text{Eng}} \text{Eng}$$

$$\text{d}e = \omega - \frac{\text{Eng}_v}{2 \text{Eng}} \text{Eng}$$

$$\text{d}v = \omega$$

$$\text{d}u = \omega$$.

(3.16)

with

$$e_{ABC} = e_{ABC} e^A e^B e^C$$

and

$$a^A = e^A e^B e^C$$.

Eq. (3.16) is in agreement with Ref. 7.

The connections for the $Q$ manifold are given by

$$\text{d}e = p_a \text{Eng}_v + q_j \text{Eng}_v + \xi_j \text{Eng}_v + \zeta_i \text{Eng}_v$$

$$\text{d}v = 0$$

$$\text{d}u = 0$$

(3.17)

where

$$p = \frac{1}{4} (v-v) + \frac{1}{4} \text{Eng}_v$$

$$q = \frac{-3}{4} (v-v) - \frac{1}{4} \text{Eng}_v$$

$$-\omega$$

$$\text{Eng}_v$$

$$\omega$$

$$-\text{Eng}_v$$

(3.18)

$$p$$ is the Sp($l$) connection.

$$q = \frac{-3}{4} (v-v) - \frac{1}{4} \text{Eng}_v$$

$$\text{Eng}_v$$

$$\omega$$

$$-\text{Eng}_v$$

(3.19)

The $Sp(l) \times Sp(n+1)$ connection is most easily seen by defining a $4(n+1)$ component vierbein

$$\gamma^{\alpha} = e^{\alpha} a^{\alpha}$$

(3.20)

The flat space metric is $\frac{1}{2} \varepsilon^{\alpha \beta} \varepsilon^{\beta \gamma}$, with $\varepsilon = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$ and $\rho = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$. $\gamma^{\alpha}$ is covariantly constant.

$$\gamma^{\alpha}$$

(3.21)

$\gamma^{\alpha}$ correspond to $\gamma^i \text{Eng}_v$ in the notation of Ref. 13. They satisfy the reality condition $\gamma^{\alpha} = \frac{1}{2} (\gamma^{\alpha})^*$. 
with connection

\[ \Omega = p \times 1_{2(n+1)} + \frac{1}{2} \times \begin{pmatrix} q & \xi \\ -\xi & -q^2 \end{pmatrix}, \]

\[ p^t = -p, \quad q^t = -q, \quad \xi^t = -\xi, \]

\[ \Omega^t = -\Omega, \]  

(3.23)

where the two terms respectively correspond to the Sp(1) and Sp(n+1) connections. The curvature two-form for the Q-manifold is

\[ R = d\Omega + \Omega \wedge \Omega = -i J \times 1_{2(n+1)} + \frac{1}{2} \times \tilde{R} \]  

(3.24)

where \( \tilde{R} \) is the Sp(n+1) curvature.

We are mainly interested in the Sp(1) curvature

\[ (-i J)^p = -V^Q \quad \psi^Q. \]  

(3.25)

\( J^p \) define the three "complex structures" satisfying the quaternionic algebra given by Eq. (3.1). The construction of the three two-forms given by Eq. (3.25) is the final proof that \( Q \) is a quaternionic manifold of quaternionic dimension \( n+1 \).

It is of interest to give the expression for the \( \text{Sp}(n+1) \) curvature as well. This is a \( (2(n+1) \times 2(n+1)) \) matrix valued two-form

\[ \tilde{R} = \begin{pmatrix} r & r^t \\ -r^t & -r \end{pmatrix} \]

in which \( r, r^t \) are two \( (n+1) \times (n+1) \) matrix valued two forms. Their expression is

\[ \tilde{r}^\alpha_\beta = -\frac{3}{2} \left( u^\alpha + v_\beta \right) - \frac{3}{2} \left( u_\alpha + v^\beta \right) \]

\[ \tilde{r}^\alpha_\beta = -\frac{1}{2} \left( A^A \right)^\alpha_\beta - \frac{1}{2} \left( A_\alpha^A \right) \]

\[ = \frac{1}{2} \left( \tilde{r}^A_{BCD} (\tilde{r}^{AB} + \tilde{r}^{BC}) \right) \]

\( \frac{1}{16(n+2)^2} (\xi^\alpha_\beta + \xi_\alpha^\beta) \)

\[ + \frac{1}{16(n+2)^2} \tilde{r}_{ABC}^{\alpha} \tilde{r}_{BDE}^{\beta} e^{A}_C e^{B}_D e^{C}_E + \frac{1}{16n+2} \tilde{r}_A \tilde{r}_B \]

\[ \xi_{\alpha}^{\beta} + \xi_{\beta}^{\alpha} \]

\[ X_{abcd} = \tilde{r}_{abcd} + \frac{1}{2} \tilde{r}_{abed} + \frac{1}{2} \tilde{r}_{abed} + \frac{1}{2} \tilde{r}_{abed} + \frac{1}{2} \tilde{r}_{abed} + \frac{1}{2} \tilde{r}_{abed} \]  

(3.26)

The Sp(n+1) curvature can alternately be written as
where $\Omega_{\Gamma^0,\Gamma^1,\Gamma^2}$ is completely symmetric due to the Bianchi identity $R^\Lambda = 0$.

The quaternionic manifold, which is always an Einstein space, has scalar curvature given by

$$ R = -8(n+1)(n+3) $$

(3.28)

in agreement with Ref. 13. We remark that the $Q$-curvature, unlike the $K$ curvature (given by Eq. (3.15), depends, through $r'$, up to the fourth derivative of the holomorphic function $f(z)$.

4. Some Examples.

In Section 2 we have seen that for a quadratic holomorphic function $F(x)$ the $Q$ manifolds become Kahler. This result can be obtained in a more general way by observing that the two-form

$$ a \circ \bar{a} + a \bar{b} + b \bar{b} $$

(4.1)

is always closed. However, this does not mean in general that the manifold is Kahler unless the vierbeins are holomorphic. The vierbein are holomorphic if and only if $dF$ is holomorphic which is actually the case only for quadratic $F$ functions. On the other hand the non-holomorphic part of the $Q$-connection given by $b$ in Eq. (3.24) is proportional to the third derivative of $f$ which consistently shows that for quadratic $f$, the connection becomes a $SU(2) \times SU(n+1) \times U(1)$ connection as appropriate for the Kahler quaternionic manifold.

$$ SU(2) \times SU(n+1) \times U(1) $$

In view of the relation between the third derivative of $f$ and the Yukawa couplings in heterotic superstrings the previous result implies that those moduli fields which correspond to vanishing Yukawa coupling correspond to a $SU(1,n+1)/U(1) \times SU(n+1)$ restricted Kahler sub-manifold whose $s$-map is the quaternionic Kahler manifold

$$ SU(2, n+1) $$

$$ SU(2) \times SU(n+1) \times U(1) $$

(4.2)

The general form of the $Sp(n+1)$ connection in Eq. (3.24) actually tells us that the above manifolds are the only $Q$ manifolds which are also Kahler.

The Kahler potential of these manifolds is (see Section 2)

$$ K_Q = K + \hat{K} = -8n \log \left( 1 - e^{\frac{2}{n+2}} \right) - 8n \left[ \frac{1}{2} (CH)^{-1} (CH) \right] $$

(4.3)
with

\[
\gamma^{-1} = \frac{2}{1 - \gamma^2} \left[ \begin{array}{cc} 1 + \gamma^a \gamma^b & \gamma^a \gamma^b \\ \gamma^a \gamma^b & \gamma^a (1 - \gamma^2) + \gamma^b \gamma^b + \gamma^b \gamma^b \end{array} \right].
\] (4.4)

The other example we would like to mention is the s-map for the one-dimensional Kahler manifolds SU(1,1)/U(1) corresponding respectively to holomorphic functions

\[
f_1(s) = 1 - s^2
\] (4.5)

and

\[
f_2(s) = is^3.
\] (4.6)

These Kahler manifolds only differ from the value of the scalar curvature (2 and -2/3, respectively); however, this s-map gives rise to completely different two-dimensional Q spaces.

According to Ref. 1, the dual quaternionic manifolds should be \(SU(2,2)/G_2\) and \(SU(2) \times SU(2) \times U(1)\) respectively.

In order to understand this point, it is sufficient to see the holonomy group in these different cases. For the quadratic case we already know that the holonomy group is \(SU(2) \times SU(2) \times U(1)\). For the cubic case, we have a non-vanishing \(\gamma\) matrix. The \(\gamma\) matrix is given in terms of 2 X 2 matrices \(q, \gamma\) (see Eq. (3.34) given by

\[
q = \begin{bmatrix} 3X & \gamma^a \\ -E & X \end{bmatrix}, \quad \gamma = -\frac{1}{4} (\gamma^4 - 5) \gamma^5
\] (4.7)

The \(Sp(2)\)-connection is given by a 4 X 4 matrix-valued one-form which only depends on the three differential forms

\[
x, \ E, \ E^\dagger.
\] (4.8)

This is the 4-dimensional (spin-3/2) representation of the \(Sp(1)\) algebra with defining (spin-1/2) representation given by

\[
\begin{bmatrix} x & \frac{E}{\sqrt{3}} \\ \frac{E}{\sqrt{3}} & -X \end{bmatrix}
\] (4.9)

To realize the above statement it is sufficient to set \(E = 0\) (Cartan sub-algebra \(l_3\)) and to see that the \(Sp(2)\) matrix has diagonal entries given by \((3X, X, -3X, -X)\) which are the eigenvalues of \(l_3\) in the spin-3/2 representation.

In agreement with Ref. 13 we see that the holonomy group is \(Sp(1) \times Sp(1)\) in this case, corresponding to the exceptional quaternionic space \(G_2/Sp(4)\).
5. Conclusions.

In this paper we have worked out the geometry of the dual quaternionic manifolds which describe the effective interactions of the moduli fields of (2,2) superstring vacua and of their (Ramond-Ramond) scalar superpartners in type II strings. These results have been obtained by space-time supersymmetry arguments but it is likely that they can be reproduced with a S-matrix approach as discussed recently by Dixon, Kaplunovsky and Louis. 2

Quaternionic spaces which correspond to non-vanishing Yukawa couplings have a complicated structure unless the Yukawa couplings are independent of the moduli. In the latter case, for untwisted sectors of orbifold compactifications, one may even get symmetric spaces of the type discussed in Refs. 1 and 8. However, when twisted fields are involved or general Calabi-Yau compactification is considered, those spaces are not symmetric nor seem to be homogeneous. For those moduli fields which correspond to vanishing Yukawa couplings, the quaternionic sub-spaces are both Kahler and symmetric. More generally, at each point of the moduli space, the pure Ramond-Ramond sector is a SU(1,n+2)/U(1) x SU(n+2) sub-manifold. The results we described can alternatively be obtained by a Kaluza-Klein compactification of the 10D N = 2 supergravity on Calabi-Yau spaces. In that case the extra Ramond-Ramond scalars come, in the case of type IIA supergravity, from the (2,1) and (3,0) internal components of the three form \( A_{ijk} \). The third and fourth derivative of the \( F \) holomorphic function, which appear in the \( Q \)-curvature (Eqs. (3.24) and (3.26)), are related to the following overlapping integrals on Calabi-Yau spaces: 3, 4

\[
\int_R \Omega \wedge \frac{\partial \Omega}{\partial x_1 \partial x_2 \partial x_3} \quad \text{and} \quad \int_R \Omega \wedge \frac{\partial \Omega}{\partial x_1 \partial x_2 \partial x_3 \partial x_4}
\]
References.


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