LHC Optics Measurement with Proton Tracks Detected by the Roman Pots of the TOTEM Experiment

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Abstract

Precise knowledge of the beam optics at the LHC is crucial to fulfill the physics goals of the TOTEM experiment, where the kinematics of the scattered protons is reconstructed with the near-beam telescopes – so-called Roman Pots (RP). Before being detected, the protons’ trajectories are influenced by the magnetic fields of the accelerator lattice. Thus precise understanding of the proton transport is of key importance for the experiment. A novel method of optics evaluation is proposed which exploits kinematical distributions of elastically scattered protons observed in the RPs. Theoretical predictions, as well as Monte Carlo studies, show that the residual uncertainty of the optics estimation method is smaller than 2.5‰.
1 Introduction

The TOTEM experiment at the LHC is equipped with near beam movable insertions – called Roman Pots (RP) – which host silicon detectors to detect protons scattered at the LHC Interaction Point 5 (IP5) [1, 2]. This note reports the results based on data acquired with a total of 12 RPs installed symmetrically with respect to IP5. Two units of 3 RPs are inserted downstream of each outgoing LHC beam: the “near” and the “far” unit located at $s = \pm 214.63\text{ m}$ and $s = \pm 220.00\text{ m}$, respectively, where $s$ denotes the distance from IP5. The arrangement of the RP devices along the two beams is schematically illustrated in Fig. 1.

Each unit consists of 2 vertical, so-called “top” and “bottom”, and 1 horizontal RP. The two diagonals top left of IP–bottom right of IP and bottom left of IP–top right of IP, tagging elastic candidates, are used as almost independent experiments. The details of the set-up are discussed in [1].

![Figure 1: Schematic layout of the LHC magnet lattice at IP5 up to the “near” and “far” Roman Pot units.](image)

Each RP is equipped with a telescope of 10 silicon microstrip sensors of $66\mu\text{m}$ pitch which provides spatial track reconstruction resolution $\sigma(x, y)$ of $11\mu\text{m}$ [3]. Given the longitudinal distance between the units of $\Delta s = 5.372\text{ m}$ the proton angles are measured by the RPs with an uncertainty of $2.15\mu\text{rad}$. During the measurement the detectors in the vertical and horizontal RPs overlap, which enables a precise relative alignment of all the three RPs by correlating their positions via common particle tracks. The alignment uncertainty better than $10\mu\text{m}$ is attained, the details are discussed in [3, 4].

The proton trajectories, thus their positions observed by RPs, are affected by magnetic fields of the accelerator lattice. The accelerator settings define the machine optics which can be characterized with the value of $\beta^*$ at IP5. In the following sections we will analyze two representatives, the $\beta^* = 3.5\text{ m}$ and $90\text{ m}$ optics [2, 5].

In order to precisely reconstruct the scattering kinematics, an accurate model of proton transport is indispensable. TOTEM has developed a novel method to evaluate the optics of the machine by using angle-position distributions of elastically scattered protons observed in the RP detectors. The method, discussed in detail in the following sections, has been successfully applied to data samples recorded in 2010 and 2012 [7–11].

Section 2 introduces the so-called transport matrix, which describes the proton transport through the LHC lattice, while machine imperfections are discussed in Section 3. The proposed novel method for optics evaluation is based on the correlations between the transport matrix elements. These correlations allow the estimation of those optical functions which are strongly correlated to measurable combinations, estimators, of matrix elements. Therefore, it is fundamental to study these correlations in details, which is the subject of Section 4. The applied eigenvector decomposition gives an insight into the obtainable uncertainties of optics estimation, and provides the theoretical baseline of the method.

Section 5 brings the theory to practice, by specifying the estimators, obtained from elastic track distributions measured in RPs. Finally, the applied optics estimation algorithm is discussed in Section 6. The uncertainty of the method was estimated with Monte Carlo simulations, described in detail in Section 7.
2 Proton transport model

Scattered protons are detected by the Roman Pots after having traversed a segment of the LHC lattice containing 29 main and corrector magnets per beam, shown in Fig. 1.

Suppose that the proton has the following kinematical properties at IP5:

- transverse positions $\mathbf{d}^*(x^*, y^*)$
- angles $(\Theta^*_x, \Theta^*_y)$

The trajectory of such proton is described approximately by a linear formula

$$\mathbf{d}(s) = T(s) \cdot \mathbf{d}^*, \tag{1}$$

where $\mathbf{d} = (x, \Theta_x, y, \Theta_y, \Delta p/p)^T$, $p$ and $\Delta p$ denote the nominal beam momentum and the proton longitudinal momentum loss, respectively. The single pass transport matrix

$$T(s) = \begin{pmatrix}
v_x & L_x & m_{13} & m_{14} & D_x \\
\frac{dv_x}{ds} & \frac{dL_x}{ds} & m_{23} & m_{24} & \frac{dD_x}{ds} \\
m_{31} & m_{32} & v_y & L_y & D_y \\
m_{41} & m_{42} & \frac{dv_y}{ds} & \frac{dL_y}{ds} & \frac{dD_y}{ds} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{2}$$

is defined by the optical functions [12]. The horizontal and vertical magnifications

$$v_{x,y} = \sqrt{\beta_{x,y}/\beta^*} \cos \Delta \mu_{x,y} \tag{3}$$

and the effective lengths

$$L_{x,y} = \sqrt{\beta_{x,y}/\beta^*} \sin \Delta \mu_{x,y} \tag{4}$$

are functions of the betatron amplitudes $\beta_{x,y}$ and the relative phase advance

$$\Delta \mu_{x,y} = \int_{IP}^{RP} \frac{ds}{\beta_{x,y}}, \tag{5}$$

and are of particular importance for proton kinematics reconstruction. The $D_x$ and $D_y$ elements are the horizontal and vertical dispersion, respectively.

Elastically scattered protons are relatively easy to distinguish due to their scattering angle correlations. In addition, these correlations are sensitive to the machine optics. Therefore, elastic scattering of protons is an ideal process to study the LHC optics.

In case of the LHC nominal optics the coupling coefficients are, by design, equal to zero

$$m_{13},...,m_{42} = 0. \tag{6}$$

Moreover, for elastically scattered protons the contribution of the vertex position $(x^*, y^*)$ in Eq. (1) is canceled due to the anti-symmetry of the elastic scattering angles of the two diagonals. Also, those terms of Eq. (1) which are proportional to the horizontal or vertical dispersions $D_{x,y}$ vanish, since $\Delta p = 0$ for elastic scattering. Furthermore, the horizontal phase advance $\Delta \mu_x = \pi$ at 219.59 m, shown in Fig. 2, and consequently the horizontal effective length $L_x$ vanishes close to the far RP unit, as it is shown in Fig. 3. Therefore, in the proton kinematics reconstruction $dL_x/ds$ is used.

\footnote{The '*' superscript indicates that the value is taken at the LHC Interaction Point 5.}
Figure 2: The horizontal $\beta_x$ and vertical betatron amplitude $\beta_y$ for the LHC $\beta^* = 3.5$ m optics. The horizontal $\mu_x$ and vertical phase advance $\mu_y$ are also shown, these functions are normalized to $2\pi$. The plot shows that the horizontal phase advance $\Delta \mu_x = \pi$ close to the far RP unit.

Figure 3: The horizontal effective length $L_x$ and its derivative $dL_x/ds$ with respect to $s$ as a function of the distance $s$ in case of the LHC $\beta^* = 3.5$ m optics. The evolution of the optical functions is shown starting from IP5 up to the Roman Pot stations.
Figure 4: The evolution of the vertical effective length $L_y$ and its derivative $dL_y/ds$ for the LHC $\beta^*=3.5$ m optics between IP5 and the location of the Roman Pot stations.

Figure 5: The horizontal $v_x$ and vertical magnification $v_y$ in case of the LHC $\beta^*=3.5$ m optics.

In summary, the kinematics of elastically scattered protons at IP5 can be reconstructed on the basis of RP proton tracks using Eq. (1):

$$\Theta_y^* \approx y L_y, \quad \Theta_x^* \approx \frac{1}{dL_x/ds} \left( \Theta_x - \frac{dv_x}{ds} x^* \right), \quad x^* = \frac{x}{v_x}. \quad \text{(7)}$$

The vertical effective length $L_y$ and the horizontal magnification $v_x$ are applied in Eq. (7) due to their sizeable value, shown in Figs. 4 and 5. The value of $x^*$ is computed at the place...
s along the accelerator where $L_x = 0$ with $x$ being interpolated from $x_N$ and $x_F$ – the proton positions detected in the 'near' and 'far' RPs. As the values of the reconstructed angles are inversely proportional to the optical functions, the errors of the optical functions define the systematic errors of the final physics results.

Figure 6: Schematic connection layout and data flow among the several databases and tools which are used to define the machine settings $\mathcal{M}$ for the optics determination in the TOTEM experiment. The connection with the TOTEM proton reconstruction and with the optics calibration loop, or matching loop, is also indicated.

The proton transport matrix $T(s; \mathcal{M})$, calculated with MAD-X [13], is defined by the machine settings $\mathcal{M}$, which are obtained on the basis of several data sources. The version V6.5 of the LHC sequence is used to describe the magnet lattice, while the nominal magnet strength file for a given beam optics is always updated using measured data: the currents of the magnet’s power converters are first retrieved using TIMBER [14] – an application to extract data from heterogeneous databases containing information about the complete LHC infrastructure. The retrieved currents are then converted to magnet strengths with the LHC Software Architecture (LSA) [15] which employs for this purpose the conversion curves described by the Field Description for the LHC (FIDEL) [16].

<table>
<thead>
<tr>
<th>Magnet</th>
<th>Power converter</th>
<th>$I_{\text{ref}}$ [A]</th>
<th>$k_{\text{LSA}} \times 10^3$</th>
<th>$k_{\text{nom}} \times 10^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MQXA.1R5</td>
<td>RPHFC.UJ56.RQX.R5</td>
<td>3244.27</td>
<td>8.71154</td>
<td>8.71153</td>
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<tr>
<td></td>
<td>RPHGC.UJ56.RTQX2.R5</td>
<td>2319.88</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>RPMBC.UJ56.RTQX1.R5</td>
<td>-0.96</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MQY.4R5.B1</td>
<td>RPHH.RR57.RQX.R5B1</td>
<td>885.37</td>
<td>-3.39670</td>
<td>-3.39682</td>
</tr>
</tbody>
</table>

Table 1: The reference currents $I_{\text{ref}}$, the converted $k_{\text{LSA}}$ and nominal magnet strength $k_{\text{nom}}$ for two example quadrupole magnets for $\beta^* = 3.5$ m LHC optics. The time window of the sample started at 2010-10-30 02:30:00 finished at 2010-10-30 02:50:00 (the currents were stable along the run). The difference between the calculated LSA and nominal values is below 1.3 $\%$ for each analyzed quadrupole magnet of Beam 1, listed explicitly in Table 3. In case of their counterparts of Beam 2 the difference is below 2 $\%$, except for MQY.4L5.B2, where it is 7 $\%$. 

6
<table>
<thead>
<tr>
<th>Magnet</th>
<th>Power converter</th>
<th>$I_{\text{ref}}$ [A]</th>
<th>$k_{\text{LSA}} \times 10^3$</th>
<th>$k_{\text{nom}} \times 10^3$</th>
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<tbody>
<tr>
<td>MQXA.1R5</td>
<td>RPHFC.UJ56.RQX.R5</td>
<td>3276.25</td>
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<td>7.53637</td>
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<td>RPHGC.UJ56.RTQX2.R5</td>
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<td>-0.62509</td>
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<td></td>
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<tr>
<td>MQY.4R5.B1</td>
<td>RPHH.RR57.RQ4.R5B1</td>
<td>185.78</td>
<td>-0.62508</td>
<td>-0.62509</td>
</tr>
</tbody>
</table>

Table 2: The reference currents $I_{\text{ref}}$, the converted $k_{\text{LSA}}$ and nominal magnet strength $k_{\text{nom}}$ for two example quadrupole magnets for $\beta^* = 90 \text{ m}$ optics. The time window begins at 2012.07.12 18:45:00 and finishes at 2012.07.13 02:30:00. The difference between calculated LSA and the nominal strength $k$ is below 1 permil for each analyzed quadrupole magnet in both LHC beams.

The WISE database [17] contains the measured imperfections (field harmonics, magnets displacement, rotations) included in $\mathcal{M}$, as well as the tolerances of the non-measured parameters. Its alignment uncertainties are obtained form the mechanical and magnetic axes measurements. It also contains other quantities of interest such as measured relative and absolute hysteresis errors or power converters’ accuracy.

### 3 Machine imperfections

The real LHC machine [2] is subject to additional imperfections $\Delta \mathcal{M}$, not measured well enough so far, which alter the transport matrix by $\Delta T$:

$$T (s; \mathcal{M}) \rightarrow T (s; \mathcal{M} + \Delta \mathcal{M}) = T (s; \mathcal{M}) + \Delta T.$$  

(8)

The most important are:

- magnet current–strength conversion error: $\sigma(k)/k \approx 10^{-3}$
- beam momentum offset: $\sigma(p)/p \approx 10^{-3}$.

Their impact on the most relevant optical functions $L_y$ and $dL_x/ds$ is presented in Table 3. It is clearly visible that the imperfections of the inner triplet (the so called MQXA and MQXB magnets) are of high influence on the transport matrix while the optics is less sensitive to the strength of the quadrupoles MQY and MQML.

Other imperfections are of lower, but not negligible, significance:

- magnet rotations: $\delta \phi \approx 1 \text{ mrad}$
- beam harmonics: $\delta B/B \approx 10^{-4}$
- power converter errors: $\delta I/I \approx 10^{-4}$
- magnet positions: $\delta x, \delta y \approx 100 \mu m$.

Generally, as can be seen in Table 3 for high-$\beta^*$ optics the magnitude of $\Delta T$ is sufficiently small from the viewpoint of data analysis and the estimation of $\Delta T$ from the data is not substantial. However, the low-$\beta^*$ optics’ sensitivity to the machine imperfections is significant and cannot be neglected.
\[ \delta L_{y,b_1,\text{far}} / L_{y,b_1,\text{far}} \% \quad \delta \left( \frac{dL_{x,b_1}}{ds} \right) / \frac{dL_{x,b_1}}{ds} \% \]

<table>
<thead>
<tr>
<th>Perturbed element</th>
<th>( \beta^* = 3.5 , \text{m} )</th>
<th>( \beta^* = 90 , \text{m} )</th>
<th>( \beta^* = 3.5 , \text{m} )</th>
<th>( \beta^* = 90 , \text{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MQXA.1R5</td>
<td>0.98</td>
<td>0.14</td>
<td>-0.46</td>
<td>-0.42</td>
</tr>
<tr>
<td>MQXB.A2R5</td>
<td>-2.24</td>
<td>-0.24</td>
<td>0.33</td>
<td>0.31</td>
</tr>
<tr>
<td>MQXB.B2R5</td>
<td>-2.42</td>
<td>-0.25</td>
<td>0.45</td>
<td>0.42</td>
</tr>
<tr>
<td>MQXA.3R5</td>
<td>1.45</td>
<td>0.20</td>
<td>-1.14</td>
<td>-1.08</td>
</tr>
<tr>
<td>MQY.4R5.B1</td>
<td>-0.10</td>
<td>-0.01</td>
<td>-0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>MQML.5R5.B1</td>
<td>0.05</td>
<td>0.04</td>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>( \Delta p_{b_1} / p_{b_1} )</td>
<td>-2.19</td>
<td>0.01</td>
<td>-0.79</td>
<td>0.71</td>
</tr>
<tr>
<td>( \Delta \phi_{\text{quadrupoles}} )</td>
<td>0.01</td>
<td>3 \cdot 10^{-3}</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>( (\Delta x, \Delta y)_{\text{quadrupoles}} )</td>
<td>6 \cdot 10^{-6}</td>
<td>1 \cdot 10^{-5}</td>
<td>3 \cdot 10^{-5}</td>
<td>2 \cdot 10^{-5}</td>
</tr>
<tr>
<td>Total sensitivity</td>
<td>4.33</td>
<td>0.43</td>
<td>1.57</td>
<td>1.46</td>
</tr>
</tbody>
</table>

Table 3: Sensitivity of the vertical effective length \( L_{y,b_1} \) and \( dL_{x,b_1} / ds \) to 1 \( \% \) deviations of magnet strengths or beam momentum for low- and high-\( \beta^* \) optics of the LHC beam 1. The total sensitivity to the perturbations of the quadrupole magnets’ transverse position \( (\Delta x, \Delta y) = 1 \, \text{mm} \) and rotation \( (\Delta \phi = 1 \, \text{mrad}) \) is also included. The subscript \( b_1 \) indicates Beam 1.

The proton reconstruction is based on Eq. (7). Thus it is necessary to know the vertical effective length \( L_y \) with an uncertainty better than 1–2 \( \% \) in order to measure the total cross-section \( \sigma_{\text{tot}} \) with the aimed uncertainty of [18]. The currently available \( \Delta \beta / \beta \) beating measurement with an error of 5 – 10 \( \% \) does not allow to estimate \( \Delta T \) with the uncertainty, required by the TOTEM physics program [19]. However, as it is shown in the following sections, \( \Delta T \) can be determined well enough from the proton tracks in the Roman Pots, by exploiting the properties of the optics and those of the elastic \( pp \) scattering.

4 Correlations in the transport matrix

The transport matrix \( T \) defining the proton transport from IP5 to the RPs is a product of matrices describing the magnetic field of the lattice elements along the proton trajectory. The imperfections of individual magnets alter the cumulative transport function. It turns out that independently of the origin of the imperfection (strength of any of the magnets, beam momentum offset) the transport matrix is altered in a similar way, as can be described quantitatively with eigenvector decomposition, discussed in Section 4.1.

4.1 Correlation matrix of imperfections

Assuming that the imperfections discussed in Section 2 are independent, the covariance matrix describing the relations among the errors of the optical functions can be calculated:

\[ V = \text{Cov}(\Delta T_r) = E(\Delta T_r \Delta T_r^T), \quad (9) \]

where \( T_r \) is the most relevant 8-dimensional subset of the transport matrix

\[ T_r^T = (v_x, L_x, \frac{dv_x}{ds}, \frac{dL_x}{ds}, v_y, L_y, \frac{dv_y}{ds}, \frac{dL_y}{ds}), \quad (10) \]

which is presented as a vector for simplicity.
The optical functions contained in $T_r$ differ by orders of magnitude and, moreover, are expressed in different physical units. Therefore, a normalization of $V$ is necessary and the use of the correlation matrix $C$, defined as

$$C_{i,j} = \frac{V_{i,j}}{\sqrt{V_{i,i} \cdot V_{j,j}}}, \quad (11)$$

is preferred. An identical behaviour of uncertainties for both beams was observed and therefore it is enough to study the Beam 1. In case of the $\beta^* = 3.5\,\text{m}$ optics the following error correlation matrix is obtained:

$$C = \begin{pmatrix}
1.00 & 0.74 & -0.42 & -0.80 & -0.51 & -0.46 & -0.61 & -0.44 \\
0.74 & 1.00 & -0.63 & -1.00 & -0.25 & -0.30 & -0.32 & -0.29 \\
-0.42 & -0.63 & 1.00 & 0.62 & 0.03 & 0.07 & 0.01 & 0.08 \\
-0.80 & -1.00 & 0.62 & 1.00 & 0.29 & 0.33 & 0.37 & 0.32 \\
-0.51 & -0.25 & 0.03 & 0.29 & 1.00 & 0.99 & 0.98 & 0.98 \\
-0.46 & -0.30 & 0.07 & 0.33 & 0.99 & 1.00 & 0.96 & 1.00 \\
-0.61 & -0.32 & 0.01 & 0.37 & 0.98 & 0.96 & 1.00 & 0.95 \\
-0.44 & -0.29 & 0.08 & 0.32 & 0.98 & 1.00 & 0.95 & 1.00
\end{pmatrix}. \quad (12)$$

The non-diagonal elements of $C$, which are close to $\pm 1$, indicate strong correlations between the elements of $\Delta T_r$. Consequently, the machine imperfections alter correlated groups of optical functions.

This observation can be further quantified by the eigenvector decomposition of $C$, which yields the following vector of eigenvalues $\lambda(C)$ for the $\beta^* = 3.5\,\text{m}$ optics:

$$\lambda(C) = (4.9, 2.3, 0.53, 0.27, 0.01, 0.01, 0.00, 0.00). \quad (13)$$

Since the two largest eigenvalues $\lambda_1 = 4.9$ and $\lambda_2 = 2.3$ dominate the others, the correlation system is practically two dimensional with the following two eigenvectors

$$v_{4,9} = (0.35, 0.30, -0.16, -0.31, -0.40, -0.41, -0.41, -0.40), \quad (14)$$

$$v_{2,3} = (-0.26, -0.46, 0.47, 0.45, -0.29, -0.27, -0.25, -0.28).$$

Therefore, contributions of the individual lattice imperfections cannot be evaluated. On the other hand, as the imperfections alter approximately only a two-dimensional subspace, a measurement of a small set of weakly correlated optical functions would theoretically yield an approximate knowledge of $\Delta T_r$.

### 4.2 Error estimation of the method

Let us assume for the moment that we can precisely reconstruct the contributions to $\Delta T_r$ of the two most significant eigenvectors while neglecting the others. The error of such reconstructed transport matrix can be estimated by evaluating the contribution of the remaining eigenvectors:

$$\delta \Delta T_{r,i} = \sqrt{E_{i,i} \cdot V_{i,i}}, \quad (15)$$

where

$$E = N \cdot \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \lambda_8
\end{pmatrix} \cdot N^T \quad (16)$$

and $N = (\nu_1, \ldots, \nu_8)$ is the basis change matrix composed of eigenvectors $\nu_t$ corresponding to the eigenvalues $\lambda_t$. 
The relative optics uncertainty before and after the estimation of the most significant eigenvectors is summarized in Table 4

<table>
<thead>
<tr>
<th>$T_{r,i}$</th>
<th>$v_{x,far}$</th>
<th>$L_{x,far}$</th>
<th>$\frac{dv_{x}}{ds}$</th>
<th>$\frac{dl_{x}}{ds}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.1</td>
<td>-1.32 \cdot 10^{-1} m</td>
<td>3.1 \cdot 10^{-2} m^{-1}</td>
<td>-3.21 \cdot 10^{-1}</td>
</tr>
<tr>
<td>$\sqrt{V_{r,i}}$</td>
<td>2.0 \cdot 10^{-1}</td>
<td>3.4 \cdot 10^{2}</td>
<td>4.2 \cdot 10^{-1}</td>
<td>1.6</td>
</tr>
<tr>
<td>$\delta T_{r,i}$</td>
<td>9.5 \cdot 10^{-2}</td>
<td>9.1 \cdot 10^{1}</td>
<td>2.6 \cdot 10^{-1}</td>
<td>3.4 \cdot 10^{-1}</td>
</tr>
</tbody>
</table>

Table 4: Nominal values of the optical functions $T_{r,i}$ and their relative uncertainty before ($\sqrt{V_{r,i}}/|T_{r,i}|$) and after ($\delta T_{r,i}/|T_{r,i}|$) the determination of the two most significant eigenvectors ($\beta^* = 3.5 \text{ m}$, Beam 1).

According to the table, even if we limit ourselves only to the first two most significant eigenvalues, the uncertainty of optical functions due to machine imperfections drops significantly. In particular, in case of $dl_x/ds$ and $L_y$ a significant error reduction down to a per mil level is observed. Unfortunately, due to $\Delta \mu_x = \pi$ (Fig. 3), the uncertainty of $L_x$, although importantly improved, remains very large and the use of $dl_x/ds$ for proton kinematics reconstruction should be preferred.

In the following sections a practical numerical method of inferring the optics from the RP proton tracks is presented and its validation with Monte Carlo calculations is reported.

5 Optics estimators from proton tracks measured by Roman Pots

5.1 $\beta^* = 3.5 \text{ m at } \sqrt{s} = 7 \text{ TeV}$ ($E_{\text{beam}} = 3.5 \text{ TeV}$)

The TOTEM experiment can select the elastically scattered protons with high purity and efficiency [7,8]. The RP detector system, due to its high resolution ($\sigma(x,y) \approx 11 \mu m$, $\sigma(\Theta_{x,y}) \approx 2.9 \mu rad$), can measure very precisely the proton angles, positions and the angle-position relations on an event-by-event basis. These quantities can be used to define a set of estimators characterizing the correlations between the elements of the transport matrix $T$ or between the transport matrices of the two LHC beams. Such a set of estimators $R_1,...,R_{10}$ (defined in the next sections) is exploited to reconstruct, for both LHC beams, the imperfect transport matrix $T(M) + \Delta T$ defined in Eq. 3.

In this Section the optics estimators are presented for the $\beta^* = 3.5 \text{ m}$ LHC machine optics used in a data taking of October 2010 at beam energy $E_{\text{beam}} = 3.5 \text{ TeV}$.

5.1.1 Correlations between the two beams

Since the momentum of the two LHC beams is identical, the elastically scattered protons will be deflected symmetrically from their nominal trajectories of Beam 1 and Beam 2:

$$\Theta_{x,b_1}^* = -\Theta_{x,b_2}^* , \Theta_{y,b_1}^* = -\Theta_{y,b_2}^*$$ (17)
which allows to compute ratios $R_{1,2}$ relating the effective lengths at the RP locations of the two beams. From Eqs. (1) and (17) we obtain:

$$R_1 \equiv \frac{\Theta_{x,b_1}}{\Theta_{x,b_2}} \approx \frac{dL_{x,b_1}}{ds} \cdot \frac{\Theta^*_{x,b_1}}{\Theta^*_{x,b_2}} = -\frac{dL_{x,b_1}}{ds}, \quad (18)$$

$$R_2 \equiv \frac{y_{b_1,\text{far}}}{y_{b_2,\text{far}}} \approx \frac{L_{y,b_1,\text{far}}}{L_{y,b_2,\text{far}}}, \quad (19)$$

where the subscripts $b_1$ and $b_2$ indicate Beam 1 and 2, respectively. Approximations present in Eqs. (18) and (19) represent the impact of statistical effects such as detector resolution, beam divergence and primary vertex position distribution. The estimators $\hat{R}_1$ and $\hat{R}_2$ are finally obtained from the $(\Theta_{x,b_1}, \Theta_{x,b_2})$ and $(y_{b_1,\text{far}}, y_{b_2,\text{far}})$ distributions and are defined with the help of the distributions’ principal eigenvector, illustrated in Figs. 7 and 8. The width of the distributions is determined by the beam divergence and the vertex contribution, which leads to 0.5% uncertainty on the eigenvector’s slope parameter.

![Figure 7: Beam 1 and 2 elastic scattering angle correlation in the horizontal plane $(\Theta_{x,b_1}, \Theta_{x,b_2})$ of protons detected by the Roman Pots.](image)

5.1.2 Single beam correlations

The distributions of proton angles and positions measured by the Roman Pots define the ratios of certain elements of the transport matrix $T$, defined by Eq. (1) and (2). First of all, $dL_y/ds$ and $L_y$ are related by

$$R_3 \equiv \frac{\Theta_{y,b_1}}{y_{b_1}} \approx \frac{dL_{y,b_1}}{ds} \cdot \frac{\Theta^*_{y,b_1}}{L_{y,b_1}}, \quad R_4 \equiv \frac{\Theta_{y,b_2}}{y_{b_2}} \approx \frac{dL_{y,b_2}}{ds} \cdot \frac{\Theta^*_{y,b_2}}{L_{y,b_2}}. \quad (20)$$

The corresponding estimators $\hat{R}_3$ and $\hat{R}_4$ can be calculated with an uncertainty of 0.5% from the distributions as presented in Fig. 9.

Similarly, we exploit the horizontal dependencies to quantify the relations between $dL_x/ds$ and $L_x$. As $L_x$ is close to 0, see Fig. 3 instead of defining the ratio we rather estimate the
Figure 8: Correlation between positions (vertical projections) of elastically scattered protons detected in Beam 1 and 2. The sharp edges are due to the vertical acceptance limits of the detectors.

Figure 9: Correlation between vertical position and angle of elastically scattered protons at the RP of Beam 1.

position $s_0$ along the beam line (with the uncertainty of about 1 m), for which $L_x = 0$. This is accomplished by resolving

$$
\frac{L_x(s_0)}{dL_x(s)/ds} = \frac{L_x(s_1)}{dL_x(s)/ds} + (s_0 - s_1) = 0, \quad (21)
$$

for $s_0$, where $s_1$ denotes the coordinate of the Roman Pot station along the beam with respect to IP5. Obviously, $dL_x(s)/ds$ is constant along the RP station as no magnetic fields are present at the RP location. The ratios $L_x(s_1)/\frac{dL_x(s_1)}{ds}$ for Beam 1 and 2, similarly to the vertical constraints
\( R_3 \) and \( R_4 \), are defined by the proton tracks:

\[
\frac{L_x}{ds} = \frac{x}{\Theta_x},
\]

which is illustrated in Fig. [10]. In this way two further constraints and the corresponding estimators (for Beam 1 and 2) are obtained:

\[
R_5 \equiv s_{b_1} \quad \text{and} \quad R_6 \equiv s_{b_2}.
\]

Figure 10: Correlation between the horizontal angle and position of elastically scattered protons at the RP of Beam 1.

5.1.3 Coupling / rotation

In reality the coupling coefficients \( m_{13}, \ldots, m_{42} \) cannot be always neglected, as it is assumed by Eq. \([6]\). RP proton tracks can help to determine the coupling components of the transport matrix \( T \) as well, where it is especially important that \( L_x \) is close to zero at the RP locations. Always based on Eq. \([1]\) and \([2]\), four additional constraints (for each of the two LHC beams and for each unit of the RP station) can be defined:

\[
R_{7,\ldots,10} \equiv \frac{x_{\text{near (far)}}}{y_{\text{near (far)}}} \approx \frac{m_{14,\text{near (far)}}}{L_{y,\text{near (far)}}}.
\]

The subscripts “near” and “far” indicate the position of the RP along the beam with respect to the IP. Geometrically \( R_{7,\ldots,10} \) describe the rotation of the RP scoring plane about the beam axis. Analogously to the previous sections, estimators \( \hat{R}_{7,\ldots,10} \) are obtained from track distributions as presented in Fig. [12] and an uncertainty of 3% is achieved.
5.1.4 Constraints’ values

Table 5 reports the values of the estimators defined in the previous sections. The estimators $\hat{R}_7$ and $\hat{R}_8$ are calculated for the ‘near’ and ‘far’ RPs of Beam 1, while $\hat{R}_9$ and $\hat{R}_{10}$ represent the values for Beam 2.

<table>
<thead>
<tr>
<th>$i$ of $R_i$</th>
<th>Value</th>
<th>Uncertainty [%]</th>
<th>$i$ of $R_i$</th>
<th>Value</th>
<th>Uncertainty [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9895</td>
<td>0.5</td>
<td>6</td>
<td>217.6</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>-1.0954</td>
<td>0.2</td>
<td>7</td>
<td>0.0436</td>
<td>1.6</td>
</tr>
<tr>
<td>3</td>
<td>0.00388</td>
<td>0.4</td>
<td>8</td>
<td>0.0390</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>0.00262</td>
<td>1.5</td>
<td>9</td>
<td>0.0369</td>
<td>2.4</td>
</tr>
<tr>
<td>5</td>
<td>217.0</td>
<td>0.2</td>
<td>10</td>
<td>0.0316</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Table 5: The measured value and uncertainty of the optics constraints $\hat{R}_i$ in case of the $\beta^* = 3.5$ m optics.

5.2 $\beta^* = 90$ m at $\sqrt{s} = 8$ TeV ($E_{beam} = 4$ TeV)

In a similar way as for $\beta^* = 3.5$ m (Section 5.1), the estimators were determined for $\beta^* = 90$ m optics runs. This optics allowed for direct determination of

$$R_5 = \frac{L_{x,b_1}}{dL_{x,b_1}/ds}, \quad R_6 = \frac{L_{x,b_2}}{dL_{x,b_2}/ds}. \quad (25)$$

(contrary to the indirect method used for $\beta^* = 3.5$ m optics given in Eq. 23). The measured values are reported in Table 6.
<table>
<thead>
<tr>
<th>$i$ of $R_i$</th>
<th>Value of the constraint</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.0152</td>
<td>$5 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>-0.9915</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>0.697</td>
<td>$8 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>1.116</td>
<td>$8 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 6: Optics constraints $\hat{R}_1, \hat{R}_2, \hat{R}_5$ and $\hat{R}_6$ for $\beta^* = 90$ m run.

5.2.1 $\hat{R}_3$ and $\hat{R}_4$ estimators

For the $\hat{R}_3$ and $\hat{R}_4$ estimators the obtained values slightly differ depending on the diagonal (see Table 7 and 8). This is most likely caused the sextupole errors altering the magnets’ gradient fields and it is consistent with the sextupole error tolerances.

<table>
<thead>
<tr>
<th>$i$ of $R_i$</th>
<th>Value of the constraint</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0180220</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>0.0180316</td>
<td>$5 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 7: Optics constraints $\hat{R}_3$ and $\hat{R}_4$ in case of the $\beta^* = 90$ m optics, case (a).

<table>
<thead>
<tr>
<th>$i$ of $R_i$</th>
<th>Value of the constraint</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0180685</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>0.0179958</td>
<td>$8 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 8: Optics constraints $\hat{R}_3$ and $\hat{R}_4$ in case of the $\beta^* = 90$ m optics, case (b).

6 Optical functions estimation

6.1 Definition of the $\chi^2$ function

The machine imperfections $\Delta \mathcal{M}$, leading to the transport matrix change $\Delta T$, are in practice determined with the $\chi^2$ minimization procedure:

$$\widehat{\Delta \mathcal{M}} = \arg \min \chi^2,$$

defined on the basis of the estimators $\hat{R}_1...\hat{R}_{10}$, where the arg min function gives the phase space position where the $\chi^2$ is minimized. As it was discussed in Section 4.1 although the overall alteration of the transport matrix $\Delta T$ can be determined precisely based on a few optical functions’ measurements, the contributions of individual imperfections cannot be established. In terms of optimization, such a problem has no unique solution and additional constraints, defined by the machine tolerance, have to be added.

Therefore, the $\chi^2$ function is composed of the part defined by the Roman Pot tracks’ distributions and the one reflecting the LHC tolerances:

$$\chi^2 = \chi^2_{\text{Design}} + \chi^2_{\text{Measured}}.$$
The design part

\[
\chi^2_{\text{Design}} = \sum_{i=1}^{12} \left( \frac{k_i - k_{i,\text{MAD-X}}}{\sigma(k_i)} \right)^2 + \sum_{i=1}^{12} \left( \frac{\phi_i - \phi_{i,\text{MAD-X}}}{\sigma(\phi_i)} \right)^2 + \sum_{i=1}^{2} \left( \frac{p_i - p_{i,\text{MAD-X}}}{\sigma(p_i)} \right)^2
\]  

(28)

where \(k_i\) and \(\phi_i\) are the nominal strength and rotation of the \(i\)th magnet, respectively. Thus Eq. (28) defines the nominal machine \((k_i, \phi_i, p_i)\) as an attractor in the phase space. Both LHC beams are treated simultaneously. Only the relevant subset of machine imperfections \(\Delta M\) was selected. The obtained 26-dimensional optimization phase space includes the magnet strengths (12 variables), rotations (12 variables) and beam momentum offsets (2 variables). Magnet rotations are included into the phase space, otherwise only the coupling coefficients \(m_{13}, ..., m_{42}\) could induce rotations in the \((x, y)\) plane Eq. (24), which would bias the result.

The measured part

\[
\chi^2_{\text{Measured}} = \sum_{i=1}^{10} \left( \frac{\hat{R}_i - R_{i,\text{MAD-X}}}{\sigma(\hat{R}_i)} \right)^2
\]  

(29)

contains the track-based estimators \(\hat{R}_1 \ldots \hat{R}_{10}\) (discussed in detail in Section 5) together with their uncertainty. The subscript “MAD-X” defines the corresponding values evaluated with the MAD-X software during the \(\chi^2\) minimization.

6.2 Interplay between the detector alignment and the optics matching procedure

Alignment procedures are practically independent from the optics imperfections. The relative RP alignment within a single arm is obtained solely on the basis of local proton tracks, by means of top and bottom RP overlaps with the horizontal devices. This procedure does not involve any optics assumption and the optics has no influence on it.

The further alignment of the above system with respect to the beam is performed with RP distributions of elastically scattered protons. The key cuts of elastic proton tagging (collinearity of left-right arm protons) require no prior optics knowledge while for the remaining cuts an indicative nominal optics knowledge is sufficient as they show insensitivity to expected optics errors [10]. Furthermore, the alignment techniques applied in TOTEM [4] rely only on the hit distribution symmetries. Although the optics imperfections may change the RP hit distributions, their symmetries are preserved making the alignment procedure immune to optics imperfections.

The estimators \(\hat{R}_1, \hat{R}_5\) and \(\hat{R}_6\) are insensitive to misalignment. The \(\hat{R}_2, \hat{R}_3\) and \(\hat{R}_4\) constraints are, in principle, very sensitive to relative top-bottom RP misalignment. However, such misalignment is very precisely determined by means of the relative alignment procedure.

Finally, there is an interplay between the RP unit rotation misalignment and the optics \(x\)-\(y\) coupling due to rotation misalignments of the lattice magnets. The quadrupole rotation misalignments can induce an \(x\)-\(y\) coupling, which provokes a RP \(x\)-\(y\) scoring plane rotation, consider Eq. (24). For low-\(\beta^*\) optics \((\beta^* = 3.5 \, \text{m})\) the uncertainty of this rotation is 35 mrad when nominal LHC uncertainties are applied. Compared to an expected 1 mrad rotation alignment uncertainty of a RP unit, the lattice related effect is clearly larger and can be estimated.

However, large-\(\beta^*\) optics is characterized by large insensitivity to quadrupole magnet rotation misalignments. For \(\beta^* = 90 \, \text{m}\) the RP \(x\)-\(y\) scoring plane rotation uncertainty of 1.8 mrad is expected which is compatible to the alignment uncertainty. Therefore in this case lattice rotation imperfections cannot be distinguished from RP rotation misalignment.
6.3 $\beta^*=3.5 \text{ m at } \sqrt{s} = 7 \text{ TeV (} E_{\text{beam}} = 3.5 \text{ TeV)}$

The optics estimation procedure with measured optics estimators described in Section 5 and $\chi^2$ function described in Section 6 has been applied for the $\beta^* = 3.5 \text{ m LHC machine optics used in a data taking of October 2010 at beam energy } E_{\text{beam}} = 3.5 \text{ TeV}$.

The statistical quality of the estimation procedure is reported in Table 9. $\chi^2/NDF = 25.82/10$ gives a p-value of $CL = 4 \times 10^{-1} \%$ of the estimated parameters. Table 10 presents the values of the estimated optical functions.

Table 9: The statistical quality of the optics estimation procedure in case of the $\beta^* = 3.5 \text{ m optics}$. The contribution of Beam 1 and 2 and the constraints $\hat{R}_1$ and $\hat{R}_2$, involving both beams, are reported.

<table>
<thead>
<tr>
<th>Constraint(s)</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_1$</td>
<td>0.23</td>
</tr>
<tr>
<td>$\hat{R}_2$</td>
<td>0.31</td>
</tr>
<tr>
<td>Beam 1</td>
<td>7.25</td>
</tr>
<tr>
<td>Beam 2</td>
<td>18.04</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>25.82</strong></td>
</tr>
</tbody>
</table>

The obtained value of the effective length $L_y$ of Beam 1 is close to the nominal one, while Beam 2 shows a significant change. The same pattern applies to the values of $dL_x/ds$. The error estimation of the procedure is discussed in Section 7.
\[ \beta^* = 3.5 \text{ m} \]

<table>
<thead>
<tr>
<th>( L_{y,b_1,\text{far}} ) [m]</th>
<th>( dL_{x,b_1}/ds )</th>
<th>( L_{y,b_2,\text{far}} ) [m]</th>
<th>( dL_{x,b_2}/ds )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>22.4</td>
<td>18.4</td>
<td>-3.29 \times 10^{-1}</td>
</tr>
<tr>
<td>Estimated</td>
<td>22.6</td>
<td>20.7</td>
<td>-3.15 \times 10^{-1}</td>
</tr>
</tbody>
</table>

Table 10: The horizontal and vertical effective lengths of both LHC beams for the \( \beta^* = 3.5 \text{ m} \) optics, obtained with the estimation procedure, compared to their nominal values.

<table>
<thead>
<tr>
<th>( v_{x,b_1} )</th>
<th>( v_{y,b_1} )</th>
<th>( v_{x,b_2} )</th>
<th>( v_{y,b_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>-3.10</td>
<td>-4.28</td>
<td>-3.11</td>
</tr>
<tr>
<td>Estimated</td>
<td>-3.11</td>
<td>-4.27</td>
<td>-3.12</td>
</tr>
</tbody>
</table>

Table 11: The horizontal and vertical magnifications of both LHC beams at the far RP unit for the \( \beta^* = 3.5 \text{ m} \) optics, obtained with the estimation procedure.

6.4 \( \beta^* = 90 \text{ m at } \sqrt{s} = 8 \text{ TeV} (E_{beam} = 4 \text{ TeV}) \)

Two optics estimation iterations (one per each diagonal) were performed with two different sets of \( \hat{R}_3 \) and \( \hat{R}_4 \) estimators, as discussed in Section 5.2. Using the constraints of Tables 6 and 7 the statistical quality of the estimation procedure is \( \chi^2/NDF = 11.9/6 \) with a confidence level of \( CL = 6.4\% \); the other diagonal leads to consistent results.

It is very important that from a measurement point of view the two diagonals represent (almost) independent experiments. Therefore, taking into account the resolution of the procedure and the required symmetry of the diagonals, the final optical functions are taken as the average of the two results. The result is reported in Table 12.

<table>
<thead>
<tr>
<th>( L_{y,b_1,\text{far}} ) [m]</th>
<th>( dL_{x,b_1}/ds )</th>
<th>( L_{y,b_2,\text{far}} ) [m]</th>
<th>( dL_{x,b_2}/ds )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>263.2</td>
<td>263.2</td>
<td>-5.36 \times 10^{-1}</td>
</tr>
<tr>
<td>Estimated</td>
<td>264.1</td>
<td>266.3</td>
<td>-5.17 \times 10^{-1}</td>
</tr>
</tbody>
</table>

Table 12: Selected optical functions of both LHC beams for the \( \beta^* = 90 \text{ m} \) optics, obtained with the estimation procedure, compared to their nominal values.

7 Monte Carlo estimation of the errors

7.1 \( \beta^* = 3.5 \text{ m at } \sqrt{s} = 7 \text{ TeV} (E_{beam} = 3.5 \text{ TeV}) \)

In order to demonstrate that the proposed \( \hat{R}_i \) optics estimators are effective the method was validated with Monte Carlo simulations. The error of the procedure can be also determined from these simulations.

The nominal machine settings \( M \) were altered by simulated machine imperfections \( \Delta M \), applied within their tolerances using Gaussian distributions, in order to provide a model for the LHC imperfections. The simulated elastic proton tracks were used afterwards to calculate the estimators \( \hat{R}_1...\hat{R}_{10} \). The study included the impact (within their tolerances) of

- magnet strengths,
beam momenta,
magnet displacements, rotations and harmonics,
settings of kickers,
measured proton angular distribution.

The error distributions of the optical functions $\Delta T$ obtained for $\beta^* = 3.5 \text{ m}$ and $E_{\text{beam}} = 3.5 \text{ TeV}$ are presented in Fig. 14 and Table 13 while the $\beta^* = 90 \text{ m}$ results at $E_{\text{beam}} = 4 \text{ TeV}$ are shown in Fig. 21 and Table 14.

<table>
<thead>
<tr>
<th></th>
<th>Simulated optics distribution</th>
<th>Reconstructed optics error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta L_{y,b_{1, far}}/L_{y,b_{1, far}}$</td>
<td>Mean 0.39 %, RMS 4.2 %</td>
<td>Mean -0.97 %, RMS 1.6 %</td>
</tr>
<tr>
<td>$\Delta L_{y,b_{2, far}}/L_{y,b_{2, far}}$</td>
<td>Mean -0.14 %, RMS 4.9 %</td>
<td>Mean 0.21 %, RMS 1.6 %</td>
</tr>
</tbody>
</table>

Table 13: The Monte-Carlo study of the impact of the LHC imperfections $\Delta M$ on selected transport matrix elements $dL_x/ds$ and $L_y$ for $\beta^* = 3.5 \text{ m}$ at $E_{\text{beam}} = 3.5 \text{ TeV}$. The LHC parameters were altered within their tolerance. The relative errors of $dL_x/ds$ and $L_y$ (mean value and RMS) characterize the optics uncertainty before and after optics estimation.

Figure 13: The MC error distribution of $\beta^* = 3.5 \text{ m}$ optical functions $L_y$ and $dL_x/ds$ for Beam 1 at $E_{\text{beam}} = 3.5 \text{ TeV}$, before and after optics estimation.
Figure 14: The MC error distribution of $\beta^* = 3.5$ m optical functions $L_y$ and $dL_x/ds$ for Beam 2 at $E_{\text{beam}} = 3.5$ TeV, before and after optics estimation.

First of all, the impact of the machine imperfections $\Delta M$ on the transport matrix $\Delta T$, as shown by the MC study, is identical to the theoretical prediction presented in Table 4. The bias of the simulated optics distributions is due to magnetic field harmonics as reported by the LHC imperfections database [17]. The final value of mean after optics estimation procedure contributes to the total uncertainty of the method.

However, on the contrary, the errors of the reconstructed optical functions are significantly smaller than evaluated theoretically in Section 4.2. This results from the larger number of constraints, design and measured constraints Eq. (27), employed in the numerical estimation procedure of Section 6. In particular, the collinearity of elastically scattered protons was exploited in addition. Finally, the achieved uncertainties of $dL_x/ds$ and $L_y$ are both lower than 2% for both beams.

Figure 15: The distribution of the quadrupole magnets’ strength after optics estimation for Beam 1 (left panel) and 2 (right panel) using the $\beta^* = 3.5$ m optics at $E_{\text{beam}} = 3.5$ TeV.
Figure 16: The distribution of the quadrupole magnets’ rotation after optics estimation for Beam 1 (left panel) and 2 (right panel) using the $\beta^* = 3.5$ m optics at $E_{\text{beam}} = 3.5$ TeV.

Figure 17: The distribution of the offset of the beam momentum for Beam 1 and 2 after matching for $\beta^* = 3.5$ m at $E_{\text{beam}} = 3.5$ TeV.

### 7.2 $\beta^* = 90$ m at $\sqrt{s} = 8$ TeV ($E_{\text{beam}} = 4$ TeV)

#### 7.2.1 Results without harmonics

In this section the Monte Carlo estimation of the uncertainties of the optical functions’ in case of the $\beta^* = 90$ m optics is reported. The covariance matrix of the optical functions’ error is also calculated.

Monte Carlo studies were carried out with the following settings: the beam energy $E_{\text{beam}} = 4$ TeV, $\beta^* = 90$ meters, $10^3$ jobs. The perturbed machine elements and their perturbation are:

- strength of the quadrupole magnets ($\sigma(\Delta k) = 1\%$)
- azimuthal rotation of quadrupole magnets (1 mrad)
- quadrupole magnets’ transverse position $(x, y)$ (1 mm)
- momentum offset of the two beams (1%)

Due to their small impact on the estimated values, the kicker magnets were not perturbed in the estimation procedure, only the quadrupole magnets of both beams². The effect of the magnetic field harmonics is studied in a dedicated analysis, which described in the next section.

²The mentioned quadrupole magnets are listed in Table for Beam 1, for Beam 2 we take their counterparts.
As before 26 parameters are used during the optimization. These are the quadrupole magnets’ strength in both beams (12 parameters) together with their azimuthal rotation (12 parameters) and the momentum offsets of the two beams in addition (2 parameters).

The results of the optics optimization are presented on Figs. 18-19.

Figure 18: The MC error distribution of the vertical effective length $L_y$, in case of the $\beta^* = 90$ m optics, after optics estimation for Beam 1 (left) and Beam 2 (right) at the near RP unit without field harmonics.

Figure 19: The MC error distribution of the derivative $dL_x/ds$ of the horizontal effective length, in case of the $\beta^* = 90$ m optics, after optics estimation for Beam 1 (left) and Beam 2 (right), without field harmonics.

7.2.2 Correlation matrix after optics estimation

The covariance and correlation matrices are determined for the $\beta^* = 90$ m optics, for 16 optical functions of the $4 \times 4$ transport matrix, taken at the near and far RP unit in both beams. The dimension of the matrices is $64 \times 64$. The correlation matrix is illustrated in Fig. 7.2.2.
The largest eight eigenvalues of the correlation matrix (out of 64) were found to be $\lambda = \{23.45, 17.13, 13.78, 2.78, 2.50, 1.98, 1.83, 0.29\}$, which shows that there are 3 major eigenmodes. Roughly speaking, the eigenvector of the largest $\lambda_1$ eigenmode describes beam-beam correlations, while $\lambda_2$ and $\lambda_3$ steer Beam 2 and Beam 1 correlations, respectively.

### 7.2.3 Results with magnet field harmonics

In addition to the MC settings and parameters discussed in the previous section, each job is characterized by different field harmonics errors generated\(^3\) according to the WISE model. The MC procedure sometimes terminated with an outlier due to the involvement of the field harmonics. Such cases were removed with a confidence level requirement $CL > 0.1\%$. With $NDF=6$ the requirement was translated to $\chi^2 < 22.4$.

The results, in case of the most relevant optical functions, are reported in Table[14] and also shown in Fig. [21] for Beam 1.

---

\(^3\)The WISE tool, a Visual BASIC application, originally did not support optics with $\beta^*$ larger than 10 meters. The particular case of $\beta^* = 90$ m optics was implemented by P. Hagen using LSA settings.
Table 14: The Monte-Carlo study of the impact of the LHC imperfections $\Delta M$ on selected transport matrix elements $dL_x/ds$ and $L_y$ for $\beta^* = 90$ m at $E = 4$ TeV. The LHC parameters were altered within their tolerances. The relative errors of $dL_x/ds$ and $L_y$ (mean value and RMS) characterize the optics uncertainty before and after optics estimation.

<table>
<thead>
<tr>
<th>Relative optics distribution</th>
<th>Simulated optics distribution</th>
<th>Reconstructed optics error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta L_{y,b1,far}$</td>
<td>Mean [%]</td>
<td>Mean [%]</td>
</tr>
<tr>
<td>$L_{y,b1,far}$</td>
<td>2.2 \times 10^{-2}</td>
<td>5.8 \times 10^{-2}</td>
</tr>
<tr>
<td>$\delta dL_{x,b1}/ds$</td>
<td>6.7 \times 10^{-3}</td>
<td>-6.4 \times 10^{-2}</td>
</tr>
<tr>
<td>$\delta L_{y,b2,far}$</td>
<td>-5 \times 10^{-3}</td>
<td>5.8 \times 10^{-2}</td>
</tr>
<tr>
<td>$L_{y,b2,far}$</td>
<td>1.8 \times 10^{-2}</td>
<td>-7 \times 10^{-2}</td>
</tr>
</tbody>
</table>

In total the following scale factor is found:

$$\text{RMS}_{\text{with harmonics}} \approx 1.35 \cdot \text{RMS}_{\text{no harmonics}}$$

This scale factor is applied to scale the covariance matrix, which was calculated in the previous section without field harmonics. In this way, the effect of the field harmonics is safely taken into account with an overall scale factor, without involving software induced correlations coming from the WISE field model.

7.2.4 Additional studies with field harmonics

The elastically scattered protons, pass through and probe different parts of the LHC magnetic fields, before they could reach the RP detectors of the TOTEM experiment.
With no field harmonics present, the proton transport is well described by the linear matrix, see Eq. [2]. In the presence of field harmonics the proton transport becomes a function of the proton’s kinematical variables: the four-momentum transfer squared $|t|$ and the azimuthal scattering angle of the proton $\phi$. Consequently, the reconstruction of these kinematical variables is also affected, consider Eq. [7].

We tested the effect of the field harmonics on the proton transport: the four-momentum transfer squared $t$ of the scattering process has been varied together with the azimuthal scattering angle $\phi$. The corresponding calculated kinematical variables are chosen as MAD-X reference orbit, and then the proton transport matrix $T$ is calculated with respect to this reference orbit.

In the high energy limit, by means of $s \to \infty$, the transverse momentum of the reference orbit for a given $|t|$ value is

$$|p_\perp| \simeq \sqrt{-t},$$

and its horizontal and vertical components are computed with

$$p_{\perp,x} = |p_\perp| \cdot \cos \phi,$$

$$p_{\perp,y} = |p_\perp| \cdot \sin \phi.$$  \(31\)

The MAD-X reference orbit is calculated with Eqs. (30) and (31). The change of the vertical effective length $L_y$ with respect to its nominal value, as a function of the kinematical variables, is shown in Fig. 22 for two different field harmonics settings generated with the WISE tool. According to the results, the change of $L_y$ with the azimuthal angle $\phi$ remains within 1.5% at each tested $|t|$ value. The constant shift between the results shown in Fig. 22 can be determined using the described optics estimation method.

![Figure 22: $\phi$ dependence of $L_y$ using two different field harmonics settings. The change of $L_y$ with the azimuthal angle $\phi$ remains within 1.5% at each tested $|t|$ value. The constant shift between the results shown in Fig. 22 can be determined using the described optics estimation method.](image)

The effect on the horizontal effective length $L_x$ and on its derivative is shown in Fig. 23. Note that for $dL_x/ds$ the variation with $\phi$ remains within 5%.

By comparing Figs. 22 and 23 one can observe that the azimuthal angle $\phi$ dependence of the vertical and horizontal effective lengths is shifted with respect to each other with about $\pi$. Consequently, in the reconstruction formulae Eq. [7] the vertical and horizontal effects approximately compensate each other.

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Note, that without magnetic field harmonics the dependence of the optical functions on the reference orbit is practically negligible. In case of the vertical effective length the variation of $L_y(\lvert t \rvert)$ is smaller than $10^{-4}$ \% with respect to its nominal value.

The effect of the field harmonics on the proton’s kinematics reconstruction can be also tested directly: $10^7$ simulated protons are transported into the detectors and then reconstructed with Eq. 7 using the optics altered by field harmonics. The comparison of the originally generated and reconstructed $\lvert t \rvert$ values is shown in Fig. 24. The result shows that, up to statistical fluctuations, no systematic deviation can be observed in the generated and reconstructed $\lvert t \rvert$ value.

Figure 24: Direct test of the influence of the magnetic field harmonics on the $\lvert t \rvert$ reconstruction. The result shows that, up to statistical fluctuations, no systematic deviation can be observed in the generated and reconstructed $\lvert t \rvert$ value.

### 7.2.5 Stability of each power converters at LHC

The $pp$ elastic scattering data recorded over the period of 12 hours revealed the time-dependence of the beam position. The stability of the magnet currents during the run had to be verified in order to exclude any optics modification during data taking. All the LHC magnet currents were extracted from the TIMBER database and tested: only the MCBV.14L5.B2 kicker showed (a negligible) change along the data taking.
8 Conclusions

TOTEM has proposed a novel approach to estimate the optics at LHC. The method, based on the correlations of the transport matrix, consists in the determination of the optical functions, which are strongly correlated to measurable combinations of the transport matrix elements.

At low-β* LHC optics, where machine imperfections are more significant, the method allows to determine the real optics with a per mil level uncertainty, also permitting to assess the transport matrix errors from the tolerances of various machine parameters. In case of high-β* LHC optics, where the machine imperfections have smaller effect on the optical functions, the method remains effective and reduces the uncertainties to the desired per mil level. The method has been validated with the Monte Carlo studies both for high- and low-β* optics and was successfully used in the TOTEM experiment to estimate the real optics for TOTEM physics runs.

References

[16] FIDEL – The Field Description for the LHC, LHC-C-ES-0012 ver.2.0.