ON THE FREQUENCY FUNCTION OF THE PRODUCT OF TWO UNCORRELATED NORMAL VARIABLES

1. INTRODUCTION

Some time ago, my friend T. Lindelöf has proposed to me the following problem: Let $x$ and $y$ be two independent normally distributed variables, with frequency functions:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}$$  \hspace{1cm} (1.1)

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-y_0)^2}{2\sigma_y^2}}$$  \hspace{1cm} (1.2)

Find the frequency function $P(z)$ for the variable $z = xy$.

He had tried to solve it by constructing the characteristic function but soon found that the method led to complicated integrals and was abandoned.

This problem has been discussed already by statisticians and no simple solution has been found. C.C. Craig has put $P(z)$ in a closed form as the difference of two integrals which he has solved by expanding in an infinite series of powers of $z$, $x_0/\sigma_x$, $y_0/\sigma_y$ and Bessel functions. The series converges with sufficient speed to be of practical use for numerical computations.

The purpose of this note is to bring to the attention of those who are interested on this problem what is already known about the function $P(z)$; to point out that some of its properties can be established easily also without giving an explicit analytical form; and to show that $P(z)$ can be calculated from an integral expression using a simple program, to a degree of accuracy which is certainly adequate to all practical purposes.

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2. GENERAL PROPERTIES OF P(z)

From equations (1.1) and (1.2) it follows that

\[ P(z) = \frac{1}{2\pi \sigma_x \sigma_y} \int_{xy=z} e^{-L^2} \, dx \, dy \]  

(2.1)

where

\[ L^2 = \frac{1}{2} \left( \frac{(x-x_0)^2}{\sigma_x^2} + \frac{(y-y_0)^2}{\sigma_y^2} \right) \]  

(2.2)

Introducing the variables

\[ \zeta = \frac{z}{z_0}, \quad \zeta = \frac{x}{x_0}, \quad \eta = \frac{y}{y_0} \]

then

\[ L^2 = \frac{1}{2} \left( \frac{x_0^2 (\zeta - 1)^2}{\sigma_x^2} + \frac{y_0^2 (\eta - 1)^2}{\sigma_y^2} \right) \]  

(2.3)

and

\[ P(\zeta) \, d\zeta = d\zeta \frac{z_0}{2\pi \sigma_x \sigma_y} \int e^{-L^2} \frac{d\eta}{\eta} \]  

(2.4)

The mean value of \( \zeta \) is

\[ \langle \zeta \rangle = \int_{-\infty}^{+\infty} P(\zeta) \, d\zeta = \frac{z_0}{2\pi \sigma_x \sigma_y} \int_{-\infty}^{+\infty} e^{-k_x^2 (\zeta - 1)^2 \sigma_x^2} \, d\zeta = \frac{z_0}{2\pi \sigma_x \sigma_y} \int_{-\infty}^{+\infty} e^{-k_y^2 (\eta - 1)^2 \sigma_y^2} \, d\eta = 1 \]

where we have put \( k_x = \frac{x_0}{\sqrt{2} \sigma_x} \) and \( k_y = \frac{y_0}{\sqrt{2} \sigma_y} \).

Then

\[ \langle z \rangle = x_0 \quad y_0 \quad \langle \zeta \rangle = x_0 \quad y_0 = z_0 \]  

(2.5)

The standard deviation \( \sigma_z \) can be found in a similar way

\[ \sigma_z^2 = \langle (\zeta - 1)^2 \rangle = \frac{z_0}{2\pi x y} \int_{-\infty}^{+\infty} (\zeta - 1)^2 P(\zeta) \, d\zeta \]
or

\[ \sigma_z = (\sigma_x^2 \sigma_y^2 + \sigma_y^2 \sigma_x^2 + \sigma_x^2 \sigma_y^2)^{1/2} \]

The third moment is

\[ \mu_3 = \langle (\xi - 1)^3 \rangle = \frac{3}{2\pi \sigma_x \sigma_y} \int (\xi - 1)^3 P(\xi) \, d\xi \]

\[ = 6 \, z_0 \sigma_x^2 \sigma_y^2 \neq 0 \]

which means that \( P(z) \) is (positively) skewed with a skewness

\[ \gamma_1 = \frac{\mu_3}{\sigma^3_z} = \left( \frac{6 \, x_0 \sigma_x^2 \sigma_y^2}{(\sigma_x^2 \sigma_y^2 + \sigma_y^2 \sigma_x^2 + \sigma_x^2 \sigma_y^2)^{3/2}} \right) \]

Only for \( x_0 \) or \( y_0 \) (or both) \( \rightarrow 0 \), \( \gamma_1 \rightarrow 0 \), i.e. the frequency function \( P(z) \) tends to be symmetric. Even so, in general it is not Gaussian. It tends to a Gaussian distribution when one of the two mean values, say \( x_0 \), is limited and the other \( (y_0) \) tends to infinity.

3. CRAIG'S SOLUTION FOR \( P(z) \)

Let us introduce the variables and parameters

\[ u = \frac{x}{\sigma_x} \quad w = \frac{xy}{\sigma_x \sigma_y} \quad m = \frac{x_0}{\sigma_x} \quad n = \frac{y_0}{\sigma_y} \]

Then

\[ I^2 = \frac{1}{2} \left[ (u - m)^2 + (w - n)^2 \right] \]

and

\[ P(w) = e^{- \frac{1}{2} \left[ \frac{(m^2 + n^2)}{u^2} \right]} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(w, u) \, du \, dv \]

where

\[ \Phi(w, u) = - \frac{1}{2} \left[ \frac{u^2 + w^2}{u^2} \right] + \left[ \frac{mu + wh}{u} \right] \]

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The function
\[ \frac{mu + n \frac{w}{u}}{n} \]
can be expanded in a Laurent series of powers of \( u \)
\[ \sum_{\ell} f_{\ell} (u) = \frac{mu + n \frac{w}{u}}{n} \]
i.e.
\[ \psi_+ (w) = \int_0^\infty f_{\ell} (u) \frac{du}{u} \]
\[ = \int_0^\infty e - \frac{1}{2} \left( u^2 + \frac{w^2}{u^2} \right) \sum_{\ell} f_{\ell} (u) \, du \]
Using the Bessel functions of second kind of imaginary argument
\[ K_{\ell} (w) = w^{-1/2} e^{-w} \int_0^\infty e - \frac{1}{2} (u^2 + \frac{w^2}{u^2}) \frac{du}{u} \]
and remembering that the \( f_{\ell} \)'s are of an expansion in series of \( u \), one obtains
\[ \psi_+ (w) = S_0 K_0 + (m + n) w^{1/2} S_1 K_{1/2} + (m^2 + n^2) \frac{w^2}{2} S_2 K_1 + \ldots \]
where
\[ S_{k} = \sum_{r=0}^{\infty} \frac{(mnw)^r}{r (k+r)^r} \]
In a similar way one can expand \( \psi_- \). Here the sign of the even terms is reversed.

In conclusion,
\[ P(z) = \frac{\frac{1}{2} (m^2 + n^2)}{\pi \sigma_x \sigma_y} \left[ S_0 K_0 + (m^2 + n^2) \frac{w}{2} S_1 K_{1/2} + (m^4 + n^4) \frac{w^2}{4} S_2 K_1 + \ldots \right] \]
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For \( m = n = 0 \) the function \( P(z) \) reduces to

\[
P(z) = \frac{1}{\pi \sqrt{x} \sqrt{y}} K_0 \left( \frac{z}{2} \right)
\]

a result which can be obtained from direct integration of eq. 2.4.

4. NUMERICAL VALUES FOR A SPECIAL CASE, AS AN EXAMPLE

Values of \( P(z) \) for \( x_0/\sigma_x = y_0/\sigma_y = 0 \) have been derived integrating numerically eq. 2.4 using a very simple program. The result is given in Fig. 1 and it is compared to that of eq. 3.2. As one can see the approximation is good enough for all practical purposes.

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REFERENCES

1) C.C. Craig - Annals of Math. Stat., 7, 1, 1936


FIGURE CAPTIONS

Fig. 1  $P(z)$ for $x_0 = y_0 = 0$. Full curve: from ref. (1); dotted curve: present calculation.
$\sigma_x = \sigma_y = 1$
$X_0 = Y_0 = 0$