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A RESONANCE INDUCED BY SPACE-CHARGE FORCES

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1) INTRODUCTION

In April 1965 we put into the C.P.S. two sets of new quadrupoles, placed in 25 equally spaced mid-F straight sections and 25 equally spaced mid-D straight sections, and intended for the adjustment of the Q values at injection. It was opedh that the effect of these lenses would be slightly better than that of the two sets of ten which they were intended to replace. In fact the contrary was found; it became noticea[illegible] more difficult to find the conditions for ac[illegible]rating the maximum intensity.

Such an arrangement of lenses int[illegible]ces a substantial 25th harmonic perturbation. With nominal Q values of 6.25 this is the appropriate harmonic for exciting the quarter-integer resonance, but normally one would only expect this to occur if the lenses contained a substantial cubic (octupole) term in their field, and these new quadrupoles were sufficiently linear to make this mechanism seem unlikely. We therefore asked ourselves whether a linear 25th harmonic perturbation, in the presence of space-charge forces, can excite a $6^1/_4$ sub-resonance. The small-amplitude analysis that follows shows that this can occur even for a beam of uniform density (so that the space-charge force is linear in the particle displacement): the dependence of space-charge force on the beam diameter plays an essential part in the process. A computer programme has been written for investigating the large-amplitude behaviour.

2) SMALL-AMPLITUDE ANALYSIS

We follow Lloyd Smith (1) in considering a round beam of uniform density, and measuring its radius in units of that of a matched, unperturbed beam. The envelope equation is then:

$$\rho'' + Q^2(\rho - \rho^{-3}) = 2Q\,\Delta Q\,\rho^{-1} - V\cos n\theta\cdot\rho \qquad [1]$$

where ΔQ is the space-charge Q-depression for a single particle in a beam constrained to $\rho = 1$. For $V = 0$ this equation has a constant solution. ρ_o, so that we change the variable to

$$\xi = \rho/\rho_o - 1\,,$$

then [1] becomes

$$\xi'' + Q^2\xi + Q^2\rho_o^{-4}\,[1-(1+\xi)^{-3}] + $$
$$+ 2Q\Delta Q\,\rho_o^{-2}\,[1-(1+\xi)^{-1}] + V\cos n\theta\cdot\xi = -V\cos n\theta \qquad [2]$$

Treating the linear periodic term $V\cos n\theta\,.\,\xi$ and all non-linear terms as perturbations, the unperturbed equation has a periodic solution

$$\xi_o = \frac{V}{n^2 - 4\,Q_1^2}\cos n\theta \qquad [3]$$

where Q_1 is the effective Q, including space-charge, for oscillations of the beam envelope. It is assumed that the denominator $n^2 - 4\,Q_1^2$ is not too small, i.e. that we are not too close to an ordinary linear integral of half-integral stop-band. After a certain amount of manipulation one obtains the equation for small free oscillations about the solution [3], accurate to first order in V and in ΔQ:

$$(\Delta\xi)'' + \left[(2Q_1)^2 + \left(1 - \frac{12\,Q_1^2 - 8\,Q\Delta Q}{n^2 - 4\,Q_1^2}\right) V \cos n\theta\right] \Delta\xi = 0 \qquad [4]$$

If we put n equal to $4Q_1$, this becomes a Mathieu equation at its lowest stop-band,

$$(\Delta\xi)'' + \left[(2\,Q_1)^2 + \frac{2}{3}\frac{\Delta Q}{Q_1} V \cos n\theta\right] \Delta\xi = 0 \qquad [5]$$

and the numerical values in the C.P.S. at injection, with the new quadrupoles in operation, give growth by a factor e in about a millisecond. It should perhaps be recalled that this is a factor e not in the beam diameter, but in the amplitude of free oscillations of the beam envelope, initiated by any mismatch between the injected beam and the shape of the synchrotron's acceptance ellipse; provided the beam can be injected very close to matched, a rather large growth factor will be required before any particles are lost.

3) COMPUTER PROGRAMME

The behaviour of these oscillations for finite amplitudes and the generalization to the cases where the radial and vertical oscillations are not necessarily the same are problems that are conveniently studied numerically. The computer programme which we have written for this purpose is perhaps of some interest in that we work by matrix multiplication rather than with the differential equations corresponding to [1].

The shape of the ellipse occupied by the beam in the phase-plane for the vertical motion is represented by the matrix

$$\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \qquad [6]$$

which is, by definition, a matrix which transforms the unit circle into the ellipse in question. Another matrix represents the radial ellipse in the same way. In the absence of space-charge the evolution of these beam-shape ellipses is rather straight-forward to compute, for it is evident that they should just be multiplied (from the left) by the matrix belonging to each machine element (magnet sector or quadrupole lens) which the beam traverses. Space-carge can conveniently be introduced by treating it as short defocusing lenses in every straight-section

$$\begin{vmatrix} 1 & 0 \\ D & 1 \end{vmatrix} \qquad [7]$$

with the strength D depending on the beam dimensions by a factor

$$\frac{1}{\rho_z(\rho_z + \rho_y)} \qquad [8]$$

for the vertical (z) motion, and correspondingly for the radial. Each time, the beam dimensions are obtained from the matrix representing the ellipse shape by way of

$$\rho_z^2 = V_{11}^2 + V_{12}^2\text{, etc.} \qquad [9]$$

With this programme we have confirmed the two factors,1/2 and 3/4, found by Lloyd Smith (1) in the space-charge frequency shifts for small symmetric and antisymmetric envelope oscillations, and begun an investigation of the behaviour near $4Q_1 = 25$ in the C.P.S.

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