ON A DEFINITE INTEGRAL OF
THE PRODUCT OF TWO POLYLOGARITHMS

K.S. Kölblig

Invited talk given at the
Fourth Int. Conf. on Computer Algebra in Physical Research,
Dubna, USSR, 22–26 May 1990
ON A DEFINITE INTEGRAL OF THE PRODUCT OF TWO POLYLOGARITHMS

K.S. Kölblig

CERN, 1211 Geneva 23, Switzerland

1. Introduction

Indefinite integration of elementary functions is now, due to the powerful Risch algorithm, an established part of most symbolic algebra systems. For the integration of special functions, however, in both the indefinite and definite case, only limited progress towards symbolic evaluation has been achieved. For example, Piquette and Van Buren [7] and Piquette [6] have presented a technique for the evaluation of indefinite integrals containing products of Bessel, Legendre, and Hermite functions which can be used for symbolic computation. Geddes and Scott [2] have discussed the evaluation of definite integrals which can be expressed as repeated derivatives of products of quotients of gamma and zeta functions. Other results in this area have been presented by the present author [3], [4].

Taking into account the many disparate methods which are used for the evaluation of definite integrals, it is hardly surprising that their application in symbolic algebra systems is still only just beginning. Serious obstacles are encountered when analysing systematically the known results for certain integrals with a view to their suitability for symbolic algebra; in particular when these results contain "infinite" expressions- for example power series with parameters. The following integral illustrates some of these problems.

2. The integral

We consider the integral

\[ I_{mn}(\alpha, \sigma, \omega, r) = \int_0^\infty x^{\alpha-1} Li_n(-\sigma x)Li_m(-\omega x^r) dx \] (2.1)

where

\[ Li_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k} \quad (k \geq 2, |z| \leq 1) \] (2.2)

is the polylogarithm of order \( k \), with
\[ L_i(z) = -\ln(1-z) \]  

(2.3)

as a degenerate case. This integral can be transformed into a Mellin-Barnes integral which can 
be evaluated by residue techniques \([1]\). We give here the result for convenience:

**Theorem.** Let \( r \neq 0 \) be real, \( \alpha, \sigma \neq 0, \omega \neq 0 \) be complex with \( |\arg \alpha| < \pi, |\arg \omega| < \pi \) and let 
\( \zeta = \sigma \omega^{-1}r \). Further, with \( \mathbb{N} = \{1,2,3,\ldots\} \), let \( \mathbb{N}_0 = \{0, \mathbb{N}\} \),

\[
\begin{align*}
    h(x) &= \begin{cases} 
    1 & \text{if } x \in \mathbb{N}, \\
    0 & \text{otherwise,}
    \end{cases} \\
    \bar{h}(x) &= 1 - h(x); \\
    H(x) &= \begin{cases} 
    1 & \text{if } x > 0, \\
    0 & \text{if } x < 0,
    \end{cases}
\end{align*}
\]

and let \( k, l, K \in \mathbb{N} \).

Then, for \( |\zeta| \neq 1 \) and

\[
-1 - r < \Re \alpha < 0 \quad \text{if } r > 0, \\
-1 < \Re \alpha < -r \quad \text{if } r < 0,
\]

the integral (2.1) can, for \( \alpha \neq 0 \), be expressed in the form

\[
\begin{align*}
    &\int_0^\infty x^{\sigma - 1} L_i(-\sigma x) L_i(-\alpha x) dx \\
    &= (1)\sum_{|r|K \neq 0} (-1)^{r} \sum_{|r|K \neq 0} (-1)^{r} \csc \left[ \pi (\alpha \pm |r|K)(\alpha \pm |r|K) \right]^{n} \\
    &+ (1)\sum_{|r|K \neq 0} (-1)^{r} \sum_{|r|K \neq 0} (-1)^{r} \csc \left[ \pi (\alpha \pm |r|K)(\alpha \pm |r|K) \right]^{n} \\
    &+ (1)\sum_{|r|K \neq 0} (-1)^{r} \sum_{|r|K \neq 0} (-1)^{r} \csc \left[ \pi (\alpha \pm |r|K)(\alpha \pm |r|K) \right]^{n} \\
    &+ (1)\sum_{|r|K \neq 0} (-1)^{r} \sum_{|r|K \neq 0} (-1)^{r} \csc \left[ \pi (\alpha \pm |r|K)(\alpha \pm |r|K) \right]^{n} \\
\end{align*}
\]

(2.4)
\( + H(\pm 1)(-1)^{m'} \omega^{-s} \pi^{n+1} r^{m_n - m} \)
\[
\left\{ \ln \left( -\frac{\alpha}{|r|} \right) (-1)^{n + n'} \sum_{n_{1}} \frac{1}{n_{1}!} \left( -\frac{1}{\pi \ln \zeta} \right)^{n_{1}} \sum_{\begin{smallmatrix} n_{2} = 0 \\ m_{2} = 0 \end{smallmatrix}} \left( \begin{array}{c} n_{2} + m_{2} - 1 \\ m_{2} - 1 \end{array} \right) (\pi \alpha)^{n_{2}} \sum_{\begin{smallmatrix} n_{3} = 0 \\ m_{3} = 0 \end{smallmatrix}} r^{-n_{3}} B^*_{n_{3}} B^*_{m_{3} + n_{1} - n_{2} - n_{3}} \right) \left( -\frac{\alpha}{r} \right)^{n_{2}} \sum_{n_{3} = 0} (r^{-n_{3}} B^*_{n_{3}} C_{n_{1} - n_{2} - n_{3}} \left( \frac{\alpha}{r} \right)) \right\},
\]

and for \( \alpha = 0 \) (which implies \( r < 0 \)), in the form

\[
\left[ \int_{0}^{\infty} Li_{n}( - \sigma x) Li_{m}( - \omega x) \frac{dx}{x} \right] = - (\mp 1)^{m + n} \left\{ |r|^{-n} \sum_{k \in \mathbb{N}} (\pi |r|) k^{-m + n} \zeta^{-1} r^{k} \right. \\
+ |r|^{-m - 1} \sum_{l \in \mathbb{N}} (-1)^{l} \csc \left( \pi \frac{l}{|r|} \right) r^{-l - m + n} \zeta^{-l} r^{l} \\
- |r|^{-n} \sum_{K \in \mathbb{N}} (-1)^{K+1} K^{-l} k^{-m + n} \zeta^{-1} r^{-K} \left( \frac{m + n}{|r|} \pm \ln \zeta \right) \left\} \right.
\]
\[
+ H(\pm 1)(-1)^{n + 1} \pi^{m + n + 1} r^{m_{1}} \sum_{m_{1} = 0} \frac{1}{m_{1}!} \left( -\frac{1}{\pi \ln \zeta} \right)^{m_{1}} \sum_{m_{2} = 0} r^{-m_{2}} B^*_{m_{2}} B^*_{m + n + 1 - m_{1} - m_{2}},
\]

where

\[
B^*_{j} = \frac{|2^j - 2|}{j!} |B_{j}|,
\]

\( B_{j} \) are Bernoulli numbers, where

\[
C_{j}(x) = \left. \frac{1}{j!} \frac{d^{j}}{ds^{j}} \csc (s + x) \right|_{s = 0},
\]

and where the logarithm is defined on its principal sheet.

In all the expressions above, the upper sign corresponds to \( |\zeta| > 1 \), the lower sign to \( |\zeta| < 1 \).

Although the Theorem is here given with the restriction \( |\zeta| \neq 1 \), it has been shown [1] that, at least for \( \sigma = \omega = 1 \) (and hence \( \zeta = 1 \)), the integral is continuous. When looking at this formula,
two facts become apparent: The formula is involved, but it is, at least in certain sections, suited for symbolic evaluation. More problematic, however, is the appearance of three rather complicated infinite series over \( k, \ i, \) and \( K \) which \textit{a priori} seem to defeat all attempts at systematic treatment. In order to evaluate the integral completely by symbolic algebra using the above formula, a program would have to be written which could recognize whether or not, for a given set of parameters, the above series can be expressed in terms of a finite number of known functions. This would certainly be difficult, and might even be impossible.

In the absence of such a general approach, the only way to obtain specific results from (2.4) and (2.5) consists of trying to evaluate the series analytically, possibly using symbolic algebra for subsidiary calculations. Incidentally, proceeding in this way may yield mathematical insight or results which would be lost by a completely automatic approach. Thus, special cases discussed in [1] demonstrate that quite interesting results can be extracted from (2.4) and (2.5). For example,

\[
\int_0^\infty x^{-3} Li_n(-x)Li_n(-x^2)dx = 2e^{-2\pi} \quad (n \geq 1)
\]

and

\[
\int_0^\infty Li_n(-x)Li_m(-x^{-2}) \frac{dx}{x} = 2^{- (n + 2)} \frac{\pi^{n + m + 1}}{(n + m + 1)!} \left\{ |E_{n + m - 1}| - \frac{2^{n + m + 1} - 2}{n + m + 1} |B_{n + m + 1}| \right\}
\]

\((n, m \in \mathbb{N}, n + m \text{ odd}).\)

3. Special cases

The special cases \( r = -2, \ \alpha = 0; \ r = 2, \ \alpha = -2; \ r = 1, \ \alpha = -\frac{1}{2} \) have already been discussed in [1]. Here we consider some additional special cases. In order to simplify the formulae we shall set, at a certain stage, \( \sigma = \omega = 1, \) and thus \( \zeta = 1. \)

3.1 \( r = -1, \ \alpha = \frac{1}{4}. \) The summation conditions for the infinite series become

\( k \pm \frac{1}{4} \notin \mathbb{N}_0, \ i \notin \frac{1}{4} \notin \mathbb{N}_0, \ K \pm \frac{1}{4} \in \mathbb{N} \)

which permit all \( k \) and \( i, \) but exclude all \( K. \) Thus the series in (2.4) can be written as

\[
\sum_{k=1}^{\infty} \zeta^k k^{-m(4k \pm 1)^{-n}}, \quad \sum_{l=1}^{\infty} \zeta^l l^{-m(4l \pm 1)^{-n}}.
\]  

(3.1)

Starting from the formulae [8] (No. 5.2.5.11, 13)

\[
\sum_{k=0}^{\infty} \frac{x^{4k+v}}{4k+v} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} x \arctan x
\]
(ν = 1, 3; ε₁ = 1, ε₃ = -1) we obtain by repeated integration, using (2.2), for j = 1, 2, 3, ..., 

\[
\sum_{k=1}^{\infty} \frac{x^{4k}}{(4k \pm 1)} = \frac{1}{4} x^T \left( L_j(x) - L_j(-x) \pm 2 T_j(x) \right) \bigg|_0^1
\]

where \( T_j(x) \) is the generalized arctangent integral (Lewin [5], p. 299):

\[
T_i(x) = \int_0^x t^{-1} \arctan(t) \, dt, \quad T_j(x) = \int_0^x t^{-1} T_{j-1}(t) \, dt.
\]

By decomposing the series (3.1) into partial fractions, and using (2.2), we can express these series in closed form. This procedure is tedious, but is well-suited to a system like REDUCE. For reasons of simplicity, we have constructed such a program for \( j \leq 2 \) only. Setting \( \sigma = \omega = 1 \), and noting that \( L_2(1) = \pi^2/6, L_2(-1) = -\pi^2/12 \), we find

\[
\int_0^{\infty} x^{-3/4} \ln(1 + x) \ln(1 + \frac{1}{x}) \, dx = 8\pi \sqrt{2} (3 \ln 2 - 2)
\]

\[
- \int_0^{\infty} x^{-3/4} \ln(1 + \frac{1}{x}) L_1(-x) \, dx = 2\pi \sqrt{2} \left( \frac{5}{3} \pi^2 - 16(3 \ln 2 + G - 2) \right)
\]

\[
- \int_0^{\infty} x^{-3/4} \ln(1 + x) L_2(-\frac{1}{x}) \, dx = 2\pi \sqrt{2} \left( \frac{5}{3} \pi^2 + 16(3 \ln 2 + G - 4) \right)
\]

\[
\int_0^{\infty} x^{-3/4} L_2(-x) L_2(-\frac{1}{x}) \, dx = 256\pi \sqrt{2} (3 - 3 \ln 2 - G)
\]

where

\[
G = T_i(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2} = 0.9159655941...
\]

is Catalan's constant.

3.2 \( r = -1, \alpha = \frac{1}{2} \). The summation conditions become

\[
k \pm \frac{1}{2} \notin \mathbb{N}_0, \quad l \mp \frac{1}{2} \notin \mathbb{N}_0, \quad K \pm \frac{1}{2} \in \mathbb{N}
\]

which permit all \( k \) and \( l \), but exclude all \( K \). The series can be written as

\[
\sum_{k=1}^{\infty} \zeta^T k^{-m(2k \pm 1)^{-n}}, \quad \sum_{l=1}^{\infty} \zeta^T l^{-n(2l \mp 1)^{-m}}. \quad (3.2)
\]
We find for $j = 1, 2, 3, ...$

\[
\sum_{k=1}^{\infty} \frac{x^{2k}}{(2k \pm 1)!} = \frac{1}{2} x^{\mp 1} (L_i(x) - L_i(-x)) - \begin{cases} 1 \\ 0 \end{cases}
\]

and the series (3.2) can be obtained in closed form by decomposition into partial fractions. For $j \leq 2$, $\sigma = \omega = 1$, we find

\[
\begin{align*}
\int_0^\infty x^{-1/2} \ln(1 + x) \ln(1 + \frac{1}{x}) dx &= 4\pi(2 \ln 2 - 1) \\
- \int_0^\infty x^{-1/2} \ln(1 + \frac{1}{x}) Li_2(-x) dx &= 2\pi \left( \frac{1}{3} \pi^2 - 8 \ln 2 + 4 \right) \\
- \int_0^\infty x^{-1/2} \ln(1 + x) Li_2(-\frac{1}{x}) dx &= 2\pi \left( \frac{1}{3} \pi^2 + 8 \ln 2 - 8 \right) \\
\int_0^\infty x^{-1/2} Li_2(-x) Li_2(-\frac{1}{x}) dx &= 16\pi(3 - 4 \ln 2).
\end{align*}
\]

### 3.3 $r = 1, \alpha = -1$. The summation conditions become

\[
k \mp 1 \not\equiv 0, \quad l \pm 1 \not\equiv 0, \quad K \equiv 1 \in \mathbb{N}
\]

which exclude all $k$ and $l$, but permit all $K \geq 2$ if $|\zeta| > 1$, and all $K$ if $|\zeta| < 1$. Because of the continuity of the integral at $\zeta = 1$, we set $\sigma = \omega = 1$ for simplicity, and obtain (choosing arbitrarily the case $|\zeta| > 1$) the series

\[
- m \sum_{K=1}^{\infty} K^{-n}(K + 1)^{-(m+1)} - n \sum_{K=1}^{\infty} K^{-(n+1)}(K + 1)^{-m}.
\]

Using formula [8] (No. 5.1.24.8), which expresses these series in closed form, we obtain by using REDUCE the following examples:

\[
\begin{align*}
\int_0^\infty [x^{-1} \ln(1 + x)]^2 dx &= \frac{1}{3} \pi^2 \\
- \int_0^\infty x^{-2} \ln(1 + x) Li_2(-x) dx &= \frac{1}{3} \pi^2 + 2\zeta(3) \\
- \int_0^\infty x^{-2} \ln(1 + x) Li_3(-x) dx &= \frac{1}{30} \pi^4 + \frac{1}{3} \pi^2 + 2\zeta(3) \\
- \int_0^\infty x^{-2} \ln(1 + x) Li_4(-x) dx &= \frac{1}{30} \pi^4 + \frac{1}{3} \pi^2 + 4\zeta(5) + 2\zeta(3) \\
\int_0^\infty x^{-2} Li_2(-x) Li_3(-x) dx &= \frac{1}{30} \pi^4 + \pi^2 + 6\zeta(3) \\
\int_0^\infty x^{-2} Li_3(-x) Li_4(-x) dx &= \frac{2}{15} \pi^4 + \frac{10}{3} \pi^2 + 4\zeta(5) + 20\zeta(3).
\end{align*}
\]
Using well-known properties of $L^a(z)$, one finds by partial integration that for $r = 1, \alpha = -1,$ and all $\sigma, \omega$

$$I_{n,m} = I_{n-1,m} + I_{n,m-1}.$$  

In particular,

$$I_{n,n} = 2I_{n-1,n}.$$  

**3.4** $r = -p = -1, -2, -3, \ldots, \alpha = 0$. The summation conditions become

$$pk \not\in \mathbb{N}, \ l/p \not\in \mathbb{N}, \ pK \in \mathbb{N}$$

which exclude all $k$, but permit all $l$ which are not multiples of $p$, and all $K$. The case $p = 1$ can be handled with polylogarithms; in particular for $\zeta = 1$ we obtain

$$\int_0^\infty Li_n(-x)Li_m\left(\frac{-1}{x}\right) \frac{dx}{x} = (m+n)\zeta(m+n+1).$$

The case $p = 2$ has been discussed in [1]. For all $p$, the series over $K$ can be expressed in terms of polylogarithms. The series over $l$, however, leads for $p = 3$ and $\zeta = 1$ to

$$\sum_{j=0}^{\infty} (3j+1)^{-1} (m+n) - \sum_{j=0}^{\infty} (3j+2)^{-1} (m+n). \quad (3.4)$$

Considering [8] (No. 5.2.4.9, 10)

$$\sum_{j=0}^{\infty} \frac{x^{3j+1}}{3j+1} = \frac{1}{6} \ln(1 + x + x^2) - \frac{1}{3} \ln(1 - x) + \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{x + 2},$$

(and a similar formula for $3j+2$), it does not seem likely that (3.4) can be expressed in terms of elementary functions for $m + n \geq 2$. The same is probably true for $p \geq 4$.

**3.5** $r = -2, \alpha = -\frac{3}{2}$. The summation conditions become

$$2k \mp \frac{3}{2} \not\in \mathbb{N}_0, \ \frac{1}{2} \left(l \pm \frac{3}{2}\right) \not\in \mathbb{N}_0, \ 2K \mp \frac{3}{2} \in \mathbb{N}$$

which permit all $k$ and $l$, but exclude all $K$. The series over $k$ is

$$\sum_{k=0}^{\infty} (-1)^k \zeta^{2k} k^{-m} (4k \mp 3)^{-n},$$
which, because of [8] (No. 5.2.4.12, 14)

\[ \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+v}}{4k+v} = \frac{1}{2\sqrt{2}} \arctan \frac{x\sqrt{2}}{1-x^2} + \frac{1}{4\sqrt{2}} \varepsilon_v \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}, \]

\( (v = 1, 3; \varepsilon_1 = 1, \varepsilon_3 = -1) \) is unlikely to be expressible in closed form. The sine factor in the series over \( l \) gives rise to a sequence \( ++--... \) of signs for consecutive terms which probably excludes closed summation for this series too.

The last two examples (and others) show that the infinite series in (2.4) and (2.5) are likely to be representable in terms of known functions only for a very restricted set of parameters \( r \) and \( \alpha \). Further investigation will be necessary in order to get a more general picture.

References


