Computing Wake Functions of Convex Structures

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Abstract

A time-saving method of time domain computation of the wake fields is proposed for convex structures, like collimators, which have an aperture smaller than the attached beam pipe radius.

Geneva, Switzerland
July 1990

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When we compute the wake function for concave structures like rf cavities using time-domain computer codes such as TBCI[1], we normally integrate the field along $r = a$, $a$ being the radius of the attached beam pipe, because the integrand becomes zero after the bunch gets out of the cavity. (Whenever we say 'bunch' in this paper, it includes the location of the test particle which can be far behind the real bunch.) Since the longitudinal wake function $W_z(r, z)$ is exactly proportional to $r^m$ where $m$ is the azimuthal mode number, we can easily find the wake function by this recipe.

It cannot be applied, however, to a convex structure, like collimators, which has an aperture smaller than the attached beam pipe radius. In this case we have to make the final beam pipe very long at the expense of a long computing time.

The present note provides a method of integrating the effect of infinitely long final beam pipe. Aharonian, Meller and Sieman[2] have considered the same problem and given a prescription of the integration. We shall give different formulas which are much simpler. We consider ultrarelativistic particles only.

First, let us consider the longitudinal wake function, which is defined by (positive for acceleration)

$$W_z(r, \zeta) = \int_{-\infty}^{\infty} dz E_z(r, z, z + \zeta) = \int_{-\infty}^{\infty} dt E_z(r, t - \zeta, t)$$  \hspace{1cm} (1)

where $E_z$ is the longitudinal electric field. (We measure the time $t$ in units of length by multiplying the velocity of light.) In this paper we consider the radiated field only. The direct source field does not contribute to the wake function for ultrarelativistic particles. Suppose that we have integrated this expression upto time $t_0$ by a time domain code. What we want now is the rest of the integration;

$$\Delta W_z(r, \zeta) = \int_{t_0}^{\infty} dt E_z(r, t - \zeta, t).$$  \hspace{1cm} (2)

We assume that, at $t = t_0$, the entire bunch is already in the final beam pipe, i.e.,

$t_0 - \zeta > z_0$, $z_0$ being a point somewhere in (preferably near the beginning of) the final beam pipe. The field at $z < z_0$ cannot contribute to $\Delta W_z$ because we are assuming ultrarelativistic particles. Therefore, we expect that $\Delta W_z(r, \zeta)$ can be expressed as a convolution of the field at $(r', z', t_0)$ ($z' > t_0 - \zeta$) with an appropriate kernel $K(r, \zeta, r', z').$

Since we are considering the fields for $z > z_0$, we expand the field using Laplace transformation with respect to $z$ (more rigorously, $z - t$ is better);

$$E_z(r, z, t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} dq e^{q(z-z_0)} \sum_{m, \pm} \sum_{q} R_{m, \pm}(\frac{r}{a}) e^{\mp i\omega_s(q)(t-t_0)} F^{\pm}_m(q)$$  \hspace{1cm} (3)

Here,

$$\omega_s(q) = \sqrt{-q^2 + (j_s/a)^2}$$  \hspace{1cm} (4)
\[ R_{m,s}(x) = \frac{\sqrt{2} J_m(j_s x)}{J_m'(j_s)} \quad (0 \leq x \leq 1), \quad (5) \]

where \( m \) is the azimuthal mode number, \( j_s \) the \( s \)-th zero of the Bessel function \( J_m(z) \). We omit the factor \( \cos m \phi \). The convergence coordinate \( \lambda \) can be taken to be 0—, assuming that the integrand has no singularity in \( \Re q \geq 0 \). (As will be seen later, the branch point due to \( \omega_s(q) \) disappears after summing over \( \pm \).) By using the orthogonality condition of the function \( R_{m,s} \)

\[ \int_0^1 xdx R_{m,s}(x) R_{m,s'}(x) = \delta_{s,s'}, \quad (6) \]

and the Laplace transformation, we can express the coefficient \( F_s^\pm \) in terms of the field at \( t = t_0 \) as

\[ F_s^\pm(q) = \int_{s_0}^\infty dz e^{-q(z-s_0)} \int_0^\infty \frac{rdr}{a^2} R_{m,s}(\frac{r}{a}) \frac{1}{2} \left[ E_z(r, z, t_0) \pm \frac{i}{\omega_s(q)} \hat{E}_z(r, z, t_0) \right], \quad (7) \]

where the dot denotes the partial time derivative. There is no convergence problem of the integration to \( z = \infty \) even for \( \Re q < 0 \) because there is no field at sufficiently large \( z \) at \( t = t_0 \).

Substituting eq.(3) and eq.(7) into eq.(2), integrating over \( t \) and summing over \( \pm \), we get

\[ \Delta W_s(r, \zeta) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} dq \int_{s_0}^\infty dz \int_0^a r'dr' \sum_s \frac{1}{j_s^2} R_{m,s}(\frac{r}{a}) R_{m,s}(\frac{r'}{a}) \]
\[ \times e^{q(t_0-\zeta)} \left[ -q E_z(r, z, t_0) + \hat{E}_z(r, z, t_0) \right]. \quad (8) \]

After integrating over \( q \) and \( z \) successively, we obtain a simple formula

\[ \Delta W_s(r, \zeta) = \int_0^a r'dr' K_m(\frac{r}{a}, \frac{r'}{a}) \left[ -\frac{\partial E_z}{\partial t} + \frac{\partial E_z}{\partial t} \right](r', t_0-\zeta, t_0), \quad (9) \]

with

\[ K_m(x, y) = \sum_s \frac{1}{j_s^2} R_{m,s}(x) R_{m,s}(y). \quad (10) \]

Because of the completeness of the orthonormal basis \( R_{m,s} \), we have

\[ \sum_s R_{m,s}(x) R_{m,s}(y) = \frac{1}{x} \delta(x - y), \quad (11) \]

from which we get an explicit form of the kernel \( K_m \);

\[ K_m(x, y) = \begin{cases} -\log \max(x, y) & (m = 0) \\ \frac{1}{2m} \left[ \left( \frac{\min(x, y)}{\max(x, y)} \right)^m - (xy)^m \right] & (m \geq 1) \end{cases} \quad (12) \]
Note that \( z = t_0 - \zeta \) is just the \( z \)-coordinate at which the time domain integration for \( \zeta \) was quitted. The partial time derivative in eq.(9) can be evaluated either by taking difference from the previous time step or by using \( \dot{E}_z = (\text{rot} \mathbf{B})_z \).

As is expected, \( \Delta W_z \) does not depend on the fields at \( z < t_0 - \zeta \) because they cannot catch up the beam. What is surprising, however, is that it does not depend on the fields at \( z > t_0 - \zeta \) either. We have only to integrate over \( r' \). Physically speaking, a field ahead of the beam should eventually have some influence on the relativistic beam as long as the field has non-vanishing transverse component of the wave number. This apparent contradiction may be explained as follows. If there is a wave packet (consisting of wave guide modes) ahead of the beam at \( t = t_0 \) and it has no tail at the position of the test particle, the particle will go through the entire wave packet eventually so that the integrated effect is zero after all. The point is that the non-zero contribution to \( \Delta W_z \) comes from integration over half infinite interval.

The transverse wake can be obtained by the formula derived from the Panofsky-Wenzel theorem\[3]\:

\[
W^{(m)}(r, \zeta) = -\frac{1}{r} \int_\zeta^z W^{(m)}(r, \zeta) d\zeta. \tag{13}
\]

This formula is not valid, however, in some cases (e.g., when the initial and the final beam pipes have different radii). We need direct extrapolation formulas for the transverse wakes

\[
W_r(r, \zeta) = \int_{-\infty}^{\infty} dt \left[ E_r(r, t - \zeta, t) - E_\phi(r, t - \zeta, t) \right] \tag{14}
\]

\[
W_\phi(r, \zeta) = \int_{-\infty}^{\infty} dt \left[ E_\phi(r, t - \zeta, t) + B_r(r, t - \zeta, t) \right]. \tag{15}
\]

We will describe their derivation only briefly because it is quite similar to that of \( \Delta W_z \). The longitudinal magnetic field can be expanded as

\[
B_z(r, z, t) = \frac{1}{2\pi} \int_{\lambda - i\infty}^{\lambda + i\infty} dq e^{i(z - z_0)} \sum_s \sum_{\pm} S_m(s) \frac{r}{a} e^{\mp i \kappa_s(q)(t - t_0)} G^\pm_s(q) \tag{16}
\]

with

\[
\kappa_s(q) = \sqrt{-q^2 + (\mu_s/a)^2} \tag{17}
\]

\[
S_m(s)(x) = \frac{\sqrt{2} J_m(\mu_s x)}{J_m(\mu_s) \sqrt{1 - m^2/\mu_s^2}} \quad (0 \leq x \leq 1, m \geq 1), \tag{18}
\]

where \( \mu_s \) is the \( s \)-th zero of \( J_m'(x) \) and \( S_m(s) \) satisfies the same normalization equation eq.(6) as \( R_m(s) \). The coefficient \( G^\pm_s \) can be written as eq.(7) with \( R_m(s), E_z \) and \( \omega_s(q) \) replaced by \( S_m(s), B_z \) and \( \kappa_s(q) \).

By using \( F^\pm_s \) and \( G^\pm_s \), which represent TM and TE modes respectively, other components of the radiated field can be written in the form (apart from factors \( \sin m\phi \)
and \( \cos m\phi \)

\[
\frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} dq e^{q(z-z_0)} \sum_{\pm} \sum_{s} \left[ P F_s^\pm(q) e^{\mp i\omega_s(q)(t-t_0)} + Q G_s^\pm(q) e^{\mp i\kappa_s(q)(t-t_0)} \right]
\]

where

\[
P = \left( qa/j_s^2 \right) R_{s,m,s} \quad Q = \pm \left( i\kappa_s(q) ma^2 / \mu_s^2 r \right) S_{s,m,s} \quad \text{(for } E_r \text{)}
\]

\[
-\left( q m a^2 / j_s^2 r \right) R_{s,m,s} \quad \mp \left( i\kappa_s(q) a / \mu_s^2 r \right) S_{s,m,s} \quad \text{(for } E_\phi \text{)}
\]

\[
\pm \left( i\omega_s(q) ma^2 / j_s^2 r \right) R_{s,m,s} \quad + \left( qa/j_s^2 r \right) S_{s,m,s} \quad \text{(for } B_r \text{)}
\]

\[
\pm \left( i\omega_s(q) a / j_s^2 r \right) R_{s,m,s} \quad + \left( q m a^2 / \mu_s^2 r \right) S_{s,m,s} \quad \text{(for } B_\phi \text{)}
\]

Here, \( R_{s,m,s} \) and \( S_{s,m,s} \) are evaluated at \( r/a \).

Using these formulas, we obtain the extrapolation formulas for the transverse wakes;

\[
\Delta W_r(r, \zeta) \equiv \int_{t_0}^\infty dt (E_r - B_\phi)(r, t-\zeta, t)
\]

\[
= \int_0^a r' dr' \left[ -\frac{1}{a} K'_m(-\frac{r}{a}, \frac{r'}{a}) E_z(r', z, t_0) + \frac{m}{r} L_m(-\frac{r}{a}, \frac{r'}{a}) B_z(r', z, t_0) \right]_{z=t_0-\zeta}
\]

\[
\Delta W_\phi(r, \zeta) \equiv \int_{t_0}^\infty dt (E_\phi + B_r)(r, t-\zeta, t)
\]

\[
= \int_0^a r' dr' \left[ \frac{m}{r} K_m(-\frac{r}{a}, \frac{r'}{a}) E_z(r', z, t_0) - \frac{1}{a} L'_m(-\frac{r}{a}, \frac{r'}{a}) B_z(r', z, t_0) \right]_{z=t_0-\zeta}
\]

where the kernel \( L_m \) is defined by

\[
L_m(x, y) = \sum_{s} \frac{1}{\mu_s^2} S_{s,m,s}(x) S_{s,m,s}(y)
\]

and the primes on \( K_m \) and \( L_m \) are the derivative with respect to the first argument. Explicit forms of the kernels are \((m \geq 1)\)

\[
K'_m(x, y) = \frac{1}{2x} \left[ -\text{sgn}(x-y) \left( \frac{\min(x,y)}{\max(x,y)} \right)^m - (xy)^m \right]
\]

\[
L_m(x, y) = \frac{1}{2m} \left[ \left( \frac{\min(x,y)}{\max(x,y)} \right)^m + (xy)^m \right]
\]

\[
L'_m(x, y) = \frac{1}{2x} \left[ -\text{sgn}(x-y) \left( \frac{\min(x,y)}{\max(x,y)} \right)^m + (xy)^m \right].
\]

The formula for \( \Delta W_r \) is useful in the case of integrating on the pipe wall too, because, in TBCI, \( E_r \) and \( B_\phi \) are not evaluated exactly on the wall but inside the pipe by a half mesh. Eq.(21) can be used for the correction.
Fig. 1 shows an example of a longitudinal wake \((-W_z)\) of a rectangular cavity computed by the same method as TBCI. The beam pipe radius is \(a=1\) mm, the cavity radius 2 mm, the cavity length 5 mm, the r.m.s. bunch length 0.1 mm and the mesh size is 0.02 mm. The dotted line is the charge distribution. The dashed line is the integration along \(r=a\) and the dot-dash is the integration along \(r=0.5\) mm until the entire bunch gets out of the cavity. (Final beam pipe length is 2.9 mm.) The solid line is the corrected wake function using eq.(9).

Fig. 2 shows the transverse wake \((m=1)\) for the same structure. Eq.(13) was used to compute \(W_z\). Each line has the same meaning as in Fig. 1.

Fig. 3 is the transverse wake for a collimator with the aperture radius \(b=1\) mm, length 5 mm and the beam pipe radius 2 mm. The dashed line is the integration with the final pipe length of 10 mm (a smaller mesh size of 0.01 mm was used for this case) and the dot-dash line with the pipe length of 2.9 mm. The solid line was obtained from the dot-dash line using eq.(13) and eq.(9).

In all these cases the agreements between the dashed and the solid lines are perfect.

Finally, let us mention the case where the time domain computation does not stop at a given time \(t_0\) but stops at a given position \(z = z_0\). The latter is the case discussed in [2]. In fact, the latter is advantageous from the view point of the computing time because actually no computation in the final beam pipe is needed even if the test particle is far behind the bunch.

In this case, \(\Delta W_z\) is defined by

\[
\Delta W_z(r, \zeta) = \int_{z_0}^{\infty} E_z(r, z, z - \zeta) dz
\]

instead of eq.(1). However, the extrapolation formula for \(\Delta W_z\) of this definition is exactly the same as eq.(9) except that the quantity in the square brackets is to be evaluated at \((r', z_0, z_0 - \zeta)\). This is also true for the extrapolation of the transverse wakes eqs.(21) and (22).

Acknowledgements
The author is thankful to Drs. K. Bane and K. Satoh for informing me the earlier paper[2] treating the same problem.

References


Fig. 1 Longitudinal Wake of a Cavity

Fig. 2 Transverse Wake of a Cavity
Fig. 3  Transverse Wake of a Collimator