Abstract
This brief introduction to Quantum Field Theory and the Standard Model contains the basic building blocks of perturbation theory in quantum field theory, an elementary introduction to gauge theories and the basic classical and quantum features of the electroweak sector of the Standard Model. Some details are given for the theoretical bias concerning the Higgs mass limits, as well as on obscure features of the Standard Model which motivate new physics constructions.

1 Introduction
The development of Quantum Field Theory and the raise of the Standard Model remains as one of the most fascinating adventures of fundamental science of the twentith century. Indeed, despite the seemingly great difference between the strength, action range and the different role played in the birth and the evolution of our universe by the electromagnetic, weak and strong interactions, we know that all three interactions are based on the gauge principle, which seems to be a fundamental principle of nature. Amazingly enough, gauge theories with or without spontaneous symmetry breaking are also renormalizable, in the leading expansion in the dimension of operators in quantum field theory. There is nothing inconsistent from the modern perspective in non-renormalizable theories, the prominent and most important example of this type being Einstein gravity. However, renormalizability renders a theory highly predictive up to high energy scales. This allowed highly precise tests of quantum electrodynamics (QED) like for example the computation of the electron anomalous magnetic moment or the running with the energy of the fine-structure constant. That’s why we can talk today about the precision tests of the Standard Model, possible deviations from it, if found experimentally, having to be interpreted unambiguously as signatures of new physics.

These lectures contain an introduction to the basic features of quantum field theory and the electroweak sector of the Standard Model. They are organized as follows. Section 2 introduces symmetries and the Noether theorem. Section 3 introduces perturbation theory, first time-dependent perturbation theory in quantum mechanics, followed by perturbation theory in quantum field theory. Section 4 is an introduction to abelian and non-abelian gauge theories and elements of their quantization. Section 5 describes spontaneous symmetry breaking, Goldstone theorem and the Higgs mechanism. Section 6 introduces the classical aspects of the electroweak sector of the standard sector. Section 7 discusses renormalizability and examples of energy evolution of couplings in the $\phi^4$ scalar theory and QED. Section 8 contains some simple applications and constraints coming from global and gauge anomalies. Section 9 enters into the Higgs physics and some theoretical arguments in favor of a light Higgs boson. As well known, Higgs searches are presently the main goal of the Large Hadron Collider (LHC) at CERN. (Very) preliminary LHC results seem to validate the theoretical picture pioneered long-time ago by Higgs and by Brout–Englert [1] and the more recent theoretical arguments pointing in favor of a light scalar Higgs boson. We end up with brief standard arguments in favor of the Standard Model as an effective theory, to be completed beyond some unknown energy scale with an underlying microscopic theory.

They notes were intended to be necessary ingredients for the lectures on Quantum Chromodynamics (QCD) given by Fabio Maltoni in this School [2], Heavy Ion Collisions lectured by Edmond
Iancu [3], the lectures on Cosmology by Valery Rubakov [4], on Neutrino Physics by Boris Kayser [5] and Beyond the Standard Model (BSM) by Bogdan Dobrescu [6].

2 Fields, Symmetries and the Noether theorem

Symmetries are fundamental in our understanding in nature. Classic examples are:

– Continuous *spacetime symmetries*, for example space rotations.

– *Discrete symmetries* are fundamental in classification and properties of crystals.

– *Continuous and discrete* internal symmetries in particle physics.

**Example** the eightfold way: Flavor SU(3)$_f$, used by M. Gell-Mann in his famous classification of hadrons which also led to the introduction of color as a new quantum number and to the modern theory of strong interactions, the QCD (see F. Maltoni’s lectures [2]).

The importance of symmetries in nature is to a large extent due to the **Noether theorem**: *To any continuous symmetry of a physical system, it corresponds a conserved current and an associate conserved charge.*

Examples of conserved charges associated to continuous symmetries are:

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Conserved charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time translation</td>
<td>Energy</td>
</tr>
<tr>
<td>Space translation</td>
<td>Momentum</td>
</tr>
<tr>
<td>Rotations</td>
<td>Angular momentum</td>
</tr>
<tr>
<td>Phase rotations wave function</td>
<td>Electric charge</td>
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For the case of internal symmetries which are our primary goal here, the proof of the Noether theorem goes as follows. Consider a field theory with $\phi$ denoting collectively all the fields of the theory, of Lagrangian $\mathcal{L}(\phi, \partial_m \phi)$. The field transformations generated by infinitesimal parameters $\alpha_a$

$$\delta \phi = \alpha_a(x) T^a \phi,$$

lead to a new Lagrangian

$$\mathcal{L}(\phi, \partial_m \phi) \rightarrow \hat{\mathcal{L}}(\phi, \alpha_a, \partial_m \phi, \partial_m \alpha_a).$$
The variation of the action functional $S(\phi, \partial_m \phi)$ under field variations (1) is

$$\delta S = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \alpha^a} \alpha_a + \frac{\partial \mathcal{L}}{\partial (\partial_m \alpha^a)} \partial_m \alpha_a \right] = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \alpha^a} - \partial_m \frac{\partial \mathcal{L}}{\partial (\partial_m \alpha^a)} \right] \alpha_a,$$

where to get the result in the last line we performed an integration by parts. By defining the currents

$$J^m_a = \frac{\partial \mathcal{L}}{\partial (\partial_m \alpha^a)},$$

we find that the variation of the action (3) vanishes if

$$\partial_m J^m_a = \frac{\partial \mathcal{L}}{\partial \alpha^a}. \tag{5}$$

In the particular case where the field variation is a symmetry of the Lagrangian, we immediately find the conservation law

$$\partial_m J^m_a = 0 \Rightarrow \frac{dQ^a}{dt} = \int d^3x \partial_m J^m_a = 0,$$

where $Q^a = \int d^3x J^0_a$. \( \tag{6} \)

is a conserved charge. It is straightforward and is left to the reader as an exercise to show that the conserved current can also be computed according to the following formulae

$$\delta \mathcal{L} = J^m_a \partial_m \alpha_a, \quad \text{or} \quad J^m_a = \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \frac{\delta \phi}{\delta \alpha^a}. \tag{7}$$

Through the Noether theorem, continuous symmetries lead to conserved charges that are manifest in the spectrum and interactions. As known already from quantum mechanics\(^1\) their study greatly simplifies the dynamics.

As we will see later on, the local (space-time dependent) symmetries determine the structure of all the fundamental interactions in nature! Indeed, all four fundamental interactions, the electromagnetism, the weak and strong forces and (in a somewhat different way) the gravitational one can be found as consequences of local symmetries called gauge symmetries.

3 Quantization and perturbation theory

The second quantization of fields and perturbation theory lead to precise formulae for scattering amplitudes which led to the Feynman diagrams, that are crucial for computing cross sections and other physical observables. The appropriate formalism uses the Heisenberg or interaction picture in quantum mechanics, that we first review, before introducing the corresponding quantum field theory formalism.

3.1 Time-dependent perturbation theory in quantum mechanics

Let’s start from Schrodinger versus interaction/Heisenberg picture in Quantum Mechanics.

$$H = H_0 + H_{int}, \tag{8}$$

where $H_0$ is the free hamiltonian and $H_{int}$ is the interaction. The Schrodinger equation is

$$i \frac{d}{dt} \left| \Psi_S(t) \right> = (H_0 + H_{int}) \left| \Psi_S(t) \right>. \tag{9}$$

\(^1\)For example the conservation of angular momentum greatly simplifies the study of hydrogen atom.
In the interaction (or Heisenberg) picture
\[ |\Psi_I(t)\rangle = e^{iH_0t} |\Psi_S(t)\rangle , \quad H_{int}(t) = e^{iH_0t} H_{int}(t) e^{-iH_0t} \] (10)
the Schrödinger equation becomes (Exercise:)
\[ i \frac{d}{dt} |\Psi_I(t)\rangle = H_{int}(t) |\Psi_I(t)\rangle. \] (11)

We define the evolution operator \( U(t, t_i) \) by
\[ |\Psi_I(t)\rangle = U(t, t_i) |\Psi_I(t_i)\rangle , \quad U(t_i, t_i) = 1. \] (12)

Exercise: Check that \( U \) satisfies the equation
\[ i \frac{\partial U(t, t_i)}{\partial t} = H_{int}(t) U(t, t_i). \] (13)

It can be shown that (Exercise:)
\[ U(t, t_i) = T e^{-\int_{t_i}^{t} dt' H_{int}(t')} , \] (14)
where the time-ordered product of the operators \( A \) and \( B \) is defined as
\[ TA(t_1)B(t_2) = \theta(t_1 - t_2) A(t_1)B(t_2) + \theta(t_2 - t_1) B(t_2)A(t_1). \] (15)

The S-matrix is defined as
\[ S = \lim_{t_i \to \infty} U(t, t_i) = T e^{-\int dt H_{int}(t)} \] (16)

The states in the far past, before the interaction process are free wave packets and are denoted by \( | p_1 \cdots p_n, \text{in} \rangle \), where \( p_i \) are the momenta of the incident particles. Similarly, the states in the far future, after the interaction process are again free and are denoted by \( | p'_1 \cdots p'_m, \text{out} \rangle \), where \( p'_i \) are the momenta of the scattered particles. The transition amplitudes passing from the initial to the final state is
\[ S_{ij} = \langle \Psi_f | S | \Psi_i \rangle = \langle p'_1 \cdots p'_m, \text{in} | S | p_1 \cdots p_n, \text{in} \rangle \\
= \langle p'_1 \cdots p'_m, \text{out} | p_1 \cdots p_n, \text{in} \rangle \\
= \text{no interaction term} + i (2\pi)^4 \delta^4 \left( \sum_{j=1}^{m} p'_j - \sum_{i=1}^{n} p_i \right) \mathcal{A}_{ij}. \] (17)

The Feynman rules are usually given for the matrix \( \mathcal{A}_{ij} \).

### 3.2 Quantization of the scalar theory

Canonical quantization of Quantum field theory uses the Heisenberg (interactive) picture. Let us consider for illustration a scalar theory
\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 = \frac{1}{2} \phi^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \]
\[ = \mathcal{L}_0 + \mathcal{L}_{int} \quad \text{with} \quad \mathcal{L}_{int} = - \frac{\lambda}{4!} \phi^4. \] (18)

The metric convention throughout these lectures will be \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The conjugate momentum is \( \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \) and the hamiltonian
\[ H = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \]
Fig. 2: Scattering amplitude of $n$ initial particles and $m$ final particles. The amplitude of the process is denoted $S_{if} = \langle p'_1 \cdots p'_m, \text{out} | p_1 \cdots p_n, \text{in} \rangle$.

$$\equiv H_0 + H_{int} \quad \text{with} \quad \begin{cases} H_0 = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right], \\ H_{int} = \int d^3x \frac{\lambda}{4!} \phi^4. \end{cases}$$

(19)

The field equations and the solutions for the free-field theory are:

$$\Box \phi(x) = 0 \quad \Rightarrow \quad \phi(x) = \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left( e^{ikx} a_k^\dagger + e^{-ikx} a_k \right),$$

(20)

where $k_0 = \omega_k = \sqrt{k^2 + m^2}$. The solution $\phi(x)$ is the operator in the Heisenberg picture. Quantization proceeds as usual:

$$[a_k, a_{k'}^\dagger] = \delta^3(k - k') \quad \Rightarrow \quad [\phi(t,x,\pi(t,y))] = i\delta^3(x - y).$$

(21)

The one-particle states are defined by

$$|k\rangle = a_k^\dagger |0\rangle \Rightarrow \langle k'|k\rangle = \delta^3(k - k'),$$

(22)

and the energy/hamiltonian is

$$H_0 = \int d^3k \omega_k (a_k^\dagger a_k + \frac{1}{2})$$

(23)

and is one of a collection of quantum oscillators. Therefore (there is by definition no interaction in the asymptotic past and future)

$$\begin{cases} |\psi_i\rangle = |p_1 p_2 \cdots p_n\rangle = a_{p_1}^\dagger \cdots a_{p_n}^\dagger |0\rangle \\ |\psi_f\rangle = |p'_1 p'_2 \cdots p'_m\rangle = a_{p'_1}^\dagger \cdots a_{p'_m}^\dagger |0\rangle \end{cases}$$

(24)

3.3 Evolution operator and S-matrix in quantum field theory

We define the evolution operator by

$$\phi(x) = U^{-1}(t) \phi_{in}(x) U(t),$$

(25)

where $U(t) = U(t, -\infty)$, $\phi_{in}$ is the incoming (free) field and $\phi$ is the interacting field. As in quantum mechanics, we separate the interaction from the free hamiltonian

$$H = H_0 + H_{int}(t).$$

(26)
The evolution equations for the quantum fields are
\[ \frac{\partial \phi(x)}{\partial t} = i \{ H(\phi), \phi(x) \}, \quad \frac{\partial \phi_{in}(x)}{\partial t} = i \{ H_{0}(\phi_{in}), \phi_{in}(x) \}. \] (27)

By combining (25) and (27), we obtain the equation satisfied by the evolution operator
\[ i \frac{dU}{dt} = (H(\phi_{in}) - H_{0}(\phi_{in})) U = H_{f}(t) U, \] \quad (28)
where \( H_{f}(t) = H_{int}(\phi_{in}, \pi_{in}) \). It is easy to check that the evolution operator satisfies the integral equation
\[ U(t) = I - i \int_{-\infty}^{t} dt_{1} H(t_{1}) U(t_{1}). \] \quad (29)
This equation can be solved by iteration. It can be shown term by term in the expansion in the interaction
that the solution of Eq. (29) can be written in the compact elegant form
\[ U(t) = T e^{-i \int_{-\infty}^{t} dt' H_{f}(t')} . \] \quad (30)
Consequently, the S-matrix is given by
\[ S = \lim_{t \to \infty} U(t) = T e^{-i \int_{-\infty}^{\infty} dt' H_{f}(t')} = T e^{i \int d^{4}x \mathcal{L}_{f}} . \] \quad (31)
Whereas at first sight the last equality is true only in the absence of derivative interactions \( \mathcal{H}_{1} = -\mathcal{L}_{1} \), it is actually true in general.

### 3.4 Reduction formula, perturbation theory and Feynman diagrams

**Feynman rules** and perturbation theory follow from the expansion in powers of the interaction of S-matrix elements
\[ \langle p_{1} \cdots p_{n}, in \mid S \mid q_{1} \cdots q_{l}, in \rangle = \langle 0 \mid a_{p_{n}}^{\dagger} \cdots a_{p_{1}}^{\dagger} T e^{i \int d^{4}x \mathcal{L}_{int}(x)} a_{p_{1}} \cdots a_{p_{n}} \mid 0 \rangle . \] \quad (32)
A very important formula in S-matrix perturbation theory is the reduction or the LSZ (Lehmann–Symanzik–Zimmermann) formula, which relates S-matrix elements to the time-ordered Green functions
\[ \langle p_{1} \cdots p_{n}, out \mid q_{1} \cdots q_{l}, in \rangle = \langle p_{1} \cdots p_{n}, in \mid S \mid q_{1} \cdots q_{l}, in \rangle \]
\[ = \text{disconnected terms} + (iZ^{-1/2})^{n-l} \times \]
\[ \int d^{4}y_{1} \cdots d^{4}x_{l} e^{i \left( \Sigma_{l} p_{y_{k}} - \Sigma_{l} q_{x_{k}} \right)} \left( \Box_{y_{1}} + m^{2} \right) \cdots \left( \Box_{x_{l}} + m^{2} \right) \langle 0 \mid T \phi(y_{1}) \cdots \phi(x_{l}) \mid 0 \rangle \] \quad (33)
where \( Z \) is the wave-function renormalization for the scalar field. The central figures in perturbation theory are therefore the Green functions
\[ G(x_{1} \cdots x_{n}) = \langle 0 \mid T \phi(y_{1}) \cdots \phi(x_{n}) \mid 0 \rangle . \] \quad (34)
The Green functions of the interactive field \( \phi \) can be expressed in terms of Green functions of the free-field \( \phi_{in} \) via the crucial formula (see for example Refs. [8, 50])
\[ G(x_{1} \cdots x_{n}) = \frac{\langle 0 \mid T \phi_{in}(y_{1}) \cdots \phi_{in}(x_{n}) e^{i \int d^{4}x \mathcal{L}_{int}(\theta_{in})} \mid 0 \rangle}{\langle 0 \mid T e^{i \int d^{4}x \mathcal{L}_{int}(\theta_{in})} \mid 0 \rangle} . \] \quad (35)
Another important notion is **normal ordering**. A normal ordered operator \( \mathcal{O} \) is defined such that all creation operators are on the left and all annihilation operators are on the right. By construction then its vev vanishes \( \langle \mathcal{O} \rangle = 0 \). G. Wick found an elegant way to express free-field Green functions in terms
of normal-ordered products, by the so-called Wick theorem. The simplest example is the two-point function
\[ T \phi_{in}(x)\phi_{in}(y) = : \phi_{in}(x)\phi_{in}(y) : + D_F(x-y) , \tag{36} \]
where \( D_F(x-y) \) is the Feynman propagator. An explicit computation gives
\[ D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} . \tag{37} \]
The \( i\epsilon \) prescription in the Feynman propagator has the property of propagating the positive frequencies into the future and the negative frequencies into the past. This is precisely what will be needed later on in order to capture both particles and antiparticles propagation in a causal way. Wick theorem can be generalized to a time-ordered product of an arbitrary number of fields
\[ T \phi_{in}(x_1)\phi_{in}(x_2) \cdots \phi_{in}(x_n) = : \phi_{in}(x_1)\phi_{in}(x_2) \cdots \phi_{in}(x_n) : + \text{all possible contractions} . \tag{38} \]
Let us now discuss the Feynman diagrams for the simplest \( \phi^4 \) theory, with
\[ \mathcal{L}_{\text{int}} = -\frac{\lambda}{4!} \phi^4 . \tag{39} \]
The Feynman rules are usually formulated in the momentum space:
\[ G(p_1 \cdots p_n) = \int d^4x_1 \cdots d^4x_n \ e^{i\sum \rho n_i} G(x_1 \cdots x_n) . \tag{40} \]
Applying perturbation theory (33), we obtain the following Feynman rules:
- associate to each propagator the factor \( \frac{i}{p^2 - m^2 + i\epsilon} \).
- to each vertex the factor \( -i\lambda \).
- impose momentum conservation at each vertex.
- integrate over undetermined internal momenta \( k \), \( \int \frac{d^4k}{(2\pi)^4} \).
- each diagram is to be divided by a symmetry factor, equal to the number of ways of interchanging components without changing the diagram.
- sum the contributions of all topologically distinct connected diagrams.

It can also be shown from Eq. (33) that the denominator cancels precisely all non-connected diagrams in the Feynman diagrams of the Green functions.

Perturbation theory is now one of the cornerstones of QFT. The anomalous magnetic moment of the electron was computed for the first time by Schwinger at one-loop in 1948 [16] (the factor below, \( \frac{\alpha}{2\pi} \), is engraved on Schwinger’s tombstone). Today it is known up to four-loops!
\[ a_e = \frac{g_e^2 - 2}{2} = \frac{\alpha}{2\pi} + \cdots \]
\[ a_e^{\text{exp}} = (1159652185.9 \pm 3.8) \times 10^{-12} , \]
\[ a_e^{\text{th}} = (1159652175.9 \pm 8.5) \times 10^{-12} , \tag{41} \]
where \( g_e \) is the gyromagnetic factor of electron coupling to a magnetic field. The theoretical prediction agrees with the experimental measurements in (41) up to the eight digit! There are however still mysteries in perturbation theory. For example, for the muon magnetic moment, the measured value at BNL disagrees by 3.4 \( \sigma \) from the theoretical SM calculation
\[ a_\mu^{\text{th}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}} \]
Anomalous magnetic moment of an electron near a dispersive surface

Robert Bennett and Claudia Eberlein, Brighton BN1 9QH, UK
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boundary-dependent radiative corrections are investigated. The electromagnetic field is quantized.

Perturbation theory in the Dirac equation leads to a general formula for the magnetic moments of the surface. Analysis of the magnetic moment shift for several classes of materials yields markedly different results from the previously considered simplistic 'perfect reflector' model.

Fig. 3: Simplest Feynman diagrams contributions to the electron magnetic moment. The agreement between perturbative QED computations and the experimentally measured value agree up to the eight digit.

\[ a_{\mu}^{exp} \simeq 0.00116592089. \]

In this case, it is likely that the hadronic contribution is not known accurately enough, since the muon mass is much closer to the hadronic contributions compared to the electron one. This is a very hot research topic nowadays, since any real disagreement could be a hint for new physics contributions coming from virtual loops of new particles.

3.5 Fermions

Relativistic fermions satisfying the Pauli principle are described by spinors in quantum field theory. In particular, the relativistic spin 1/2 fermion is described by a four component spinor \( \Psi \) via the Dirac equation

\[ (i\gamma^m \partial_m - M)\Psi = 0 \] (42)

where \( \gamma_m \) are the \( 4 \times 4 \) Dirac matrices satisfying the Clifford algebra

\[ \{ \gamma^m, \gamma^\nu \} = 2\eta_{mn}. \] (43)

The Lagrangian giving the Dirac equation is

\[ \mathcal{L}_0 = \bar{\Psi}(i\gamma^m \partial_m - M)\Psi, \] (44)

where \( \bar{\Psi} = \Psi^\dagger \gamma^0 \). A particular role is played by fermions which are eigenstates of the chirality operator, satisfying

\[ \gamma^5 = i\gamma_0 \gamma_2 \gamma_3 , \quad (\gamma^5)^2 = 1 , \quad \{ \gamma^m, \gamma^5 \} = 0. \] (45)

It is then possible to define left and right-handed chirality fermions

\[ \gamma^5\Psi_L = -\Psi_L , \quad \Psi_L = \frac{1 - \gamma^5}{2} \Psi , \]

\[ \gamma^5\Psi_R = \Psi_R , \quad \Psi_R = \frac{1 + \gamma^5}{2} \Psi . \] (46)

In terms of the left/right chirality fermions, the Dirac Lagrangian is written

\[ \mathcal{L}_0 = \bar{\Psi}_L i\gamma^m \partial_m \Psi_L + \bar{\Psi}_R i\gamma^m \partial_m \Psi_R - M(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) , \] (47)

whereas the Dirac equation can be split into two equations

\[ i\gamma^m \partial_m \Psi_L - M\Psi_R = 0 , \quad i\gamma^m \partial_m \Psi_R - M\Psi_L = 0 . \] (48)
As we will see in Section 6, in Nature left and right chirality fermions have different interactions. This is related to the parity violation in the weak interactions and it is at the heart of the construction of the Standard Model.

There are two different type of fermions that could exist in nature. The fermions charged under gauge symmetries are of Dirac type, i.e. their mass (eventually after symmetry breaking, as it will be the case in the Standard Model) is of Dirac type \( (47) \). For fermions uncharged under gauge symmetries, they can be of Majorana type. In this case, the charge conjugate fermion \( \Psi^c = C \bar{\Psi}^T \), (49)

where \( C \) is the charge conjugation matrix satisfying

\[
C^{-1} \gamma_m C = - \gamma_m^T, \quad C = -C^{-1} = -C^T = -C^\dagger
\]

is self-conjugate \( \Psi^c = \Psi \), i.e. the fermion is its own antiparticle. In this case, the mass of the fermion can be written as

\[
\mathcal{L}_M = -\frac{M}{2} \Psi^T C \Psi + \text{h.c.}
\]

It is not yet known if there exist Majorana fermions in nature. One natural possibility are the neutrinos, which will be discussed in detail in the lectures by B. Kayser [5].

4 Gauge theories

The four fundamental interactions in nature have a common feature: they are gauge interactions. We will discuss here the internal symmetries which describe the electromagnetic, weak and strong interactions and elements of their quantization.

4.1 Gauge invariance of Schrödinger equation

The simplest example of gauge symmetry arises in the description of particle of mass \( m \) and charge \( q \) in quantum mechanics. The hamiltonian is

\[
H = \frac{1}{2m} (p - qA)^2 + qV
\]

Fig. 4: The four fundamental interactions in nature. From Ref. [17].
where the vector $A$ and the scalar $V$ potential are related to the electric/magnetic fields via
\[ E = -\nabla V - \frac{\partial A}{\partial t} \, , \quad B = \nabla \times A \, . \] (53)

The Maxwell equations are invariant under the gauge transformations
\[ A' = A + \nabla \alpha \, , \quad V' = V - \frac{\partial \alpha}{\partial t} \, . \] (54)

The Schrödinger equation is covariant, with $H = H(A, V)$, $H' = H(A', V')$
\[ i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \rightarrow i\hbar \frac{\partial \Psi'}{\partial t} = H'\Psi' \] (55)
if under the gauge transformations (52), the wave function transforms as
\[ \Psi'(r, t) = e^{i\alpha} \Psi(r, t) \, . \] (56)

Note that the mean value of any physically measurable quantity is gauge invariant; for ex. $P(r) = |\Psi|^2 = |\Psi'|^2$.

**Homework:** Defining the velocity operator $v = \frac{1}{m}(p - qA)$, check that $\langle \Psi | v | \Psi \rangle = \langle \Psi' | v' | \Psi' \rangle$.

**Gauge principle:** Postulate that physical laws are invariant under (54)+ (56). In this case, it can be proven that the hamiltonian is uniquely determined to be (52). Equations (56) and (54) define an $U(1)$ transformation. Therefore, $U(1)$ gauge invariance determines the electromagnetic interaction.

### 4.2 From Dirac and Maxwell equations to QED

Maxwell equations in terms of $A_\mu = (A, V)$ are invariant under the gauge transformations
\[ A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \alpha \, . \] (57)

**Gauge invariance postulate:** the physics is invariant under (57), supplemented with the phase transformation
\[ \Psi(x) \rightarrow \Psi'(x) = e^{i\alpha(x)}\Psi(x) \, . \] (58)

Then Dirac equation is not invariant under (58) unless we replace the derivative with the covariant derivative
\[ D_\mu \Psi \equiv (\partial_\mu + iqA_\mu)\Psi \rightarrow (D_\mu \Psi)' = (\partial_\mu + iqA'_\mu)\Psi' = e^{i\alpha(x)}D_\mu \Psi(x) \, . \] (59)

Dirac equation in an electromagnetic field becomes therefore
\[ (i\gamma^\mu D_\mu - M)\Psi = (i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - M)\Psi = 0 \, . \] (60)

The Dirac and Maxwell equations can be derived from the Lagrangian density
\[ \mathcal{L}_{QED} = \bar{\Psi}(i\gamma^\mu D_\mu - M)\Psi - \frac{1}{4}F^2_{\mu\nu} \] (61)

The coupled Euler–Lagrange field equations are then Eq. (60) and
\[ \partial^\mu F_{\mu\nu} = g\bar{\Psi}Y_\nu \Psi \equiv j_\nu \, , \] (62)

where $j_\nu$ is the electromagnetic current of the charged fermion. From Eq. (62) we can derive the charge conservation law
\[ \partial^\mu j_\mu = 0 \rightarrow \frac{dQ}{dt} = \int d^3x \partial^\mu j_\mu = 0 \, , \text{ where } Q = \int d^3x j_0(x) \, . \] (63)

Some comments are in order
The massless photon has two propagating degrees of freedom.
A photon mass \( \mathcal{L}_{mass} = \frac{M_0^2}{2A_\mu} \) breaks gauge invariance and describes three degrees of freedom.

The propagator of a massive photon is found from inverting the free Lagrangian
\[
\Delta_{\mu\nu}(x-y) = \frac{-i}{\mathcal{A}} \frac{\mathcal{A}}{\mathcal{M}_2} \delta^\nu \delta^\mu \delta^3(x-y) \quad (71)
\]
\[
\Delta_{\mu\nu}(k) = -i \frac{\bar{\delta} \cdot k}{k^2 - M_2^2} \quad (65)
\]

Therefore, in momentum space (Homework)

Notice that due to the current conservation \( \partial_\mu j_\mu = 0 \), the longitudinal polarization does not contribute to amplitudes. Therefore, the UV properties of the massless and massive photon theories are the same. On the other hand, experimentally the photon is massless to a high accuracy. Indeed, the present experimental limit on the photon mass is \( m_\gamma \leq 10^{-18} \text{ eV} \).

### 4.3 Fermions and the quantization of the Dirac field

According to perturbation theory, we start from the free Dirac Lagrangian
\[
\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - M)\Psi \quad (66)
\]
Conjugate momentum is \( \pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = \gamma^\nu \dot{\Psi} \). The free-field hamiltonian is then
\[
H_0 = \int d^3x \bar{\Psi}(i\gamma^\nu \partial_\nu + M)\Psi = \int d^3x \bar{\Psi}(i\alpha\nabla + \beta M)\Psi \quad (67)
\]
where \( \gamma = \beta \alpha, \gamma_0 = \beta \) and in the last parenthesis we can recognize the Dirac hamiltonian of relativistic quantum mechanics. The solutions of the Dirac equation (66) are of the form
\[
\Psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{2\omega_k} \sum_{s=1,2} \left[ e^{-i\omega_k} a_k^s u^s(k) + e^{i\omega_k} b_k^{s\dagger} v^s(k) \right] \quad (68)
\]

where \( u^s(k) \) (\( v^s(k) \)) are positive (negative) frequency solutions of the Dirac equation
\[
(i\gamma^\mu k_\mu - M) u^s(k) = 0 \quad , \quad (i\gamma^\mu k_\mu + M) v^s(k) = 0 \quad (69)
\]

The Dirac equation have two independent solutions \( s = 1,2 \). The correct quantization for fermions uses anti-commutators
\[
\{ \Psi_\alpha(t,x), \Psi_\beta^\dagger(t',y) \} = \delta_{\alpha\beta} \delta^3(x-y) \quad (70)
\]

all the other anticommutators being zero. This defines the anticommutation relations
\[
\{ a^s_p, a^{s\dagger}_{q} \} = \{ b^s_p, b^{s\dagger}_{q} \} = \delta^{rs} \delta^3(p-q) \quad (71)
\]

The vacuum is defined by \( a^s_0 = b^s_0 = 0 \), whereas the hamiltonian is given by
\[
H_0 = \int d^3k \sum_{s} \omega_k \left[ a^{s\dagger}_k a^s_k + b^{s\dagger}_k b^s_k \right] \quad (72)
\]

Notice that if the theory would have been quantized with commutators, the contribution of the \( b \) oscillators would have been of opposite sign and the hamiltonian would have been unbounded from below. The electric charge operator can be defined as in (63) and equals
\[
Q = \int d^3k \sum_s \left[ a^{s\dagger}_k a^s_k - b^{s\dagger}_k b^s_k \right] \quad (73)
\]

By defining also the helicity operator, it can shown that:
\( a^s,^t \) creates fermions of energy \( \omega_k \), momentum \( k \), electric charge +1 (in units of the electron electric charge), helicity left (right) for \( s = 1 \) (\( s = 2 \)).

\( b^s,^t \) creates antifermions of energy \( \omega_k \), momentum \( k \), electric charge \(-1\) and helicity right (left) for \( s = 1 \) (\( s = 2 \)).

Similarly for the scalars case, there is a reduction/LSZ formula (32) and perturbative expansion (33) for the Green functions. The simplest and most important Green function is the fermionic Feynman propagator

\[
S_F(x-y) = \langle 0| T\Psi(x)\overline{\Psi}(y) |0\rangle = \theta(x^0-y^0)\langle 0|\Psi(x)\overline{\Psi}(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\Psi}(y)\Psi(x)|0\rangle
\]

(74)

\[
= \int \frac{d^4k}{(2\pi)^4} \frac{i(\gamma^\mu k_\mu + M)}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)} .
\]

### 4.4 Quantization of the electromagnetic field

The quantization of the free electromagnetic field is subtler than for the case of scalars and fermions. Indeed, starting from the Maxwell Lagrangian

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

(75)

the conjugate momentum is

\[
\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu0} \quad \Rightarrow \quad \pi^0 = 0,
\]

(76)

and we cannot impose canonical commutation relations. The problem can be avoided by using the non-covariant gauges, like the Coulomb gauge (\( \text{div}A = 0 \)), but it is preferable to maintain manifest Lorentz covariance. The standard option is to modify the Lagrangian by adding a \textit{gauge-fixing term} and changing the Lagrangian to

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 ,
\]

(77)

where \( \xi \) is a real arbitrary (and unphysical) parameter. In this case the field equations become

\[
\Box A_\mu - \frac{1}{\xi} \partial_\mu (\partial A) = 0 ,
\]

(78)

and the canonical momentum \( \pi^0 = -\frac{1}{\xi} (\partial_\mu A^\mu) \) does not vanishes anymore.

**Observation:** the field equations imply \( \Box (\partial A) = 0 \), which suggests that we could impose the Lorentz condition \( \partial A = 0 \). This is however incompatible with canonical quantization, since \( \pi^0 \sim \partial A \). The condition can be only be imposed on physical states \( |\psi_{ph}\rangle \)

\[
\langle \psi_{ph}| \partial_\mu A^\mu |\psi_{ph}\rangle = 0.
\]

(79)

It can be shown that physical results are independent of \( \xi \). A convenient choice for canonical quantization is \( \xi = 1 \) (Feynman gauge), in which case the electromagnetic field become a collection of four Klein–Gordon fields \( \Box A_\mu = 0 \). In this case, it can be expanded in plane waves according to

\[
A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{r=0}^3 \left[ e^{-ikx} d^r_\mu \epsilon^r_\mu(k) + e^{ikx} d^{r\dagger}_\mu \tilde{\epsilon}^r_\mu(k) \right],
\]

(80)

where here \( \omega_k = |k| = k^0 \) and \( \epsilon^r_\mu(k) \) are the polarization vectors. Canonical quantization in this case goes as follows

\[
[A_\mu(t,x), \pi_\nu(t,y)] = -i\eta_{\mu\nu} \delta^3(x-y) \quad \Rightarrow \quad [A_\mu(t,x), \dot{A}_\nu(t,y)] = -i\eta_{\mu\nu} \delta^3(x-y) .
\]

(81)
The commutation relations for the creation/annihilation operators then follow
\[
[a^+_k,a^+_q] = -\eta^{rs} \delta^3(k-q). \tag{82}
\]
Finally, the Feynman propagator in this gauge is
\[
\langle 0|TA_\mu(x)A_\nu(y)|0 \rangle = -\eta_{\mu\nu}D_F(x-y)|_{M=0} = -i\eta_{\mu\nu} \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\varepsilon}. \tag{83}
\]
Finally, we can give the Feynman rules for QED:

- associate to each fermion propagator of momentum \( p \) the factor \( \frac{i(\gamma^\mu p_\mu + M)}{p^2 - M^2 + i\varepsilon} \).
- to each photon propagator of momentum \( p \) (in the \( \xi = 1 \) gauge) the factor \( \frac{-i\eta_{\mu\nu}}{p^2 - M^2 + i\varepsilon} \).
- to each internal initial photon the factor \( w'(p) \).
- to each external final fermion the factor \( \bar{v}(p) \).
- to each external initial antifermion the factor \( v'(p) \).
- to each external final photon the factor \( \varepsilon_\mu(p) \).
- to each external final photon the factor \( \bar{\varepsilon}_\mu(p) \).

The reader can find more about the historical rise of QED in Ref. [14].

### 4.5 Non-abelian gauge theories

The action of \( U(1) \) is a particular case of unitary abelian transformation. Another case of particular interest are the non-abelian transformations. The non-abelian \( SU(n) \) transformations are described by \( n \times n \) matrices \( U \), satisfying
\[
U^\dagger U = UU^\dagger = I, \quad \det U = 1. \tag{84}
\]
The simplest case is \( SU(2) \), proposed by Yang and Mills in 1954 [18]. Its simplest representation is a doublet
\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi' = U(\theta)\Psi, \text{ where } U(\theta) = e^{ig\theta_a \tau_a}, \tag{85}
\]
where \( \tau_a \) are the Pauli matrices and \( g \) is the \( SU(2) \) gauge coupling. It turns out that the number of gauge bosons \( W^a_\mu \) equals the number of generators (three for \( SU(2) \)). The most compact notation introduces a matrix
\[
W_\mu = W^a_\mu \frac{\tau_a}{2} = \begin{pmatrix} W^3_\mu \\ W^1_\mu + iW^2_\mu \\ W^1_\mu - iW^2_\mu \\ 2W^3_\mu \end{pmatrix} \equiv \begin{pmatrix} W^3_\mu \\ \sqrt{2}W^1_\mu \\ -W^3_\mu \\ -W^3_\mu \end{pmatrix}. \tag{86}
\]

**Homework:** show that
\[
D_\mu \Psi \equiv (\partial_\mu - i g W_\mu)\Psi \to (D_\mu \Psi)' = UD_\mu \Psi \quad \text{if } W_\mu \to W'_\mu = U W_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}. \tag{87}
\]
The infinitesimal gauge variation in component form is
\[
\delta W_\mu = D_\mu \theta^a = \partial_\mu \theta^a + g \epsilon_{abc} W^b_\mu \theta^c. \tag{88}
\]
The field strength is built from
\[
[D_\mu, D_\nu] = -i g F_{\mu\nu}. \tag{89}
\]
Feynman rules for Yang–Mills theories. Solid lines are fermion propagators, wavy lines are gauge propagators, whereas the dotted ones are scalars. The structure constants are defined for an arbitrary gauge group via \( [T^a, T^b] = i f^{abc} T^c \). From Ref. [15].

**Homework**: Show that
\[
F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig [W_\mu, W_\nu] , \quad F^I_{\mu\nu} \to F^I_{\mu\nu} = UF_{\mu\nu}U^{-1} .
\] (90)

For \( SU(2) \) this implies (homework:)
\[
F^{a\mu}_{\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon_{abc} W^b_\mu W^c_\nu .
\] (91)

The Yang–Mills Lagrangian is finally given by
\[
\mathcal{L}_{YM} = - \frac{1}{4} F^{a\mu\nu}_{\mu\nu} F_{\mu\nu} = - \frac{1}{4} (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - \frac{g}{2} \epsilon_{abc} \partial_\mu W^a_\nu W^b_\mu W^c_\nu - \frac{g^2}{4} \epsilon_{abc} \epsilon_{ade} W^b_\mu W^c_\nu W^d_\mu W^e_\nu .
\] (92)

Non-abelian gauge bosons have self-interactions, unlike the photon! The full Lagrangian describing interaction of Yang–Mills fields with charged fermions is then
\[
\mathcal{L} = \bar{\Psi} (i \gamma^\mu D_\mu - M) \Psi - \frac{1}{4} F^{a\mu\nu}_{\mu\nu} F^a_{\mu\nu} + \mathcal{L}_g + \mathcal{L}_{\text{ghosts}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} .
\] (93)

In Eq. (93), \( \mathcal{L}_g \) is a gauge fixing term, whereas \( \mathcal{L}_{\text{ghosts}} \) is the Fadeev–Popov ghost Lagrangian [2, 19], coming from the covariant quantization of non-abelian theories.

**Homework**: show that for an \( SU(2) \) doublet
\[
\bar{\Psi} (i \gamma^\mu D_\mu - M) \Psi = \bar{\Psi}^k [\delta^k_l (i \gamma^\mu \partial_\mu - M) + \frac{g}{2} \gamma^\mu W^a_\mu (\tau_a)_{kl}] \Psi^l ,
\] (94)
whereas the fermion and Yang–Mills field equations are
\[(i\gamma^\mu D_\mu - M) \Psi = 0, \quad \partial_\mu F_{\mu\nu}^a + g\varepsilon_{abc} A_{b\mu}^a F_{c\nu}^a = -g \bar{\Psi} \gamma_\nu \tau^a \frac{\Psi}{2}\] (95)
where on the right hand side, we can identify the \(SU(2)\) fermionic current \(j_\mu^a = -g \bar{\Psi} \gamma_\mu \frac{\tau^a}{2} \Psi\).

Notice that the massive Yang–Mills field propagator is
\[\Delta^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{g_{\mu\nu} - k_\mu k_\nu}{M^2_A} .\] (96)
Since here \(\partial_\mu j_\mu^a \neq 0\), the longitudinal polarization does contribute to scattering amplitudes. Therefore, unlike the abelian case, here the UV properties of the massless and massive YM theories are different. This fact has various consequences:

- the theory has bad UV behaviour (uncontrolled UV divergences).
- the amplitude \(W_iW_L \to W_iW_L\), where \(W_L\) is the longitudinal component of the \(W\) gauge boson, grows with energy and invalidates perturbation theory for energies above around 1.2 TeV.

The conclusion of all these problems is that the Yang–Mills boson masses should not be added by hand, but be generated in a more subtle way. On the other hand, massless gauge fields (infinite range) cannot describe electroweak interactions, which are short range. We need therefore to give gauge bosons a mass, but we need need another way to generate gauge boson masses. This is explained via the spontaneous symmetry breaking and the Higgs mechanism, to which we now turn.

5 Spontaneous symmetry breaking
We already noticed that Symmetries, through the Noether theorem, imply the existence of conserved charges. There are however two different ways symmetries are realized in nature:

1. **Weyl–Wigner (WW) realization:** in this case the vacuum state is invariant under the symmetry. Then the symmetry is manifest in the spectrum and the interactions. Simple examples of this type are: translations (conserved charge: momentum), rotations (conserved charge: angular momentum), \(U(1)_{em}\) (conserved charge: electric charge)...

2. **Nambu–Goldstone (NG) realization:** in this case the vacuum state is not invariant under the symmetry. In this case the symmetry is not manifest in the spectrum. We talk about spontaneous symmetry breaking. Examples of this type include rotation (or parity) symmetry in ferromagnets, \(SU(2)^{\text{weak}}\), \(SU(2)_L \times SU(2)_R\) chiral symmetry of strong interactions, etc

A nice sentence summarizing the outcome of the two realizations of global symmetries is that of S. Coleman in his Erice Lectures [24]: *the symmetry of the vacuum is the symmetry of the world*.

The simplest example of the NG realization is the Ising model describing \(N\) spins in space dimension \(d\), of hamiltonian
\[H = -J \sum_{(i,j)} S_i S_j - B \sum_i S_i ,\] (97)
with \(S_i = \pm 1\) labelling the two possible values of the spin "\(i\)". For zero magnetic field \(B = 0\) the system has a \(Z_2\) symmetry which reverses the spins \(S_i \to -S_i\). As a consequence, the magnetization defined as
\[M = \lim_{B\to 0,N\to\infty} \frac{1}{N} \sum_{k=1}^N \langle S_k \rangle\]
should therefore vanish. However experimentally it is known that
\[M = 0 \quad \text{for} \quad T \geq T_c , \quad M \neq 0 \quad \text{for} \quad T < T_c, \quad \text{where} \quad kT_c = 2dJ.\] (98)
Fig. 6: Magnetization in ferromagnets. At high temperatures, spins orientations are random. At low temperatures, the alignment of spins due to spin-spin interactions breaks the rotational symmetry of the Hamiltonian.

The reason for the violation of the $Z_2$ symmetry is that at low temperatures, due to the spin-spin interactions of strength $J$, spins tend to align, such that the ground state corresponds to a state with all spins aligned. This state does violate the $Z_2$ symmetry, since the $Z_2$ transform of this ground state is the state with all spins reversed. Whereas both states (vacua) are equally possible, the transition from one to the other is highly suppressed for large $N$. So if the system is in one of the two vacua, it will stay there a time that scales as $e^N$. On the other hand, at high temperature, spins are oriented arbitrarily in order to increase the entropy (number), which wins over the higher-energy of such configurations. This phenomenon is called spontaneous symmetry breaking, since the Hamiltonian of the system respects the $Z_2$ symmetry, which is broken only by the ground state for $T < T_c$. The field theory analog of this phenomenon is described in the next paragraph.

5.1 The Goldstone theorem

In a theory with continuous symmetry, for every generator which does not annihilate the vacuum $\langle T_a \Phi \rangle \neq 0$ there is a massless, NG particle [20].

Example: One of the simplest examples is the $O(N)$ linear sigma model. Consider a theory with $N$ scalar fields $\Phi = (\phi_1, \phi_2, \cdots, \phi_N)$, with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 - V(\Phi), \quad V(\Phi) = -\frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4} (\Phi^2)^2,$$

(99)

where in our convention $\mu^2 > 0$ and where $\Phi^2 = \sum_{i=1}^N \phi_i \phi_i$. The model has a continuous $O(N)$ symmetry acting as $\Phi \rightarrow R \Phi$, with $R$ a $N \times N$ rotation matrix. The scalar potential is minimized for

$$\frac{\partial V}{\partial \phi_i} = 0 \implies \Phi_0^2 = \frac{\mu^2}{\lambda} \equiv v^2.$$

(100)

The vacuum manifold is $O(N)$ invariant. By an $O(N)$ rotation, the ground state can be chosen to be

$$\langle \Phi \rangle = \Phi_0 = (0, 0, \cdots, v),$$

(101)

preserving an $O(N-1)$ subgroup. Goldstone’s theorem tells us that we expect the model to have $N - 1$ massless particles, corresponding to the number of broken generators of the coset group $O(N)/O(N-1)$. In order to check this, we define a set of shifted fields:

$$\Phi(x) = (\pi^k(x), v + \sigma(x)) , \quad k = 1, 2, \cdots, N-1,$$

(102)

such that $\langle \pi^k \rangle = \langle \sigma \rangle = 0$. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2) - \mu^2 \sigma^2 - \sqrt{\lambda} \mu \sigma^3 - \sqrt{\lambda} \mu \pi^2 \sigma - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 ,$$

(103)
where $\pi^2 = \sum_{k=1}^{N-1} \pi^k \pi^k$. The manifest symmetry is indeed $O(N - 1)$, which rotates the "pions" $\pi$'s among themselves. The physical masses, visible from (103) are

$$m_\sigma^2 = 2\mu^2, \quad m_\pi^2 = 0.$$  

(104)

Therefore we find that the "pions" are massless; they are the $N - 1$ Nambu–Goldstone (NG) bosons of the broken symmetry. It is said that the unbroken $O(N - 1)$ symmetry is realized a la Weyl–Wigner, whereas the original $O(N)$ symmetry is realized a la Nambu–Goldstone.

**General (classical) proof of the Goldstone theorem.** Consider the scalar theory of Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_m \Phi_i)^2 - V(\Phi_i),$$  

(105)

and a global continuous symmetry group with generators $T^a$. The invariance of the scalar potential

$$V(\phi_i + \delta \phi_i) = V(\phi_i)$$  

(106)

under infinitesimal transformations $\delta \phi_i = i \theta^a T^a_{ij} \phi_j$ of parameters $\theta^a$ implies

$$\frac{\partial V}{\partial \Phi_i} T^a_{ij} \phi_j = 0.$$  

(107)

Differentiating again and taking the vacuum expectation value (vev), we get

$$\left< \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_i} T^a_{ij} \phi_j + \frac{\partial V}{\partial \Phi_i} T^a_{ik} \right> = 0.$$  

(108)

Remembering that $\mathcal{M}^2 = \left< \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_i} \right>$ is the scalar mass matrix, we obtain

$$\mathcal{M}^2 (T^a v)_i = 0.$$  

(109)

We therefore found the general form of the **Goldstone theorem**: *If the vacuum is not invariant under a symmetry generator $T^a v \neq 0$, then $T^a v$ is an eigenvector of the mass matrix $\mathcal{M}^2$ corresponding to a zero eigenvalue.*

Are there known examples of Goldstone bosons in nature? Yes, there are several, but none of them *not corresponding to a fundamental spin 0 particle!* Two well-known examples are
Magnons spin waves in ferromagnets, which are long wavelength collective spin configurations.

Pions $\pi \sim q \bar{q}$ are pseudo-Goldstones for the breaking of the chiral → vector symmetries $U(3)_L \times U(3)_R \rightarrow SU(3)_V \times U(1)_B$ (see figure 8). They are not exactly massless (therefore the name "pseudo") due to a small explicit breaking coming from quark masses. Pions are (pseudo)scalar particles, but not elementary, they are quark-antiquark bound states. In this case we talk about dynamical symmetry breaking.

**Observation:** The $U(1)_A$ symmetry is broken by quantum anomalies hence there is no corresponding goldstone boson.

A natural question arises: *What happens if the spontaneously broken symmetry is gauged?* The answer is given in the next subsection.

### 5.2 The Higgs mechanism

Let us start for simplicity with an *abelian gauge theory*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \Phi|^2 - V(\Phi),$$

with $D_\mu = \partial_\mu + ie A_\mu$, $\Phi = \frac{1}{\sqrt{2}} (\Phi_1 + i \Phi_2)$ and a scalar potential

$$V = -\mu^2 |\Phi|^2 + \lambda (|\Phi|^2)^2 = -\frac{\mu^2}{2} (\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4} (\Phi_1^2 + \Phi_2^2)^2,$$

invariant under the local $U(1)$ gauge transformations

$$\Phi \rightarrow e^{i\alpha(x)} \Phi, \quad A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha.$$  

We expand around the vacuum state

$$\Phi_0 = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}}, \quad \Phi(x) = \frac{1}{\sqrt{2}}(v + \phi_1 + i \phi_2).$$
From the quadratic mass terms in the scalar potential we find $m_1^2 = 2\mu^2$, $m_2 = 0$, therefore $\phi_2$ is the Goldstone boson. New features appear however from the kinetic term

$$|D_\mu \Phi|^2 = \frac{1}{2} (\partial_\mu \phi_1)^2 + evA_\mu \partial^\mu \phi_2 + \frac{e^2 v^2}{2} A^2_\mu + \cdots$$  \hspace{1cm} (114)$$

Indeed, it is manifest from (114) that the gauge boson acquired a mass $M_A^2 = \mu^2$. But this can only happen if the gauge field absorbed one degree of freedom, since the massive gauge field has three degrees of freedom, whereas the massless one has only two degrees of freedom. The correct counting of degrees of freedom is

$$A_\mu (M = 0) + \phi_2 \rightarrow A_\mu (M \neq 0)$$  \hspace{1cm} (115)$$

That this is indeed true can be seen in various ways:

1. The quadratic term in the Lagrangian can be diagonalized by redefining the gauge field

$$-\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} (\partial_\mu \phi_2)^2 + \sqrt{2} evA_\mu \partial^\mu \phi_2 + \frac{e^2 v^2}{2} A^2_\mu = -\frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{e^2 v^2}{2} B^2_\mu ,$$  \hspace{1cm} (116)$$

where $B_\mu = A_\mu + \frac{1}{v} \partial_\mu \theta$. Therefore $\phi_2$ disappeared from the quadratic part, and is "absorbed" into the longitudinal component of the gauge field.

2. The Goldstone can be eliminated altogether from the Lagrangian in the so-called unitary gauge. The corresponding parametrization is

$$\Phi(x) = \frac{1}{\sqrt{2}} e^{\frac{\theta(x)}{v}} (v + \rho(x))$$  \hspace{1cm} (117)$$

and the Goldstone is removed by the gauge transformation $\Phi \rightarrow \Phi' = e^{-i \theta \Phi}$, $A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{v} \partial_\mu \theta$. In the unitary gauge, the Lagrangian is (homework):

$$\mathcal{L} = -\frac{1}{4} (F_{mn}')^2 + (\partial_m - i e A'_m) \Phi' (\partial^n + i e A'^n) \Phi' - \mu^2 \Phi'^2 - \lambda \Phi'^4$$

The spectrum of the model contains therefore a massive gauge boson and the Higgs boson $\Phi'$, of mass $2\mu^2$ [1].

**The Higgs mechanism, non-abelian case.** Consider a gauge group $G$ of rank $r$ and scalar fields in some irreducible $n$-dimensional representation

$$\mathcal{L} = -\frac{1}{4} F^\mu_{\rho\nu} F^{\alpha\mu\nu} + \left| \left[ (\partial_\mu - i g T_\alpha^a A_\mu^a) \Phi \right] \right|^2 - V(\Phi) ,$$  \hspace{1cm} (118)$$

and $H \in G$ the subgroup of rank $s$ leaving the ground state invariant

$$T^a v = 0, \quad a = 1 \cdots s,$$

$$T^a v \neq 0, \quad a = s + 1 \cdots r.$$  \hspace{1cm} (119)$$

In the unitary gauge parametrization

$$\Phi(x) = e^{i \sum_{a=s+1}^r T_a \xi_a} \frac{\rho(x) + v}{\sqrt{2}},$$  \hspace{1cm} (120)$$

where $\xi_a$ are the Goldstone bosons, $\rho(x)$ the remaining scalar fields, and $\langle \xi_a \rangle = \langle \rho \rangle = 0$. The gauge transformation

$$\Phi(x) \rightarrow \Phi'(x) = U \Phi , \text{ with } U = e^{-i \sum_{a=s+1}^r T_a \xi_a}$$
\[ A_\mu \rightarrow A'_\mu = U (A_\mu + \frac{i}{g} \partial_\mu) U^{-1} \] (121)

eliminates the Goldstone bosons from the Lagrangian. The resulting mass matrix of the vector fields is then
\[ M_{ab}^2 = g^2 (T^a v)^\dagger (T^b v) . \] (122)

In this case \( r - s \) gauge bosons become massive
\[ A_\mu'' \rightarrow \xi_a \rightarrow A'_\mu = A_\mu - \frac{1}{v} D_\mu \xi^a + \cdots \] (123)

where \( A_\mu \) denote the massless gauge fields, containing two degrees of freedom, whereas \( A'_\mu \) denote the massive gauge fields.

Notice that the number of physical massive Higgs scalars is equal to the number of original scalars, minus the number of broken gauge generators. In particular:

- In the Standard Model with one Higgs doublet, there is one real Higgs scalar \( 4 - 3 = 1 \), where 4 is the number of initial real degrees of freedom contained into an \( SU(2)_L \) scalar doublet, whereas 3 is the number of broken generators in the Standard Model.
- In the Standard Model with two Higgs doublets (or the Minimal Supersymmetric Standard Model, MSSM), there are \( 8 - 3 = 5 \) physical Higgs scalars: two neutral scalars \( h \) and \( H^0 \), one pseudoscalar \( A \) and two charge ones \( H^\pm \).

The Higgs mechanism is a elegant and economical way to break electroweak symmetry. However, it has its own mysteries:

- Elementary scalars were never observed in nature until now.
- It is difficult to keep a scalar light after quantum corrections (so-called hierarchy problem). We will come back later on to quantify this problem.

Taken into account these observations, it is reasonable to ask is there are other ways of breaking a gauge (electroweak for our purposes) symmetry. The answer is yes, there are several other options. Some popular ones are:

- A new confining force (technicolor) with \( \Lambda_{TC} \sim v \). The goldstone bosons "eaten up" by the \( W \) and \( Z \) gauge bosons are called "technipions" (see Fig. 9).
- Composite Higgs models, in which Higgs is a bound state of fermions. One example is the top-antitop condensation, where the Higgs is a top-antitop bound-state \( h = \bar{t}_L t_R \).
- Symmetry breaking by boundary conditions in extra-dimensional Kaluza–Klein (string) type theories.

### 6 The electroweak sector of the Standard Model

#### 6.1 Gauge group and matter content

The Standard model is a "unified" description of weak and electromagnetic interactions. From the Fermi theory of weak interactions with \( G_F/\sqrt{2} = g^2/8M_w^2 \), we know that we need a theory that contains at least a charged gauge boson \( W^\pm_\mu \) and the photon \( A_\mu \).

Experimentally, there also exists neutral currents discovered in 1973, mediated by a neutral massive gauge boson, and also coloured strong interactions. The SM gauge group is therefore

\[ G = SU(3)_c \times SU(2)_L \times U(1)_Y \] (124)
A confining force similar to QCD called technicolor could be responsible for electroweak symmetry breaking. The electroweak vev is given by a condensate of fermions $\langle \bar{T}_L T_R \rangle \sim v^3 = M_3^3 e^{-3/(2b_0 g_T^2)}$. Taken from Ref. [23].

Fig. 10: Fermi theory of weak interactions: the beta decay $n \rightarrow p e^− \bar{\nu}_e$ at low energies $E \ll M_W$ can be described as an effective four-fermion interaction.

In addition to the gauge bosons, the SM contains matter fermions and the Higgs field, in the gauge group representations

- Leptons: $l_i = \begin{pmatrix} V_i \\ e_i \end{pmatrix}_L : (1, 2)_{Y=−1}, \text{ and } e_{iR} : (1, 1)_{Y=−2}$
- Quarks: $q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L : (3, 2)_{Y=1/3}, \text{ and } u_{iR} : (3, 1)_{Y=4/3}, \text{ and } d_{iR} : (3, 1)_{Y=−2/3}$
- Higgs field: $\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} : (1, 2)_{Y=1}$.

In the Standard Model, the Higgs doublet vev breaks the electroweak gauge sector down to the electric charge $SU(2)_L \times U(1)_Y \rightarrow U(1)_Q$. Notice that only left-handed quarks and leptons interact with $SU(2)_L$ gauge fields. The SM Lagrangian has the symbolic form\(^2\)

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{kin}} - V(\Phi) + \mathcal{L}_{\text{Yuk}},$$

\(^2\)No gluons in what follows. For strong interactions, see the heavy-ions and QCD lectures by Edmond Iancu and Fabio Maltoni.
where
\[
\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} (F_{\mu\nu}^a)^2 + |D_\mu \Phi|^2 + \bar{\Psi}_L \gamma^\mu D_\mu \Psi_L + \bar{\Psi}_R \gamma^\mu D_\mu \Psi_R ,
\] (126)
and
\[
D_\mu \Psi_L = (\partial_\mu - ig \frac{\tau^a}{2} A_\mu) \Psi_L, \quad D_\mu \Psi_R = (\partial_\mu - ig \frac{Y_R}{2} B_\mu) \Psi_R ,
\] (127)
whereas the Higgs potential has the form
\[
V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 .
\] (128)
The Yukawa sector \( \mathcal{L}_{\text{Yuk}} \) will be discussed later on. With our conventions the electric charge is related to the third component of the isospin \( T_3 \) and to the hypercharge \( Y \) via
\[
Q = T_3 + \frac{Y}{2} .
\] (129)

6.2 Weak mixing angles and gauge boson masses

With the help of an \( SO(4) \) rotation, the Higgs vev can be written as
\[
\Phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} , \text{ where } v^2 = \frac{\mu^2}{\lambda} \simeq (246 \text{ GeV})^2 \quad \text{(from experimental data)} .
\] (130)
Gauge boson masses arise from the covariant derivative (\textbf{homework:})
\[
|D_\mu \Phi|^2 \rightarrow \frac{g^2 v^2}{8} |A_\mu^1 - iA_\mu^2|^2 + \frac{v^2}{8} |gA_\mu^3 - g'B_\mu|^2 = \frac{g^2 v^2}{4} W^{\pm} W^\mu_\mu + \frac{(g^2 + g'^2)v^2}{8} Z^\mu Z_\mu ,
\] (131)
where the definitions and the masses of gauge bosons are
\[
W^{\pm}_\mu = \frac{1}{\sqrt{2}} (A_\mu^1 \pm iA_\mu^2), \quad M_W = \frac{g v}{2} ,
\]
\[
Z_\mu = \frac{gA_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}} , \quad M_Z = \frac{v}{2} \sqrt{g^2 + g'^2} ,
\]
\[
A_\mu = \frac{g'A_\mu^3 + gB_\mu}{\sqrt{g^2 + g'^2}} , \quad M_A = 0 .
\] (132)
We now introduce the \textbf{electroweak mixing angle}
\[
\cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}} = \frac{M_W}{M_Z} , \quad \tan \theta_W = \frac{g'}{g} ,
\] (133)
that rotates from the weak basis to the mass basis
\[
\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} .
\] (134)
Notice that the ratio
\[
\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1 \text{ at tree-level in the SM.}
\] (135)
The \( \rho \) parameter has quantum corrections in the SM, which are dominated by the top quark. Any experimental deviation from the SM value is a possible hint of new physics. Conversely, any model of new physics has to be able to produce a \( \rho \) parameter close to one, which is one of the \textbf{precision tests} of the Standard Model. This is a killer for most proposals of Beyond the Standard Model physics. For example, technicolor-like theories have difficulties in this respect (although there is no formal proof that they cannot accomodate precision data). Finally, the definition of the electric charge is \( e = g \sin \theta_W \).
6.3 Neutral and charged currents

The neutral and charged currents are defined as the fermion bilinears coupling to the charged (W) and neutral (Z) gauge fields. They are worked out starting from the fermionic kinetic terms.

**Homework:** With the definitions above, show that

\[ D_\mu = \partial_\mu - igA_\mu^a \frac{\tau_a}{2} - ig\frac{Y}{2} B_\mu \]

\[ = \partial_\mu - ieQ_\mu - \frac{ig}{2\sqrt{2}} (W_\mu^+ \tau_+ + W_\mu^- \tau_-) - \frac{ig}{\cos\theta_W} Z_\mu (T_3 - \sin^2\theta_W Q) \quad (136) \]

The fermionic currents are defined as

\[ \mathcal{L} = \bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i + \frac{g}{\sqrt{2}} (W_\mu^+ J_W^{\mu+} + W_\mu^- J_W^{\mu-}) + \frac{g}{\cos\theta_W} Z_\mu J_Z^\mu + eA_\mu J_{em}^\mu \quad (137) \]

**Homework:** Using the quantum numbers of the quarks/leptons, show that

\[ J_{\mu+}^W = -\bar{\nu}_L \gamma^\mu \nu^\prime_R + \bar{u}_L \gamma^\mu \bar{u}^\prime_R + \bar{d}_L \gamma^\mu \bar{d}^\prime_R \]

\[ J_{\mu-}^W = \bar{\nu}_L \gamma^\mu \nu^\prime_R + \bar{u}_L \gamma^\mu \bar{u}^\prime_R + \bar{d}_L \gamma^\mu \bar{d}^\prime_R \]

\[ J_{em}^\mu = \frac{-2}{3} \bar{\nu}_L \gamma^\mu \nu^\prime_R + \frac{1}{3} \bar{d}_L \gamma^\mu \bar{d}^\prime_R \]

\[ J_Z^\mu = \rho \frac{1}{2} \bar{\nu}_L \gamma^\mu \nu^\prime_R \]

At low energies \( E \ll M_W, M_Z \), the exchange of \( W \) and \( Z \) bosons leads to the Fermi charged current four-fermion interaction, plus a similar neutral current interaction

\[ \mathcal{L}_F = -2\sqrt{2} G_F \left[ J_{\mu+}^W J_{\mu-}^W + \rho J_Z^\mu J_{\mu}^\mu \right] \quad (139) \]

where we defined the parameter \( \rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \), which, as we noticed in the previous paragraph, equals one at tree-level in the Standard Model and plays an important role in quantum corrections and constraints on new physics.

6.4 Fermion masses and the CKM matrix

Dirac mass terms in the SM are not gauge invariant, due to the chiral nature of electroweak interactions. We can however write Yukawa-type interactions by using the Higgs field

\[ -\mathcal{L}_{\text{Yuk}} = h^u_{ij} \tilde{\nu}_L \bar{u}_R^i \Phi + h^d_{ij} \tilde{\nu}_L \bar{d}_R^i \Phi + h^e_{ij} \tilde{\nu}_L \bar{e}_R^i \Phi \quad (140) \]

where: \( \Phi = \left( \begin{array}{c} \Phi^0 \\ \Phi^+ \end{array} \right) \) is the charge-conjugate Higgs field and \( i, j = 1, 2, 3 \) are flavor indices. The Yukawa couplings generate quarks and lepton masses after the electroweak symmetry breaking:

\[ -\mathcal{L}_{\text{mass}} = m^u_{ij} \tilde{\nu}_L \bar{u}_R^i \Phi + m^d_{ij} \tilde{\nu}_L \bar{d}_R^i \Phi + m^e_{ij} \tilde{\nu}_L \bar{e}_R^i \Phi \quad (141) \]

where \( m^{u,d,l}_{ij} = h^{u,d,l}_{ij} v / \sqrt{2} \). We use in what follows for compactness a matrix notation

\[ -\mathcal{L}_{\text{mass}} = \tilde{u}_L m^u \bar{u}_R + \tilde{d}_L m^d \bar{d}_R + \tilde{e}_L m^e \bar{e}_R \quad (142) \]
Fig. 11: Diagram leading to proton decay $p \rightarrow \pi^0 e^+$ in unified theories. The superheavy $X$ particle is a GUT gauge boson.

where $m_{u,d,l}$ are $3 \times 3$ mass matrices in the flavor space.

**Observation:** The SM Lagrangian has some automatic (consequences of the gauge symmetries) global symmetries:

- baryon number $U(1)_B$
- lepton numbers $U(1)_{e}$, $U(1)_{\mu}$, $U(1)_{\tau}$.

(143)

This is actually very fortunate since there are very strong experimental constraints on baryon and lepton number violating processes, for example:

- proton lifetime $\tau_p \geq 10^{33}$ years,
- $\text{BR}(\mu \rightarrow e\gamma) < 2.4 \times 10^{-12}$, $\text{BR}(\mu^- \rightarrow e^- e^- e^+) < 10^{-10}$,
- $\text{BR}(B \rightarrow X_s \gamma) \sim 10^{-4} \Rightarrow b \rightarrow s\gamma$ should be suppressed.

(144)

These limits constrain seriously any higher-dimensional operator violating flavor, generated by eventual new physics. For example, consider the operator

$$L_{\text{eff}} \sim \frac{1}{M_X^2} (\bar{q}_R \gamma \mu u_R) (\bar{d}_R \gamma \mu d_R) .$$

(145)

The bound on the proton lifetime constrains the mass to be heavier than about $M_X \geq 3 \times 10^{16}$ GeV. There is a long list of similar effective operators that are tightly constrained by the data. Another simple example is:

$$L_{\text{eff}} \sim \frac{1}{M^2} (\bar{I}_2 \gamma \mu l_1) (\bar{I}_1 \gamma \mu l_1) \rightarrow \frac{1}{M^2} (\bar{\mu} \gamma \mu e) (\bar{e} \gamma \mu e) ,$$

(146)

that can be generated by a flavor-dependent $Z'$ gauge boson. The mass scale $M$ is constrained by the limits on $\mu^- \rightarrow e^- e^- e^+$ to be heavier than $M > 1000$ TeV. It turns out that almost all SM extension generates dangerous FCNC and/or proton decay, unless special structure. For example, in MSSM we have to impose (i) $R$-parity, (ii) flavor blindness of soft terms.

**Observation:** With the field content of the SM, there is no operator generating neutrino masses at the renormalizable level. The main effective operator in the SM leading to neutrino masses is dimension five

$$\frac{h_{ij}}{M} (\bar{I}_i \Phi) (l_j \Phi) \Rightarrow m_{ij} = h_{ij} \frac{\gamma^2}{M} .$$

(147)

Tiny values (of order $10^{-2}$ eV) neutrino masses ask for $10^{12}$ GeV $< M < 10^{15}$ GeV, see B. Kayser’s lectures.
Coming back to the quarks and charged leptons masses, we can define the \textit{mass eigenstate basis} (as compared to the \textit{weak eigenstate basis}) with the help of the $3 \times 3$ unitary transformations\footnote{These transformations are not innocent; there is a quantum anomaly that we will discuss later on.}

\begin{align}
    u_{L,R} &= V^u_{L,R}u^c_{L,R}, \quad d_{L,R} = V^d_{L,R}d^c_{L,R}, \quad e_{L,R} = V^e_{L,R}e^c_{L,R}, \tag{148}
\end{align}

such that

\begin{align}
    (V^u_L)^\dagger m^u V^u_R &= \text{diag}(m_u, m_c, m_t), \text{ etc.} \tag{149}
\end{align}

In the mass basis, the neutral and the e.m. currents remain the same, whereas the hadronic charged current becomes

\begin{align}
    (J^m_{W^0})_{\text{quarks}} &\rightarrow \frac{1}{\sqrt{2}}\bar{u}_L^c \gamma^m V_{\text{CKM}} d^c_L \equiv \frac{1}{\sqrt{2}}\bar{u}_L^c \gamma^m \tilde{d}_L, \tag{150}
\end{align}

where $V_{\text{CKM}} = (V^u_L)^\dagger V^d_L$ is the (unitary) CKM matrix\footnote{The Cabibbo matrix}. We also defined

\begin{align}
    \tilde{d}_L = V_{\text{CKM}} d^c_L \leftrightarrow \begin{pmatrix} \tilde{d}_L \\ \tilde{s}_L \\ \tilde{b}_L \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d^c_L \\ s^c_L \\ b^c_L \end{pmatrix}. \tag{151}
\end{align}

There are therefore \textit{flavor changing transitions} in the SM: $s \rightarrow uW^-$, etc. Experimental measurements give a \textit{hierarchical form} of $V_{\text{CKM}}$ of the type (Wolfenstein parametrization)

\begin{align}
    \begin{pmatrix}
        1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\
        -\bar{\lambda} & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
        A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
    \end{pmatrix}, \tag{152}
\end{align}

where $\lambda = \sin \theta_c \approx .022$ is the Cabibbo angle. N. Cabibbo wrote first in 1962 the $2 \times 2$ version of the CKM matrix

\begin{align}
    \begin{pmatrix}
        \cos \theta_c & \sin \theta_c \\
        -\sin \theta_c & \cos \theta_c
    \end{pmatrix}. \tag{153}
\end{align}
It is simple to check that, after field redefinitions, $V_{CKM}$ contain three rotation angles and one CP violating phase. Notice also that CP violation in the SM is suppressed by $\lambda^3$ in $V_{CKM}$. The unitarity of the CKM matrix

$$V_{ik}V_{kj}^\ast = \delta_{ij} \quad V_{ki}^\ast V_{kj} = \delta_{ij}$$

has various important consequences. One of them is the GIM mechanism (Glashow–Iliopoulos–Maiani, 1972), to which we now turn.

### 6.5 The GIM mechanism

The FCNC (flavor changing neutral currents) effects were measured experimentally to be small. This was puzzling in the 1970’s, but it was explained in the SM by GIM [26]. Consider for example the $K^0 - \bar{K}^0$ mixing, which can arise at the loop-level. In the limit of equal or vanishing quark masses, the amplitude vanishes due to the unitarity of $V_{CKM}$:

$$A_{K^0\bar{K}^0} \sim \frac{g^4}{M_W^4}(\sum_i V_{id}^\ast V_{is})(\sum_j V_{js}V_{jd}^\ast) = 0.$$ 

(155)

By turning on the quark masses, the main contribution turns out to be proportional to $(m_c^2 - m_u^2)^2/M_W^4$ and is in excellent agreement with the experimental result.

**Historical Remark:** In 1972, only the $u,d$ and $s$ quarks were known. The GIM mechanism is considered to be the first proof of the existence of the charm quark.

**Homework:** Write down explicitly the diagrams for the $K^0 - \bar{K}^0$ mixing in the two generation case, with $u$ and $c$ quarks in the loop.

The unitarity relation

$$V_{ud}V_{ub}^\ast + V_{cd}V_{cb}^\ast + V_{td}V_{tb}^\ast = 0$$

(156)

can be represented geometrically as a triangle in a plane, named the unitarity triangle. It is customary to rescale the length of one side, i.e. $|V_{cd}V_{cb}^\ast|$ (well-known), to 1 and to align it along the real axis. The angles are defined as

$$\beta = \arg(-\frac{V_{td}V_{tb}^\ast}{V_{cd}V_{cb}^\ast}), \quad \gamma = \arg(-\frac{V_{ud}V_{ub}^\ast}{V_{cd}V_{cb}^\ast}),$$

(157)

and the lengths are

$$R_t = |\frac{V_{td}V_{tb}^\ast}{V_{cd}V_{cb}^\ast}|, \quad R_u = |\frac{V_{ud}V_{ub}^\ast}{V_{cd}V_{cb}^\ast}|.$$ 

(158)
On the other hand, quarks, leptons masses and the CKM matrix feature strong hierarchies. For example, from neutrino masses to the top mass there are $10^{11}$ orders of magnitude $m_{\nu} \sim 10^{-2} \text{eV} \ll m_e = 0.511 \text{MeV} \ll m_t \sim 172 \text{GeV}$.

There is no hint for a solution of this flavor puzzle in the SM, since the Yukawa couplings are free-parameters and therefore are not predicted. We are clearly missing something: maybe an additional global or gauge symmetry [33] or maybe this comes from an extra dimensional localization or environmental selection.

6.6 The custodial symmetry

The tree-level relation $\rho = M_W^2 / (M_Z^2 \cos^2 \theta_w) = 1$ can be understood as the result of an (approximate) symmetry, called custodial symmetry, proven by Sikivie, Susskind, Voloshin and Zakharov [35].

**Theorem:** In any theory of electroweak interactions which conserves the electric charge and has an approximate global $SU(2)$ symmetry under which $A_m$ transform as a triplet, $\rho = 1$ at tree-level.

Here approximate means in the limit of zero hypercharge coupling $g' = 0$ and in the absence of the Yukawa couplings.

**Proof:** Under this assumption, the gauge boson mass matrix is of the form

$$
\begin{pmatrix}
M^2 & 0 & 0 & 0 \\
0 & M^2 & 0 & 0 \\
0 & 0 & M^2 & m_1^2 \\
0 & 0 & m_1^2 & m_2^2
\end{pmatrix}.
$$

(159)

The masslessness of the photon implies $M^2 m_2^2 - m_1^4 = 0$. The resulting $W - A$ mass matrix, written in terms of the $W$ and $Z$ masses, is then of the form:

$$
\begin{pmatrix}
M_W^2 \\
\pm M_W \sqrt{M_Z^2 - M_W^2} \\
\pm M_W \sqrt{M_Z^2 - M_W^2} \\
M_Z^2 - M_W^2
\end{pmatrix}.
$$

(160)

It is then easy to check that $M_W = \cos \theta_w M_Z$.

On the other hand, in the SM the Higgs potential $V(\Phi^\dagger \Phi)$ is invariant under an $SO(4)$ global symmetry. Indeed, let us write explicitly the four real components of the SM Higgs doublet

$$
\Phi = \begin{pmatrix} \Phi_1 + i \Phi_2 \\ \Phi_3 + i \Phi_4 \end{pmatrix}, \text{ then } \Phi^\dagger \Phi = \sum_{i=1}^{4} \Phi_i^2.
$$

(161)

It is then transparent that the Higgs potential and kinetic term have an $SO(4) = SU(2)_L \times SU(2)_R$ symmetry. The Higgs vev, $\langle \Phi \rangle = (0, v/\sqrt{2})$, breaks $SO(4) \rightarrow SO(3) = SU(2)_D$, which corresponds precisely to
the custodial symmetry. From these considerations, it is clear that not any Higgs representations preserve
the custodial symmetry. What happens for other Higgs representations? It can be shown that, considering
Higgs representations in weak isospin representations of isospin \( I \), the rho parameter is given by
\[
\rho = \frac{1}{2} \frac{\sum \langle I_i (I_i + 1) - I_{\rho}^2 \rangle}{\sum \langle I_i (I_i + 1) - I_{\rho}^2 \rangle} \frac{1}{\langle 0 | \phi_i | 0 \rangle} ^ 2 .
\] (162)

It is then easy to check that for an arbitrary number of singlet and higgs doublets, \( \rho = 0 \). On the other
hand, for Higgs triplets for example, the higgs vev generate the breaking \( SO(3) \rightarrow SO(2) \). In this case
there is no custodial symmetry and \( \rho \neq 1 \).

Strong interactions preserve electric charge and strong isospin. A natural choice for the custodial
symmetry is therefore the strong isospin, which then guarantees that \( \rho = 1 \) to all order in the strong
interactions.

A useful parametrization for estimating the violation of the custodial symmetry is:
\[
\mathcal{H} = (i \tau_2 \Phi^* \Phi) = \begin{pmatrix} \Phi^0_0 & \Phi^0_+ \\ -\Phi^0_+ & \Phi^0_0 \end{pmatrix} , \quad \Phi^0 \Phi = \frac{1}{2} Tr \mathcal{H}^\dagger \mathcal{H} .
\] (163)

The potential \( V(\Phi^\dagger \Phi) \) is invariant under \( \mathcal{H} \rightarrow U_L \mathcal{H} U_R^\dagger \), with \( U_{L,R} 2 \times 2 \) unitary matrices implementing
\( SU(2)_L \times SU(2)_R \) transformations. The electroweak symmetry breaking pattern is then
\[
\langle \mathcal{H} \rangle = \frac{v}{\sqrt{2}} I_2 \times 2 \text{ breaks } SU(2)_L \times SU(2)_R \rightarrow SU(2)_D .
\] (164)

As anticipated, the hypercharge gauge interactions \( U(1)_Y \) and Yukawa couplings break the custodial
symmetry. However the particular coupling
\[
\mathcal{L}_{\text{Yuk}} = h \left( i_L \bar{b}_L \right) \mathcal{H} \begin{pmatrix} t_R \\ b_L \end{pmatrix}
\] (165)

is invariant under \( SU(2)_D \). This corresponds to the limit of equal masses in the quark doublet \( h_t = h_b \).
On the other hand, \( W \) and \( Z \) boson masses have quantum corrections that lead to calculable deviations
from \( \rho = 1 \). A one-loop computation in the SM gives
\[
\delta \rho = \frac{3g^2 (m_t^2 - m_b^2)}{64 \pi^2 M_W^2} - \frac{3g^2}{32 \pi^2} \ln \frac{m_t}{M_Z} + \cdots
\] (166)
where \( \cdots \) are subleading contributions from the SM or from eventual new physics contributions (see the
lectures by B. Dobrescu) that have to be smaller than \( 10^{-3} \) in order to fit the experimental data [36].

7 Quantum corrections and renormalization

Quantum corrections through loops are subtle to incorporate, due to UV divergences appearing for large
momenta of virtual particles running in the loops. Dealing with these divergences is crucial in order to
extract physical results. This led to the program of renormalization, which was brilliantly confirmed by
various precision measurements, in particular at the LEP collider. The proof of renormalization of the
Standard Model led to the 1999 Nobel prize of G. ’t Hooft and M. Veltman [37].

7.1 UV divergences and regularization

Perturbation theory in Quantum Field Theory is plagued with UV divergences. We have to keep an UV
cutoff \( \Lambda \) (which can be implemented in various different ways) in computing physical quantities. There
are three cases that arise:
---

**Super-renormalizable theories**: In this case only a finite number of Feynman diagrams diverge. Beyond a sufficiently large number of loops, all Green functions are finite.

**Renormalizable theories**: a finite number of amplitudes/Green functions diverge, with a number of external legs below a maximal value (which is for example four for the $\phi^4$ theory, three for QED and four for Yang–Mills theories). For these amplitudes, the UV divergences arise at all orders in perturbation theory.

**Non-renormalizable theories**: All amplitudes, with an arbitrary number of legs are UV divergent at a certain order in perturbation theory.

In renormalizable and super-renormalizable theories, UV divergences can be absorbed into rescaling of fields and redefinitions of the various couplings and masses. Taking the couplings/masses from experimental data and “hiding” the UV cutoff in their redefinitions, we obtain physical quantities free of UV divergences. In this case, the theory is predictive at any energy scale. In non-renormalizable theories, we need an infinite number of couplings and masses in order to absorb the UV divergences. We would need an infinite amount of experimental data to determine all these couplings. Therefore, at high-energies $E > \Lambda$ the theory looses its predictive power. However, at low-energy the theory is perfectly predictive. The typical example of this type is the General Relativity.

### 7.2 Relevant, marginal and irrelevant couplings

Consider a scalar theory of the form

$$ S_\Lambda = \int d^4 x \left( \frac{1}{2} (\partial \phi)^2 + \frac{m^2 \phi^2}{2} + \sum_n \lambda_n \phi^n \right), $$

(167)

where $S_\Lambda$ is the euclidian action defined with a cutoff $\Lambda$. The couplings $\lambda_n$ have (classical) mass dimensions $[\lambda_n] = 4 - n$. Let us consider the theory with two different maximal euclidian cutoff momenta:

i) $0 < p < \Lambda$

ii) $0 < p < \Lambda' = \varepsilon \Lambda$, where $\varepsilon < 1$.

In case ii) the theory has therefore a lower cutoff and it is interpreted as a theory where the high-momenta of theory i) were integrated out. The theory i) has the action (167). In the theory ii) the cutoff can be redefined to be the same as in i) with the help of a *scale transformation*

$$ x' = \varepsilon x \quad , \quad p' = \varepsilon^{-1} p \quad , \quad \phi' = \varepsilon^{-1} \phi. $$

(168)

In terms of the rescaled field and coordinates, the action of theory ii) become

$$ S_\Lambda' = \int d^4 x' \left( \frac{1}{2} (\partial' \phi')^2 + \frac{m'^2 (\phi')^2}{2} + \sum_n \lambda'_n (\phi')^n \right), $$

(169)

where

$$ m'^2 = \frac{1}{\varepsilon^2} m^2 \quad , \quad \lambda'_n = \varepsilon^{n-4} \lambda_n. $$

(170)

Notice that the new mass and couplings scale with their classical dimension. We see therefore that the mass and couplings with positive dimension grow in the IR, whereas couplings with negative dimension decrease in the IR. It is said that

$$ [\lambda_n] > 0 \quad \Rightarrow \quad \text{relevant couplings}, $$

$$ [\lambda_n] = 0 \quad \Rightarrow \quad \text{marginal couplings}, $$

$$ [\lambda_n] < 0 \quad \Rightarrow \quad \text{irrelevant couplings}. $$

(171)

This point of view on renormalization was introduced by K. Wilson and is summarized, for example, in Ref. [38].
7.3 (Non)renormalizability and couplings dimensions

There is a straight connection between renormalizability and the three type of couplings previously defined:

− relevant couplings $\Rightarrow$ super-renormalizability.
− marginal couplings $\Rightarrow$ renormalizability.
− irrelevant couplings $\Rightarrow$ non-renormalizability.

It is easy to argue for this by dimensional arguments. Let us consider some simple examples, going back in Minkowski space:

1. Relevant coupling

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{m^2 \phi^2}{2} - \lambda_3 \phi^3.$$ (172)

The coupling has dimension $[\lambda_3] = +1$, so it is relevant. At one-loop, the UV divergent terms lead to new terms in the Lagrangian (homework:)

$$\delta \mathcal{L}_1 \sim \lambda_3 \Lambda^2 \phi + \lambda_3^2 \phi^2 \ln \Lambda,$$ (173)

which are both of super-renormalizable type. The first leads to a scalar tadpole, whereas the second leads to a mass renormalization. At two loops, the only UV divergences are a cosmological constant and a scalar tadpole. At three loops, there is only a log UV divergence in the cosmological constant. No UV divergences exist at higher loops. Dimensional argument: By dimensional analysis, the highest UV divergent term in the coupling is the three-loop vacuum energy

$$\lambda_3^4 \ln \Lambda.$$ (174)

Higher loops have higher powers in $\lambda_3$ and cannot contribute to the UV divergent terms in the effective Lagrangian. Observation: $1/m^2$ terms are IR, not UV contributions, so they cannot appear in UV divergent terms.

2. Irrelevant coupling

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{m^2 \phi^2}{2} - \lambda_6 \phi^6.$$ (175)

The coupling has dimension $[\lambda_6] = -2$, so it is irrelevant. At one-loop, the UV divergent terms in the eight-point amplitude lead to (homework:)

$$\Gamma^{(8)}_{1\text{-loop}}(p_i) \sim c \lambda_6^2 \ln \Lambda + \cdots.$$ (176)

To cancel this divergence, one has to add a new coupling to the original action

$$\delta \mathcal{L}_1 \sim \lambda_8 \phi^8,$$ (177)

and to adjust the coupling $\lambda_8$ such that

$$\lambda_8 + c \lambda_6^2 \ln \Lambda = \text{finite}.$$ (178)

At two-loops, we get new UV divergences, like the one in the six-point amplitude, proportional to

$$\Gamma^{(6)}_{2\text{-loops}}(p_i) \sim c'(p_i p_j) \lambda_6^2 \ln \Lambda,$$ (179)

which can be canceled by adding another coupling

$$\delta \mathcal{L}_2 \sim \lambda_8' \phi^4 (\partial \phi)^2,$$ (180)
such that
\[ \lambda' + c' \lambda^2 \ln \Lambda = \text{finite}. \]  

(181)

The UV divergences \textit{proliferate} at higher loop orders, generating an infinite tower of operators of higher and higher dimension. \textit{Dimensional argument:} Terms of the type \( \lambda^n \phi^{4+2n} \ln \Lambda \), \( \lambda^n (\partial \phi)^2 \phi^{2n} \ln \Lambda \) have the correct dimension to be generated for any \( n \). Predictivity at high-energy is \textit{lost}. Let us however define \( \lambda_0 \sim 1/M^2 \). Then: In the IR, \( E < M \), the effect of non-renormalizable operators on physical quantities is proportional to some positive power or \( E/M \) and/or \( m/M \), so their effects is negligible. Effective theories with cutoff \( \Lambda \) (example General relativity, \( \Lambda = M_P \)) are therefore predictive at energies \( E \ll \Lambda \).

Another viewpoint on this problem is the following: for \( \mathcal{L}_{\text{int}} = \sum_n \lambda_n \phi^n \), the leading cross-section for \( 2 \rightarrow 2 \) particle scattering is
\[
\sigma = \sum_n c_n \lambda_n^2 E^{2n-10} \sim \frac{1}{E^2} \sum_n c_n \left( \frac{E}{M} \right)^{2n},
\]  

(182)

for \( \lambda_n \sim 1/M^{n-4} \). Therefore the \textit{predictive power is lost} for \( E \geq M \).
7.4 Coupling constant renormalization for $\phi^4$ theory

Consider the $\phi^4$ theory of Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{m_0^2}{2} \phi^2 - \frac{\lambda_0}{4!} \phi^4 .$$

(183)

Let us compute the four-point function at one-loop. By using the Feynman rules for the $\phi^4$ theory, we find, according to the figure in the next page

$$\Gamma^{(4)}(k_1k_2k_3k_4) = -i\lambda_0 + \frac{(-i\lambda_0)^2}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_0^2} \frac{i}{(p-k_1-k_2)^2 - m_0^2}$$

(184)

$$+ \text{two crossing terms} .$$

After the Wick rotation to euclidian momenta, the result is given by

$$\Gamma^{(4)}(k_1k_2k_3k_4) = -i\lambda_0 + \frac{i\lambda_0^2}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m_0^2} \frac{1}{(p-k_1-k_2)^2 + m_0^2}$$

(185)

$$+ \text{two crossing terms} .$$

The integral is log divergent in the UV. There are various ways to "renormalize" the integral. Here is a simple way. Define

$$V(s) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m_0^2} \frac{1}{(p-k_1-k_2)^2 + m_0^2} = \int_{p^2 \geq \mu^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^4} + \text{finite} ,$$

(186)

where the energy scale $\mu$ is arbitrary. We find (homework:)

$$\Gamma^{(4)}(k_1k_2k_3k_4) = -i\lambda_0 + \frac{i\lambda_0^2}{2} [V(s) + V(t) + V(u)]$$

$$= -i\lambda_0 + \frac{3i\lambda_0^2}{16\pi^2} \ln \frac{\Lambda}{\mu} + \text{finite} = -i\lambda(\mu) + \text{finite} .$$

(187)

What is the physical interpretation of this manipulation? We can separate the answer into two separate steps:

1. $\lambda_0$ is not a physical parameter. It can be chosen to depend on $\Lambda$ such that

$$\lambda(\mu) = \lambda_0(\Lambda) - \frac{3\lambda_0^2}{16\pi^2} \ln \frac{\Lambda}{\mu}$$

(188)

is independent of $\Lambda$.

2. Any value of $\mu$ leads to the same physical result. We can find a differential equation for $\lambda$ by using the fact that $\lambda_0$ is independent of $\mu$. We obtain

$$\frac{d\lambda}{d\ln \mu} = \frac{3\lambda^2}{16\pi^2} = \beta(\lambda) ,$$

(189)

which is called the renormalization group equation (RGE) of $\lambda$ at one-loop, with $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$ being the one-loop RG beta function coefficient. The solution of (189) is (homework:)

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{16\pi^2} \ln \frac{\mu}{\mu_0}} .$$

(190)

Notice that there is an equivalent prescription: to add a local "counterterm" to the Lagrangian

$$\mathcal{L} + \delta \mathcal{L} = \mathcal{L}_0 ,$$

(191)

which cancels the UV divergence. The counterterms are treated as interactions in perturbation theory. The two points of view lead to identical results.
In renormalizable theories, a finite number of counterterms are needed in order to render the theory UV finite. For the same purpose, in non-renormalizable theories we need an infinite number of counterterms.

7.5 QED and the running of fine structure constant

We use here the counterterm method for the renormalization of QED. In this case, the initial Lagrangian, the counterterms and their sum is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - M)\Psi,$$

$$\delta \mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}^2 + (Z_2 - 1)\bar{\Psi}i\gamma^\mu \partial_\mu \Psi - (Z_1 - 1)q\bar{\Psi}\gamma^\mu A_\mu \Psi - (Z_M - 1)M\bar{\Psi}\Psi,$$

$$\mathcal{L}_0 = \mathcal{L} + \delta \mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^0)^2 + \bar{\Psi}_0(i\gamma^\mu \partial_\mu - q_0\gamma^\mu A_\mu^0 - M_0)\Psi_0.$$

(192)

The relations between the bare and renormalized quantities are then

$$A_\mu^0 = Z_3^{1/2} A_\mu, \quad \Psi_0 = Z_2^{1/2} \Psi,$$

$$M_0 = \frac{Z_M}{Z_2} M, \quad q_0 = \frac{Z_1}{Z_2Z_3^{1/2}} q,$$

(193)

where $Z_1$ comes from the one-loop vertex correction, $Z_2$ ($Z_3$) is the fermionic (photon) wave function renormalization, whereas $Z_M$ is the mass renormalization. In QED it can be shown that $Z_1 = Z_2$, the so-called Ward identity. Then charge renormalization in QED comes only from vacuum polarization $q_0 = Z_3^{-1/2} q$. The RG running can be found from

$$\mu \frac{\partial}{\partial \mu} q_0 = 0 \Rightarrow \beta(q) = \mu \frac{\partial q}{\partial \mu} = q \frac{\partial \ln Z_3^{1/2}}{\partial \ln \mu}.$$

(194)

By an explicit computation in QED with just the electron in the loop and by defining the fine-structure constant $\alpha = q^2/(4\pi)$, we find

$$Z_3 = 1 - \frac{\alpha}{3\pi} \ln \frac{\Lambda}{\mu} + \text{finite},$$

(195)

where $\mu$ is an arbitrary, renormalization scale. We then find

$$\beta(q) = \frac{q^3}{24\pi^2} \Rightarrow \frac{1}{\alpha(q)} = \frac{1}{\alpha(\mu)} - \frac{1}{3\pi} \ln \frac{Q}{\mu}.$$

(196)

We found therefore that the fine structure coupling increases with energy! This can be intuitively interpreted due to the screening of the electric charge by electron-positron pairs from the quantum vacuum (see Figure 16).

The situation for the strong coupling $\alpha_s$ is different due to the non-abelian nature of the interaction. The result is an anti-screening due to gluon self-interactions [40].

There is a tantalizing hint of unification of gauge couplings at high-energy, as seen from figure 17, that could point towards a unified gauge structure at high-energy [41]. Running couplings and renormalization are important everywhere in the SM and its applications. For example:

- In any process the couplings have to be evaluated at the relevant energy scale. Examples:
  - In $\pi^0 \to \gamma\gamma$, the fine-structure constant has to be evaluated at the pion mass $\alpha(m_\pi)$.
  - Identification of relevant momenta and RGE of operators in QCD is crucial in order to extrapolate perturbative quantities down in energy via the renormalization group.

33
Fig. 16: Screening of electric charges by vacuum polarization provides an intuitive picture of the "running" of electric charge. Figure taken from Ref. [39].

Fig. 17: Extrapolation of gauge couplings in the (minimal supersymmetric extension) of the Standard Model [42]: hint of unification of couplings at high energy?

In the study of electroweak baryogenesis, for example the scalar potential

\[ V(\Phi) \simeq -\mu^2 (|\langle \Phi \rangle|) |\Phi|^2 + \lambda (|\langle \Phi \rangle|) |\Phi|^4 \]  

is to be evaluated at the minimum at the scalar potential.

8 Global and gauge anomalies

Symmetries of the classical action can have anomalies at the quantum level, generated by one-loop triangle diagrams [27], see Fig. 18. There are two different cases to consider:

– anomalies in the conservation of a global symmetry current,
Fig. 18: Adler–Bell–Jackiw triangle anomalies.

— anomalies for gauged symmetries.

8.1 Global anomalies

For global symmetries, this does not creates consistency problems and they actually play an important role in QCD and the electromagnetic decay of the $\pi^0$ pion. Consider to start with a Dirac fermion coupled to a $U(1)$ gauge field

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu D_\mu \Psi - M \bar{\Psi} \Psi.$$  \hspace{1cm} (198)

In the massless limit $M \to 0$, the model has a vector and an axial symmetry $U(1)_V \times U(1)_A$. The corresponding Noether currents

$$J_m = \bar{\Psi} \gamma_m \Psi \ , \ J_m^5 = \bar{\Psi} \gamma_m \gamma^5 \Psi$$  \hspace{1cm} (199)

satisfy

$$\partial^a J_m = 0 \ , \ \partial^a J_m^5 = 2i M \bar{\Psi} \gamma^a \Psi - \frac{g^2}{16 \pi^2} \varepsilon^{mnpq} F_{mn} F_{pq},$$  \hspace{1cm} (200)

The last term is the quantum anomaly. Even if both currents are both classically conserved for $M = 0$, there is no regularization preserving both the vector and the axial conservation. If $U(1)_V$ is a gauge symmetry (the electromagnetism), we have therefore to choose a regularization preserving the vector current conservation. Then one is forced to accept an anomaly in the axial current. This explains actually why the $\eta'$ meson is not a pseudo-Goldstone for the dynamical chiral symmetry breaking $U(2)_L \times U(2)_L = SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A \Rightarrow SU(2)_V \times U(1)_B$ in QCD. Indeed, in this case the $U(1)_A$ axial current has the QCD anomaly

$$J_m^{U(1)_A} = \bar{u} \gamma_m \gamma^5 u + \bar{d} \gamma_m \gamma^5 d,$$

$$\partial^a J_m^{U(1)_A} = 2i (m_u \bar{u} \gamma^a u + m_d \bar{d} \gamma^a d) - \frac{g^2}{16 \pi^2} \varepsilon^{mnpq} F_{mn} F_{pq} A^A,$$  \hspace{1cm} (201)

where $F^A$ is the gluon field strength. Due to the explicit breaking and the nonperturbative instanton effects, the $\eta'$ gets a mass larger than the other pions, which are the pseudo-goldstones of the axial $SU(2)_A$ symmetry. Another manifestation of the axial anomaly is the decay $\pi^0 \to \gamma \gamma$. Let us define the $SU(2)$ currents

$$J_m = \bar{q} \gamma_m \tau^a q \ , \ J_m^5 = \bar{q} \gamma_m \gamma^5 \tau^a q.$$  \hspace{1cm} (202)

The fact that the pions are Goldstone bosons implies

$$\langle 0 | J_m^{\tau^a}(x)| \pi^b(p) \rangle = - i p_m f_\pi \delta^{ab} e^{-i px}.$$  \hspace{1cm} (203)

Axial isospin currents have no QCD anomalies, but $J_m^{\tau^a}$ has an anomaly from the electromagnetic coupling

$$\partial^m J_m^{\tau^a} = - \frac{1}{32 \pi^2} \varepsilon^{mnpq} F_{mn} F_{pq} \text{tr}(Q^2 \tau^a) = - \frac{N_c e^2}{48 \pi^2} \varepsilon^{mnpq} F_{mn} F_{pq},$$  \hspace{1cm} (204)
where $Q$ describes the quark electric charges $Q_u = 2e/3$, $Q_d = -e/3$ and $N_c = 3$ is the number of quark colors.

By using Eqs. (203) and (204) and using that under an axial $SU(2)$ with quarks transforming as

$$\delta q = i\gamma^5\tau_3 q$$

with $q = (u, d)$, the pion transforms like a Goldstone boson $\delta\pi^0 = \alpha f_{\pi}$, we obtain that the effect of the anomaly is to generate an effective pion–photon–photon coupling

$$L_{\text{eff}} = \frac{\pi^0}{f_{\pi}} \partial^m f^{5,3}_m = -\frac{N_c e^2}{48\pi^2 f_{\pi}} \pi^0 \varepsilon^{mpnq} F_{mn} F_{pq}.$$  
(205)

Using this effective coupling, the following result is obtained $\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{\alpha^2 m_{\pi}^3}{64\pi^3 f_{\pi}}$, which is in excellent agreement with the experimental branching ratio of the pion decay into two photons $\Gamma(\pi^0 \rightarrow \gamma\gamma) = (1.19 \pm 0.08) \times 10^{16}\text{s}^{-1}$.

On the other hand, there is no symmetry principle forbidding the term in the QCD Lagrangian

$$L_{\theta} = \frac{g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}),$$  
(206)

where $\theta$ is a real (angular) parameter. Even if this can be shown to be a total derivative, instanton configurations in QCD makes this term to have nontrivial consequences. Actually, the $\theta$ parameter has another contribution from the unitary redefinitions (148) that we were forced to do in order to diagonalize the quark mass matrices. Indeed, the $U(1)_{A}$ part of these transformations is anomalous, leading to a change in the theta parameter

$$\theta \rightarrow \theta - \frac{1}{2} \arg \det m^q.$$  
(207)

The theta parameter violates CP and the gluonic term generates a neutron dipole moment of order $d_n \sim \left| \theta \right| e m_{\pi}^2 / m_N^3 \sim 10^{-16} \left| \theta \right| \text{ emu}$, in conflict with the experimental data unless $\theta < 10^{-10}$. This leads to the so-called strong CP problem. The problem would be absent if the up-quark mass would be zero, since in this case the theta parameter could be shifted to zero by an up-quark chiral redefinition. The masslessness of the up-quark is however excluded by now. One of possible solutions to the strong CP problem is the axion, $a$ [28]. If

- there is a new $U(1)_{PQ}$, spontaneously broken global symmetry, with the corresponding pseudo-Goldston boson $a$, and the symmetry breaking scale $f$,
- which has triangle anomalies with the QCD gauge group $U(1)_{PQ} \times SU(3)_c$,

then the anomaly generates new couplings in the Lagrangian which shift the $\theta$ parameter

$$\frac{g^2}{32\pi^2} \frac{a(x)}{f} \varepsilon^{\mu\nu\rho\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) \Rightarrow \theta_{\text{eff}} = \theta + \frac{a}{f},$$  
(208)
where \( \xi \) is a model-dependent parameter parametrizing the strength of the axion couplings to matter. Then non-perturbative QCD instanton effects generate an axion potential of the type

\[
V(a) \sim \frac{g^2}{32\pi^2} \Lambda^4_{\text{QCD}} \left[ 1 - \cos \left( \frac{\xi a(x)}{f} + \theta \right) \right].
\]

The minimum of the scalar potential is then at \( \theta_{\text{eff}} = 0 \) and the axion mass is

\[
m_a \sim \frac{\xi g}{4\sqrt{2}\pi} \Lambda^2_{\text{QCD}} \frac{1}{f}.
\]

Axions were intensively searched since the 80’s, see Fig. 20 for recent constraints in the plane \( (g_{a\gamma}, m_a) \), where \( g_{a\gamma} \) is the axion-photon coupling. Axions are also present in \textit{most SUSY and string extensions} of the SM.

Let us finish this presentation with a comment. The axial anomaly is actually a total derivative:

\[
\varepsilon^{\mu\nu\rho\sigma} Tr(F_{\mu\nu} F_{\rho\sigma}) = \partial^\mu K_\mu,
\]

where

\[
K_\mu = 2\varepsilon_{\mu\nu\alpha\beta} \left( A_{\nu}^{\alpha} \partial^\beta A_{\alpha}^{\beta} + \frac{1}{3} f_{\alpha\beta\gamma} A_{\nu}^{\alpha} A_{\alpha}^{\beta} A_{\beta}^{\gamma} \right).
\]

Despite this fact, classical configurations generate effects like the theta angle in QCD and \( B \) and \( L \) \textit{numbers are non-conserved}. Indeed, the baryonic current for example has an \( \text{U}(1)_B \times \text{SU}(2)_L \) anomaly

\[
J^\mu_B = \frac{1}{3} \sum_i \bar{q}_i \gamma_\mu q_i, \quad \partial^\mu J_\mu_B \sim \frac{g^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} Tr(F_{\mu\nu} F_{\rho\sigma}),
\]
Fig. 21: One-loop gauge anomalies, if present, render the theory inconsistent at the quantum level.

\[ \Delta B = \int d^4\partial \mu J_\mu \sim \int d^4\varepsilon_{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) \sim \oint d\Sigma \mu dK_\mu \]  

which is different for zero for classical gauge field configurations vanishing slowly at infinity. The violation of baryonic symmetry of this type is important for generating the observed baryon asymmetry in our universe [29].

8.2 Gauge anomalies

For gauge symmetries on the other hand, anomalies, if present as in Fig. 25, generate inconsistencies [30]. Indeed, they would violate gauge invariance of the theory since the gauge variation of the Lagrangian from the Noether theorem (5) is

\[ \delta \mathcal{L} \sim \alpha_A \partial^\mu J^A_\mu . \]  

The corresponding currents are of chiral type

\[ J^A_\mu = \bar{\Psi} \gamma_\mu \gamma_5 T^a \Psi = \bar{\Psi}_R \gamma_\mu T^a \Psi_R - \bar{\Psi}_L \gamma_\mu T^a \Psi_L , \]  

and their divergences are proportional to

\[ \partial^\mu J^A_\mu = - \frac{gB}{32\pi^2} A^{ABC} \epsilon^{\mu\nu\rho\sigma} F^B_{\mu\nu} F^C_{\rho\sigma} . \]  

The anomaly coefficients that have to vanish are then

\[ A^{ABC} = \text{tr} (\{T^A, T^B\} T^C)_L - \text{tr} (\{T^A, T^B\} T^C)_R = 0 , \]  

where the trace is taken over all the fermions in the theory. For the SM, the only possible anomalies are

\[ SU(2)_L^2 U(1)_Y , \ U(1)_Y^3 \ and \ SU(3)_c^2 U(1)_Y . \]  

The results in the SM are

\[ \text{Tr} \left( \left\{ \frac{\tau^a}{2}, \frac{\tau^b}{2} \right\} Y \right)_L = \frac{1}{2} \delta^{ab} (\text{Tr} Y)_L = 3 \times (N_c \times \frac{1}{3} - 1) = 0 , \]

\[ \text{Tr}(\{Y, Y\} Y)_{L-R} = \cdots = 6(-2N_c + 6) = 0 , \]

\[ \text{Tr} \left( \left\{ \frac{\lambda^A}{2}, \frac{\lambda^B}{2} \right\} Y \right)_{L-R} = \frac{1}{3} \delta^{AB} (\text{Tr} Y)_{L-R} = \cdots = 0 . \]  

Notice that anomaly cancelation happens precisely for \( N_c = 3 ! \) This seems to provide a deep connection between quarks and leptons in the SM, and a possible hint towards Grand Unified Theories. Anomaly cancelation gives strong constraint on new chiral particles. For example, it is easy to show that (homework:)

- the only flavor-independent, anomaly-free \( Z' \) with the chiral SM spectrum is \( U(1)_{B-L} . \)
- a fourth lepton generation \( l_4, E_R \) alone is inconsistent.
9 The Higgs / Symmetry breaking sector of the Standard Model

The Higgs boson is the last building block of the Standard Model awaiting its experimental discovery. There are good theoretical reasons to and preliminary experimental hints from LHC to hope that this can happen quite soon. We review here some of the theoretical biases which make theorists to favor the existence of a light higgs scalar. The first two, the perturbativity and the stability bound, are obtained by extrapolating the Standard Model to high energy scales and imposing perturbativity of couplings and stability of the SM ground state, respectively. The third one is related to the breakdown of unitarity in the longitudinal $WW$ scattering at high-energy if the higgs is too heavy or if does not exist.

9.1 Perturbativity bounds

As shown in Section 7, in quantum field theory couplings run. The Higgs mass is then obtained by knowing the Higgs self-coupling $\lambda$ at the electroweak scale $M_h^2 = 2\lambda(v)v^2$. If this coupling is large enough, it will hit a Landau pole at a high-energy scale called $\Lambda$ in what follows. The RGE for the Higgs self-coupling in the SM is

$$16\pi^2 \frac{d\lambda}{d\ln \mu} = 24\lambda^2 - (3g'^2 + 3g^2 - 12h^2_t) \lambda + \frac{3}{8}(g'^4 + 2g^2g'^2 + 3g^4) - 6h^4_t + \cdots ,$$

where $\cdots$ denote smaller Yukawas. In the large Higgs mass limit $\lambda \gg g^2, h^2_t$, this reduces to

$$\frac{d\lambda}{\lambda^2} = \frac{3}{2\pi^2} \ln \mu \Rightarrow \frac{1}{\lambda(\mu)} = \frac{1}{\lambda(\Lambda)} + \frac{3}{2\pi^2} \ln \frac{\Lambda}{\mu} .$$

This can be interpreted in two alternative ways [44]

1. If the Higgs mass is known, the SM has a Landau pole (signal of a non-perturbative regime) $\lambda(\Lambda) \gg 1$ at an energy scale

$$\Lambda = v e^{\frac{3\lambda^2}{4\pi^2}} = v e^{\frac{4\pi^2}{3M_h^2}} .$$

2. Conversely, asking for perturbativity up to scale $\Lambda$ (say $M_{GUT}$), we obtain an upper bound on the Higgs mass (homework:)

$$M_h^2 \leq \frac{4\pi^2v^2}{3\ln \frac{\Lambda}{\tau}} .$$

9.2 Stability bounds

Standard Model has a potential instability in the small Higgs mass limit [45], since if too small at the electroweak scale, $\lambda$ can become negative at high-energy by the RG running. If $\lambda \ll h^2_t$, the relevant leading RGE’s are

$$16\pi^2 \frac{d\lambda}{d\ln \mu} = -6h^4_t , \quad 16\pi^2 \frac{dh_t}{d\ln \mu} = \frac{9h^4_t}{2} ,$$

which integrate to (homework:)

$$\lambda(\mu) = \lambda(\Lambda) + \frac{\frac{3h^4_t}{16\pi^2} \ln \frac{\Lambda}{\mu}}{1 + \frac{9h^4_t}{16\pi^2} \ln \frac{\Lambda}{\mu}} ,$$

$$h^2_t(\mu) = \frac{h^2_t(\Lambda)}{1 + \frac{9h^4_t}{16\pi^2} \ln \frac{\Lambda}{\mu}} .$$

As in the perturbativity case limit, this can be interpreted in two ways:
1. For a fixed, known value of the Higgs mass. Let us take $\mu = v$. Then, new physics should show up before the scale $\Lambda$ where $\lambda(\Lambda) = 0$, $\Lambda \leq v e^{\frac{4\pi^2 h}{2^4 \pi^2}} = v e^{\frac{4\pi^2 M^2_h}{2^4 \pi^2}}$. (226)

2. Alternatively, for a fixed $\Lambda$, we get a lower bound on the Higgs mass (homework):

   $$M_{h}^{2} \geq \frac{3h_{W}^{4}v^{2}}{4\pi^{2}} \ln \frac{\Lambda}{v} = \frac{3m_{h}^{4}}{\pi^{2}v^{2}} \ln \frac{\Lambda}{v}.$$ (227)

These theoretical Higgs mass limits are summarized in the plot in Fig. 22 which contains more accurate numerical solution to the RG equations. If the scale $\Lambda$ is very low, these bounds are very loose. On the other hand, if the SM as an effective theory is valid up to the Planck scale, we obtain a pretty tight mass range $120 \text{GeV} \lesssim M_{h} \lesssim 170 \text{GeV}$.

9.3 \textit{W W} scattering and unitarity

There is another bound on the higgs mass which does not involve extrapolations of the SM model to very high energies. It is coming from the unitarity of scattering amplitude for the longitudinal $W_{L}W_{L} \rightarrow W_{L}W_{L}$ scattering [47].

For a massive $W$ gauge particle of momentum $k$ and mass $M_{W}, A_{m} = \varepsilon_{m} e^{ikx}$, the three polarizations satisfy $\varepsilon_{m}e^{m} = -1, k_{m}e^{m} = 0$. In the rest frame $k^{m} = (E, 0, 0, k)$, they are

- transverse: $\varepsilon_{1}^{m} = (0, 1, 0, 0)$, $\varepsilon_{2}^{m} = (0, 0, 1, 0)$,
- longitudinal: $\varepsilon_{3}^{m} = \left( \frac{k}{M_{W}}, 0, 0, \frac{E}{M_{W}} \right) \sim \frac{k^{m}}{M} + \mathcal{O} \left( \frac{M_{W}}{E} \right)$,

the last expressions being valid for $k \rightarrow \infty$. Since longitudinal polarization is proportional to the energy, tree-level amplitude behaves as

$$\mathcal{A} = \mathcal{A}^{(4)} \left( \frac{E}{M_{W}} \right)^{4} + \mathcal{A}^{(2)} \left( \frac{E}{M_{W}} \right)^{2} + \cdots.$$ (229)
Actually, the diagrams a), b) and c) in Figure 27 give \( A = g^2 \left( \frac{E}{M_W} \right)^2 \). On the other hand, unitarity constrains the amplitude to stay small enough at any energy. In order to see this, let us consider the unitarity of the S-matrix \( S^\dagger S = 1 \). Then
\[
S = 1 + i\mathcal{A} \quad \Rightarrow \quad i(\mathcal{A} - \mathcal{A}^\dagger) + \mathcal{A}^\dagger \mathcal{A} = 0
\]  
(230)

By sandwiching this equation between a two-particle state \( |i\rangle \):
\[
i(\mathcal{A} - \mathcal{A}^\dagger)a + \sum_f |\mathcal{A}_f| = 0 ,
\]  
(231)

we find the optical theorem: the imaginary part of the forward amplitude of the process \( i \rightarrow i \) is proportional to the total cross section of \( i \rightarrow \) anything. Let us now decompose the scattering amplitude into partial waves
\[
\mathcal{A} = \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) a_l ,
\]  
(232)

where \( a_l \) are partial wave amplitudes of the elastic scattering of two particles. Projecting Eq. (231) into the partial wave \( l \) gives:
\[
|\text{Re} a_l| \leq 1/2 , \quad 0 \leq |\text{Im} a_l| \leq 1 \quad \Rightarrow \quad |a_l|^2 \leq 5/4 ,
\]  
(233)

which is the unitarity bound we were searching for.

In the case of the SM without the Higgs boson, diagrams a), b) and c) in Fig. 23 lead to
\[
a_0 = \frac{g^2 E^2}{M_W^2} \quad \Rightarrow \quad \text{unitarity breaks down for } \sqrt{s} \sim 1.2 \text{ TeV} .
\]  
(234)
With the Higgs boson present, amplitudes d), e) in Fig. 23 cancel the raising energy term, such that

$$a_0 = \frac{g^2 M_H^2}{4 M_W^2} \rightarrow \text{unitarity breaks down unless } M_H \leq 1.2 \text{ TeV}.$$  \hfill (235)

And by considering other channels, one get the stronger bound $M_H \leq 800 \text{ GeV}$.

**Interpretation:** If LHC finds no Higgs with a mass $M_H \leq 800 \text{ GeV}$, unitarity of S-matrix will be violated. New light degrees of freedom should exist in order to restore unitarity. Most theorists interpret this result as a *no-loose "theorem"* for LHC: either LHC finds the Higgs, or it should find the degrees of freedom replacing it in order to unitarize the $WW$ scattering.

It is important to keep in mind however that most BSM models have *invisible higgs decays*. For example, dark matter models can have higgs decays into dark matter particles $h \rightarrow DM DM$. In this case, higgs searches are more complicated: the higgs can be "hidden" due to its non-standard decays.

There are other constraints on the Higgs mass that we do not discuss here, coming from precision tests in the Standard Model (see Fig. 24). Most theories have a biased towards a *light Higgs*, since it provides a better fit for the SM precision tests.

### 9.4 Higgs and the hierarchy problem

Quantum corrections to the Higgs mass in the SM, coming from diagrams in Fig. 25, are quadratically divergent

$$\delta m_h^2 \sim \frac{3 \Lambda^2}{8 \pi^2 v^2} (4 m_t^2 - 4 M_W^2 - 2 M_Z^2 - m_h^2).$$  \hfill (236)

In a theory including gravity or GUT’s, $\Lambda$ is a physical mass scale $\Lambda = M_P, M_{\text{GUT}}$. It is then difficult to understand why

$$m_h^2 = (m_0^2) + \frac{3 \Lambda^2}{8 \pi^2 v^2} (4 m_t^2 - 4 M_W^2 - 2 M_Z^2 - m_h^2) \sim v^2 \ll \Lambda^2.$$  \hfill (237)

This is the *hierarchy problem* [49].
Fig. 25: Quadratic divergences to the Higgs mass in the SM, leading to the hierarchy problem.

Fig. 26: Possible evidence of a Higgs scalar with mass around 125 GeV from ATLAS (left) and CMS (right), presented at the 2012 Moriond conference [52].

The latest news before this School, from "Lepton-Photon" in August 2011 concerning the Higgs were that both ATLAS and CMS did exclude the SM Higgs at 95 CL for $145\text{ GeV} \leq M_H \leq 446\text{ GeV}$ except $288–296\text{ GeV}$. Before the Christmas 2012 however, some excess in the data, first at ATLAS and then at CMS, has been interpreted as the first possible evidence for a Higgs boson around 125 GeV. Figure 26 summarizes the situation in the Moriond 2012 conference [52].

10 Epilogue: Can Standard Model be the final theory?

Most people believe that Standard Model is just an effective description, for a lot of various reasons:

- There are no neutrino masses at the renormalizable level. The neutrino masses and mixings are often considered as a first hint towards a new mass scale beyond the Standard Model. The seesaw mechanism points towards heavy Majorana singlet neutrinos, maybe remnants of Grand Unified Theories.
- The mysterious hierarchies in the quarks/lepton masses and mixings. It is likely that quarks and leptons hierarchies hide the existence of new flavor symmetries or of a geometrical origin related to wave functions profiles in a higher-dimensional space.
- Standard Model has no viable Dark Matter candidate. This is currently maybe the most pressing
problem: understanding the origin and the properties of the dark matter candidate, which provides about 30% of the energy density of the Universe.

− The problem with the radiative stability of the electroweak scale (“the hierarchy problem”).
− SM has no accurate gauge coupling unification.

− The strong CP problem. The most popular solution postulates the existence of new light particles, the axions, which exist in all string theories and often play a central role in their quantum consistency.
− Gravity is not incorporated into a renormalizable framework. The only viable well-studied framework of quantum gravity to date is string theory.
− The cosmological constant problem \( \Lambda \sim 10^{-4} \text{eV}^4 \sim 10^{-120} M_p^4 \). This is certainly the biggest mystery in modern physics.

Note that the third, fourth and fifth problems find together a nice solution in low-energy supersymmetry. The elegant embedding of quarks and leptons into complete representations of \( SU(5) \) also points out towards a unified gauge group structure.

On the other hand, any theory describing nature has to be validated by experiments. For the time being, LHC found no signal of new physics. It is still early to judge the viability of low-energy supersymmetry or extra dimensional models. There are however preliminary positive LHC hints for a light Higgs boson [51]. But if no SM higgs is discovered by the end of the next year, something else must replace it in order to save the unitarity of the S-matrix in the Standard Model. It is likely in this case that new strong forces and resonances exist at TeV energies and LHC should be able to see them after a couple of years of running.

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