COMPLETENESS OF THE CAVITY KERNEL USED IN CALCULATING LONGITUDINAL COUPLING IMPEDANCE

R.L. Gluckstern*

Abstract

The question of the completeness of the Green function expansion into eigenmodes for the electromagnetic fields in a cavity is examined in detail. In particular, the eigenmode $k_1 = 0$ must be included. With this requirement in mind, the analysis for the coupling impedance, using an integral equation involving a Kernel expanded into eigenmodes, is reexamined and found to be valid.

Geneva, Switzerland
August 1990

* On leave of absence from University of Maryland, Physics Department, College Park, MD 20742
Completeness of the Cavity Kernel Used in Calculating Longitudinal Coupling Impedance

R.L. Gluckstern
SL Division, CERN

Abstract

The question of the completeness of the Green function expansion into eigenmodes for the electromagnetic fields in a cavity is examined in detail. In particular, the eigenmode with $k_z = 0$ must be included. With this requirement in mind, the analysis for the coupling impedance, using an integral equation involving a kernel expanded into eigenmodes, is reexamined and found to be valid.

I. Introduction

In the analysis of the longitudinal coupling impedance for a cavity with beam pipes, an integral equation was derived for the axial electric field at the pipe radius. The kernel of the integral equation was expressed as the sum of a pipe kernel and a cavity kernel. In the derivation of the cavity kernel, we used cavity eigenmodes which satisfied $\frac{\partial}{\partial \tan z} = 0$ on the cavity surface, including the portion of the cavity at the pipe radius where it joins the pipe.

Questions have been raised about the validity of the expansion used to calculate the cavity kernel. An attempt to answer these is included in

---

1 Work supported by the U.S. Department of Energy.
2 Permanent Address: Physics Department, University of Maryland, College Park, MD 20742.
4 H. Henke, private communication.
Appendix A of Reference 1. However, one question which persits is whether the set of functions used for $\vec{E}$ and $\vec{H}$ in the cavity are complete. This note addresses that question explicitly.

II. Cavity Eigenmodes

We expanded the fields in the cavity in terms of the normalized eigenmodes $\vec{E}_\ell$, $\vec{H}_\ell$, which satisfy

$$k_\ell \vec{E}_\ell = \nabla \times \vec{H}_\ell, \quad k_\ell \vec{H}_\ell = \nabla \times \vec{E}_\ell,$$

(2.1)

$$\int \vec{E}_\ell \cdot \vec{E}_m \; dv = \int \vec{H}_\ell \cdot \vec{H}_m \; dv = \delta_{\ell m},$$

(2.2)

and

$$\hat{n} \times \vec{E}_\ell = n \cdot \vec{H}_\ell = 0$$

(2.3)

on the entire boundary of the cavity which, in our application, is the annular cavity shown in Fig. 1, including the dashed line at $r = a$. If we write

$$\nabla \times \vec{E} = -j \omega \mu \vec{H}$$

(2.4)

and

$$\nabla \times \vec{H} = j \omega \varepsilon \vec{E}$$

(2.5)
for a cavity without charge or current in its interior, we can scalar multiply Eq. (2.4) by \( \vec{H}_\ell \) and Eq. (2.5) by \( \vec{E}_\ell \), integrating over the cavity volume, to obtain

\[
k_\ell V_\ell = -j \omega \mu I_\ell - \int dS \vec{n} \cdot \vec{E} \times \vec{H}_\ell
\]

(2.6)

and

\[
k_\ell I_\ell = j \omega \epsilon V_\ell.
\]

(2.7)

Here

\[
V_\ell = \int \vec{E} \cdot \vec{E}_\ell \, dv, \quad I_\ell = \int \vec{H} \cdot \vec{H}_\ell \, dv,
\]

(2.8)

and the surface integral contains the actual field \( \vec{E} \) on the dashed surface.

Assuming we are dealing with a complete set of functions, we can then write

\[
\vec{H}(\vec{x}) = \sum_\ell I_\ell \vec{H}_\ell(\vec{x}) = j \omega \epsilon \int dS' \ (\vec{n}' \times \vec{E}(\vec{x}')) \cdot \sum_\ell \frac{\vec{H}_\ell(\vec{x}') \cdot \vec{H}_\ell(\vec{x})}{k^2 - k_\ell^2},
\]

(2.9)
where the term involving the sum over \( \ell \) eventually becomes our cavity kernel or Green function.

In Reference 1 we argued that the Green function satisfied the required conditions

\[ (\nabla^2 + \kappa^2) \ G(\mathbf{x}', \mathbf{x}) = \delta(\mathbf{x}' - \mathbf{x}) \]  

(2.10)

and

\[ \frac{\partial G}{\partial n} = 0 \text{ on } S, \]  

(2.11)

and verified Eq. (2.9), using in our proof

\[ \sum \mathbf{H}_\ell(\mathbf{x}) \cdot \mathbf{H}_\ell(\mathbf{x}') = \delta(\mathbf{x}' - \mathbf{x}) , \]  

(2.12)

which again assumes that the \( \mathbf{H}_\ell(\mathbf{x}) \) form a complete set.

III. Completeness of the Expansion for the Green Function

There is a detailed discussion of the completeness of the expansions for \( \mathbf{E}_\ell, \mathbf{H}_\ell \) in a paper\(^5\) brought to my attention by H. Henke\(^4\). The relevant point of the paper is that, to be complete, the set of functions must include a term with \( \kappa_\ell = 0 \) corresponding to

\[ ^5\text{T. Teichmann and E.P. Wigner, J. Appl. Phys. 24, 262 (1953).} \]
\[ \nabla \times \mathbf{H} = 0 \]  

(3.1)

and

\[ \mathbf{H} \cdot \mathbf{n} = 0 \text{ on } S. \]  

(3.2)

In our case, this corresponds to an azimuthal component

\[ \left( \frac{\mathbf{H}}{r} \right) = \frac{k}{r}. \]  

(3.3)

While this was not explicitly pointed out in the earlier paper\(^3\), subsequent applications\(^6\) did indeed include this term, which played a major role in the coupling impedance for a small obstacle. Specifically, the final result for a small obstacle of width \(d\) and height \(b-a\) was given as

\[ Z_0 Y(k) = 2\pi k a \left[ \frac{-j}{k^2 g(b-a)} + \text{other terms} \right], \]  

(3.4)

where the first term in the bracket is directly the term for \(k_\ell = 0\).

In the high frequency analysis\(^3\) the important terms are those for which \(k_\ell\) is near \(k\), and the term with \(k_\ell = 0\) is unimportant.

A further demonstration that the term with \(k_\ell = 0\) is included can be made for a small pillbox cavity. In this case the normalized modes are

---

\[
\begin{align*}
\left( \hat{\mathcal{H}}_\ell \right)_\phi &= \frac{\sqrt{2} \cos \frac{m\pi (r-a)}{b-a} \cos \frac{n\pi z}{g}}{\sqrt{\pi} \text{ag}(b-a) (1 + \frac{\delta}{\text{mo}}) (1 + \frac{\delta}{\text{no}})},
\end{align*}
\]  

(3.5)

where the cavity extends from \(0 \leq z \leq g\), and where

\[
\begin{align*}
k^2_{\ell} &= \left( \frac{m\pi}{b-a} \right)^2 + \left( \frac{n\pi}{g} \right)^2.
\end{align*}
\]  

(3.6)

The Green function at \(r = r' = a\) then is

\[
\begin{align*}
G(z,z') = \frac{1}{2\pi \text{ag}(b-a)} \sum_{n=-\infty}^{\infty} \frac{\cos n\pi z}{g} \frac{\cos n\pi z'}{g} \sum_{m=-\infty}^{\infty} \frac{1}{k^2_{m} - \left( \frac{n\pi}{g} \right)^2 - \left( \frac{m\pi}{b-a} \right)^2}.
\end{align*}
\]  

(3.7)

The sum over \(m\) (including \(m = 0\)) is readily performed to yield

\[
\begin{align*}
G(z,z') &= \frac{1}{2\pi \text{ag}} \sum_{n=-\infty}^{\infty} \frac{\cos n\pi z}{g} \frac{\cos n\pi z'}{g} \frac{\cot \kappa_n(b-a)}{\kappa_n},
\end{align*}
\]  

(3.8)

where

\[
\begin{align*}
\kappa_n^2 &= k^2 - \left( \frac{n\pi}{g} \right)^2.
\end{align*}
\]  

(3.9)

This is the form obtained by using

\[
\begin{align*}
\left( \hat{\mathcal{H}}_\ell \right)_\phi &= \sum_{n=-\infty}^{\infty} A_n \cos \left( \frac{n\pi z}{g} \right) \cos \kappa_n(b-r).
\end{align*}
\]  

(3.10)
in the small cavity and matching $\frac{E_{\ell \epsilon}}{H_{\ell \phi}}$ at $r=a$, as was done by Henke.\textsuperscript{7}

IV. Conclusions

We have examined in detail the expansion of the Green function for the electric and magnetic fields in a cavity. In particular, we considered the requirement\textsuperscript{5} that the $k_\ell = 0$ term must be included in order that the set of eigenmodes for the magnetic field be complete. It was shown that this term was indeed included in a recent analysis for the coupling impedance\textsuperscript{3,6}. Therefore the results, which are in agreement with those of Henke\textsuperscript{7}, appear to be rigorously correct.

V. Acknowledgment

The author wishes to acknowledge many conversations with H. Henke and B. Zotter, and is following B. Zotter's advice to record this information as a CERN note.

\textsuperscript{7}H. Henke, "Point Charge Passing a Resonator with Beam Tubes", CERN-LEP-RF/85-41.