Abstract: The strong coupling is extracted from hadronic W boson decays using the most recent theoretical and experimental inputs, obtaining \( \alpha_s(m_Z^2) = 0.117 \pm 0.043_{\text{exp}} \pm 0.001_{\text{th}} \) (assuming CKM matrix unitarity), where a detailed estimation of the associated uncertainties is provided for the first time. Prospects for future \( \alpha_s \) determinations from W data at the LHC, and in \( e^+e^- \rightarrow W^+W^- \) collisions at FCC-ee are highlighted.

Introduction

The hadronic decay widths of the electroweak bosons (\( \Gamma_{Z,\text{had}} \), \( \Gamma_{W,\text{had}} \)) are high-precision theoretical and experimental observables from which an accurate extraction of \( \alpha_s \) can be obtained. Whereas \( \Gamma_{Z,\text{had}} \) provides, together with other Z-pole hadronic measurements, one of the most precise constraints on the current \( \alpha_s \) world-average [1], no extraction of \( \alpha_s \) from \( \Gamma_{W,\text{had}} \) has been performed so far. The reasons for that are two-fold. First, while \( \Gamma_{Z,\text{had}} \) has been experimentally measured with 0.1% uncertainties, \( \Gamma_{W,\text{had}} \) has much larger experimental uncertainties of order \( \sim 2\% \), or 0.4% in the case of the hadronic branching ratio \( BR_{W,\text{had}} \equiv (\Gamma_{W,\text{had}}/\Gamma_{W,\text{tot}}) \), and the sensitivity to \( \alpha_s \) of the W and Z hadronic decays comes only through small higher-order (loop) corrections. Secondly, although \( \Gamma_{Z,\text{had}}, \Gamma_{W,\text{had}} \) are both theoretically known up to \( \mathcal{O}(\alpha_s^4) \), a complete expression of \( \Gamma_{W,\text{had}} \) including all computed higher-order terms was lacking until recently. This situation has now been corrected with the work of [2] that obtained \( \Gamma_{W,\text{had}} \) at N3LO accuracy including so-far missing mixed QCD+electroweak \( \mathcal{O}(\alpha_s\alpha) \) corrections, improving upon the previous calculations of one-loop \( \mathcal{O}(\alpha_s) \) QCD and \( \mathcal{O}(\alpha) \) electroweak terms [3,4,5], and two-loop \( \mathcal{O}(\alpha_s^3) \) three-loop \( \mathcal{O}(\alpha_s^4) \) [6], and four-loop \( \mathcal{O}(\alpha_s^5) \) [7] QCD corrections. Despite the clear progress, the work of [2] still contains a range of approximations (e.g. one-loop \( \alpha_s \) running between \( m_W \) and \( m_Z \), massless quarks, and CKM matrix set to unity for the calculation of higher-order corrections and renormalization constants), plus no real estimation of the associated theoretical uncertainties, which hinder its use to extract \( \alpha_s \) from a comparison to the experimental data. We remove such approximations here and provide \( \alpha_s \) values extracted from current and future W boson hadronic decay data [8].

\( \alpha_s \) from current \( \Gamma_{W,\text{had}} \) and \( BR_{W,\text{had}} \) data

The theoretical expression for the hadronic decay width, including QCD terms up to order \( \mathcal{O}(\alpha_s^4) \), plus electroweak \( \mathcal{O}(\alpha) \) and mixed \( \mathcal{O}(\alpha\alpha_s) \) corrections, reads

\[
\Gamma_{W,\text{had}} = \frac{\sqrt{2}}{4\pi} G_F m_W^3 \sum_{\text{quarks}} |V_{ij}|^2 \left[ 1 + \sum_{k=1}^4 \left( \frac{\alpha_s}{\pi} \right)^k + \delta_{\text{electroweak}}(\alpha) + \delta_{\text{mixed}}(\alpha_s) \right].
\] (1)

Compared to previous works, we implement finite quark masses in QCD corrections up to order \( \mathcal{O}(\alpha_s) \), we use NNLO (instead of LO) \( \alpha_s \) running, and for the numerical evaluation we use the
2015 PDG world average values [9] for $\alpha, G_F, m_q, m_\ell, m_W, m_Z, m_H$, and CKM matrix elements $|V_{ij}|$ (summed over $q = u, c, d, s, b$ since the top-quark is not kinematically accessible). Our final theoretical result amounts to $\Gamma_{W,\text{had}} = 1428.84 \pm 22.61_{\text{par}} \pm 0.03_{\text{th}}$ MeV (using the experimentally measured $|V_{ij}|$ values), and $\Gamma_{W,\text{had}} = 1411.58 \pm 0.79_{\text{par}} \pm 0.03_{\text{th}}$ MeV (assuming CKM matrix unitarity), well in agreement with the current experimental value $\Gamma_{W,\text{had}}^{\text{exp}} = 1405 \pm 29$ MeV [9]. The (small) theoretical uncertainties quoted include the estimation of higher-order corrections, non-perturbative effects –suppressed by $\mathcal{O} \left( \Lambda_{\text{QCD}}^4/m_W^4 \right)$ power corrections– and finite quark mass (beyond LO) [8]. The parametric uncertainties quoted have been obtained by individually varying each parameter in Eq. (1) within 1-$\sigma$ of its associated error and adding all the differences from the central $\Gamma_{W,\text{had}}$ in quadrature. The dominant parametric uncertainties are those associated with the charm-strange quark mixing element $|V_{cs}|$ (which propagates into $\pm 22$ MeV in $\Gamma_{W,\text{had}}$) followed by $m_W$ (which propagates into $\pm 0.7$ MeV). The functional dependence of $\Gamma_{W,\text{had}}$ on $\alpha_s$ alone is shown in Fig. 1 (left). Setting $\Gamma_{W,\text{had}}$ in Eq. (1) to its measured experimental value $\Gamma_{W,\text{had}}^{\text{exp}}$ and fixing all other parameters (except $\alpha_s$) to their PDG values, allows one to extract the strong coupling values listed in Table 1.

![Graphs showing functional dependences of $\alpha_s$ on the W boson hadronic width $\Gamma_{W,\text{had}}$ (left) and branching ratio $\text{BR}_{W,\text{had}}$ (right), given by Eq. (1) imposing CKM unitarity (solid curves) or to their measured values (dashed curves). The vertical lines are the experimental $\Gamma_{W,\text{had}}$ and $\text{BR}_{W,\text{had}}$ values.](image)

### Table 1: Values of $\alpha_s$ and associated uncertainties

<table>
<thead>
<tr>
<th>$\Gamma_{W,\text{had}}$</th>
<th>$\alpha_s(m_W^2)$</th>
<th>$\alpha_s(m_Z^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental CKM</td>
<td>$0.069 \pm 0.065_{\text{exp}} \pm 0.051_{\text{par}}$</td>
<td>$0.068 \pm 0.064_{\text{exp}} \pm 0.051_{\text{par}}$</td>
</tr>
<tr>
<td>CKM unitarity</td>
<td>$0.107 \pm 0.066_{\text{exp}} \pm 0.002_{\text{par}}$</td>
<td>$0.105 \pm 0.065_{\text{exp}} \pm 0.002_{\text{par}}$</td>
</tr>
<tr>
<td>$\text{BR}_{W,\text{had}}$</td>
<td>$\alpha_s(m_W^2)$</td>
<td>$\alpha_s(m_Z^2)$</td>
</tr>
<tr>
<td>Experimental CKM</td>
<td>$0.0 \pm 0.038_{\text{exp}} \pm 0.495_{\text{par}}$</td>
<td>$0.0 \pm 0.038_{\text{exp}} \pm 0.454_{\text{par}}$</td>
</tr>
<tr>
<td>CKM unitarity</td>
<td>$0.119 \pm 0.043_{\text{exp}} \pm 0.001_{\text{par}}$</td>
<td>$0.117 \pm 0.043_{\text{exp}} \pm 0.001_{\text{par}}$</td>
</tr>
</tbody>
</table>
As we can see from the top rows of Table 1, the current experimental uncertainty of $\Gamma_{W,\text{had}}$ and the parametric uncertainties of Eq. (1), mostly from the $|V_{cs}|$ element, propagate into a huge uncertainty on $\alpha_s$. An alternative approach, with reduced experimental and theoretical uncertainties, is to extract $\alpha_s$ through the hadronic W branching ratio, $BR_{W,\text{had}} \equiv (\Gamma_{W,\text{had}}/\Gamma_{W,\text{tot}})$, which is experimentally known to within ±0.4% and for which the $m_W$ parametric uncertainty cancels out in the ratio. We compute $BR_{W,\text{had}}$ from the ratio of Eq. (1) to the NNLO expression for $\Gamma_{W,\text{tot}}$ from [10,11], obtaining $BR_{W,\text{had}} = 0.6824 \pm 0.0108_{\text{par}}$ (experimental CKM matrix) and $BR_{W,\text{had}} = 0.6742 \pm 0.0001_{\text{par}}$ (assuming CKM matrix unitarity), both results in very good accord with the experimental value $BR_{W,\text{had}}^{\text{exp}} = 0.6741 \pm 0.0027$. The functional dependence of $BR_{W,\text{had}}$ on $\alpha_s$ alone is shown in Fig. 1 (right). As done for $\Gamma_{W,\text{had}}$, by comparing the experimental value of $BR_{W,\text{had}}^{\text{exp}}$ to the theoretical predictions, we can extract the values of $\alpha_s$ listed in the bottom rows of Table 1. For $V_{ij}V_{jk} = \delta_{ik}$, the extracted $\alpha_s = 0.117 \pm 0.043_{\text{exp}} \pm 0.001_{\text{par}}$ has now a relative uncertainty of 37%. Thus, even if the current experimental uncertainty of $BR_{W,\text{had}}$ is lower than that of $\Gamma_{W,\text{had}}$, it still propagates into a large uncertainty on $\alpha_s$.

### $\alpha_s$ from future $\Gamma_{W,\text{had}}$ and $BR_{W,\text{had}}$ data

Future measurements at the LHC and at FCC-ee will provide $\Gamma_{W,\text{had}}$ and $BR_{W,\text{had}}$ with higher accuracy and precision. First, it is not unreasonable to reduce the $\Gamma_{W,\text{had}}$ uncertainty to approximately ±12 MeV via high-statistics LHC measurements of the large transverse mass spectra of W decays (sensitive to $\Gamma_{W,\text{tot}}$ and thereby to $\Gamma_{W,\text{had}} = BR_{W,\text{had}} \cdot \Gamma_{W,\text{tot}}$). Combining this result with an improved determination of the $|V_{cs}|$ matrix element commensurate with that of $|V_{ud}|$ (since the other diagonal CKM member, $|V_{tb}|$, is not kinematically relevant for W decays), would change the relative uncertainty of our extracted $\alpha_s$ value from 37% to 23%. Additional LHC and Tevatron W lepton data (combined with the total width) can further reduce it to 10% (first row of Table 2). At the FCC-ee, the W hadronic branching ratio would be measured in huge samples of $e^+e^- \rightarrow W^+W^-$ collisions yielding $5 \times 10^5$ W bosons (a thousand times more than the $5 \times 10^3$ collected at LEP). This would reduce the statistical uncertainty of $BR_{W,\text{had}}$ to around 0.005%. The $BR_{W,\text{had}}$ measurement at the FCC-ee would thus significantly improve the extraction of $\alpha_s$ with propagated experimental uncertainties of order 0.3%, which could be even lowered to 0.15% combining, in addition, this result with other closely related W-decay observables as done for the Z boson [1].

<table>
<thead>
<tr>
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<th>$\delta \alpha_s(m_W^2)$</th>
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<tbody>
<tr>
<td>LHC ($\delta \Gamma_{W,\text{had}} \approx 12 \text{ MeV}$, improved $BR_{W,\text{lept}}$)</td>
<td>$\pm 0.0120_{\text{exp}} \pm 0.0004_{\text{par}}$</td>
<td>$\pm 0.10%$</td>
</tr>
<tr>
<td>FCC-ee ($\delta BR_{W,\text{had}} \approx 0.005%$)</td>
<td>$\pm 0.0004_{\text{exp}} \pm 0.0004_{\text{par}}$</td>
<td>$\pm 0.3%$</td>
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Table 2: Uncertainties on the $\alpha_s$ values at the W and Z scales, extracted from future measurements of $\Gamma_{W,\text{had}}$ at the LHC (top row) and $BR_{W,\text{had}}$ at FCC-ee (bottom row), for a value of the $|V_{cs}|$ matrix element with experimental uncertainty similar to that of $|V_{ud}|$ today.

As a short side-project, we have also extracted a value of $|V_{cs}|$ comparing the experimental $\Gamma_{W,\text{had}}$ and $BR_{W,\text{had}}$ results to their theoretical predictions fixing $\alpha_s$ to the current world average [8]. The resulting $|V_{cs}|$ values are listed in Table 3 with the experimental and parametric uncertainties propagated as described before. As we can see from the values tabulated, the extracted $|V_{cs}|$ has an uncertainty of ±0.6%, which is about 3 times smaller than the ±1.6% of the current experimental measurement, $|V_{cs}|^{\text{exp}} = 0.986 \pm 0.016$ [9].
To summarize, we have extracted $\alpha_s$ from the hadronic W decay width ($\Gamma_{W,\text{had}}$) and its hadronic branching ratio ($\text{BR}_{W,\text{had}}$) using the experimental values of both quantities compared to the theoretical predictions computed at N$^3$LO and NNLO accuracies respectively. The current experimental and parametric uncertainties on both $\Gamma_{W,\text{had}}$ and $\text{BR}_{W,\text{had}}$ are too large today to extract a precise enough $\alpha_s$ value (the best result obtained is $\alpha_s(m_Z^2) = 0.117 \pm 0.043_{\text{exp}} \pm 0.001_{\text{th}}$, assuming CKM matrix unitarity). Measurements with reduced uncertainties at the LHC and, in particular, at the FCC-ee will allow one to extract $\alpha_s$ with uncertainties as low as 0.15%, providing a new independent value of the strong coupling to be weighted-averaged with all other methods discussed in this document.

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References

[1] J. Kühn and K. Chetyrkin, these proceedings, p. 103