Hyperbolic Functions in Relativistic Mechanics

1. Introduction

Numerical calculations in the field of relativistic particle dynamics can be facilitated considerably by the application of tabulated hyperbolic functions. The abstract theory behind this method is as old as the theory of special relativity. Everything which follows results from the representation of a Lorentz-Transformation in terms of an imaginary rotation in the four-dimensional Minkowski-space treated in many textbooks. The key is the identity \( i \tanh x = \tanh x \). The fact that the theory in combination with a table of hyperbolic functions offers a powerful tool for numerical calculations seems to be almost unknown. This note shows briefly the representation of relativistic dynamical quantities in terms of hyperbolic functions and demonstrates the method by a few simple examples. They are confined to problems involving colinear velocities. Problems involving divergent velocities can be treated in terms of hyperbolic and circular functions, but the simplicity of the calculations is lost and there is probably no more advantage in comparison with the conventional method.

2. Theory

The similarity between the Einstein addition law for colinear velocities

\[
\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad (1)
\]

\(\beta = \frac{v}{c} \); \(c = \) velocity of light, \(v = \) particle velocity

and the addition theorem of the hyperbolic tangent function

\[
\tanh (a_1 + a_2) = \frac{\tanh a_1 + \tanh a_2}{1 + \tanh a_1 \tanh a_2} \quad (2)
\]

suggests the substitution

\[
\beta = \tanh a \quad (3)
\]

The dimensionless quantity $\alpha$ may be called "parameter of motion".

With $E_0, \gamma$ (3) the frequently used parameter

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

becomes

$$\gamma = \frac{1}{\sqrt{1 - \tanh^2 \alpha}} = \cosh \alpha$$

and

$$\beta \gamma = \frac{\tanh \alpha}{\sqrt{1 - \tanh^2 \alpha}} = \sinh \alpha$$

Let $E_0 = mc^2$ be the rest energy of the particle under consideration.

The expressions for the total energy $E$, the kinetic energy $T$ and the momentum $p$ read:

$$E = mc^2 \gamma = mc^2 \cosh \alpha \quad (6)$$

$$T = mc^2 (\gamma - 1) = mc^2 (\cosh \alpha - 1) = mc^2 \cdot 2 \sinh^2 \frac{\alpha}{2} \quad (7)$$

$$p = mc \beta \gamma = mc \sinh \alpha = \frac{E}{c} \cdot \sinh \alpha \quad (8)$$

With eqs. (4) and (5) the matrix of a special Lorentz-transformation in the $x_0, x_1$ plane in the frame $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$ reads

$$\begin{pmatrix}
\cosh \alpha & -\sinh \alpha & 0 & 0 \\
-\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(9)
This representation simplifies many calculations in relativistic particle dynamics, because the substitution eq.(3)

i) replaces the complicated addition theorem for the $\beta_i$s by the simple addition of the $\alpha_i$s,

ii) correlates $E$, $p$ and $\beta$ by the relations between $\cosh$, $\sinh$ and $\tanh$. These functions are tabulated. Thus numerical calculations are simplified considerably.

For big values of $\alpha$ (e.g. $\alpha > 6$) one may use tables of natural logarithms, because of the approximations

$$\cosh \alpha \approx \sinh \alpha \approx \frac{1}{2} e^\alpha,$$

$$\tanh \alpha \approx 1 - 2 e^{-2\alpha},$$

which hold with a relative error of less than $10^{-5}$ for $\alpha > 6$.

Some simple examples may demonstrate the handiness of this method.

3. Examples.

1) What is the momentum $p$ and velocity $\beta$ of a 25 GeV proton (kinetic energy)?

Answer.

The proton rest mass is about 0.938 232 GeV. Hence the total energy:

$$E = 25.938 \text{ GeV}, \quad \gamma = \cosh \alpha = \frac{E}{E_0} = 27.667.$$

The table gives: $\alpha = 4.0131$,

$$p = \frac{E}{c} = 27.650,$$

$$\beta = 0.999346.$$

* e.g. L.J. Comrie, Chambers Six Figure Mathematical Tables, Vol.II, Natural Values, London 1949.*
2) What is the matrix of a Lorentz-transformation composed of two subsequent transformations with parameters of motion $\alpha_1$ and $\alpha_2$ respectively, velocities in the same direction?

**Answer:**

Let $A_1$ be the matrix corresponding to $\alpha_1$:

$$A_1 = \begin{pmatrix}
\cosh \alpha_1 & -\sinh \alpha_1 & 0 & 0 \\
-\sinh \alpha_1 & \cosh \alpha_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

The product matrix $A = A_2 A_1$ then reads

$$A = \begin{pmatrix}
\cosh_2 \cosh_1 + \sinh_2 \sinh_1 & -(\cosh_2 \sinh_1 + \sinh_2 \cosh_1) & 0 & 0 \\
-(\cosh_2 \sinh_1 + \sinh_2 \cosh_1) & \sinh_2 \sinh_1 + \cosh_2 \cosh_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$
This example again shows the replacement of the addition theorem \( E_n(\lambda) \) by the addition of the parameters of motion. The reader may solve the same problem in the conventional representation in order to see the gain in simplicity and clarity.

3) What must be the (total) energy of a conventional proton synchrotron to produce \( p - p \) collisions of the same energy as an intersecting beam machine with 9 GeV total energy of the intersecting beams.

Answer:

For the intersecting beam machine the centre of mass coordinate system (c.m. system) and the laboratory system coincide. Thus in the c.m. system the protons both have a parameter of motion \( \alpha_c \) corresponding to a total energy of 9 GeV.

For the conventional synchrotron the parameter of motion in the laboratory system of the accelerated proton is \( \alpha^1 \), that of the proton at rest \( \alpha^0 = 0 \). The Lorentz transformation into the centre of mass system results in \( \alpha^1 = \alpha^1 - \alpha_c \) and \( \alpha^0 = \alpha_c \) (cf. eq.(9) and example 2).

Since \( \alpha^1 = \alpha^0 \) in the c.m. system, one obtains

\[
\alpha^1 = 2\alpha_c
\]

Numerical evaluation:

\[
\alpha_c = \cosh^{-1} \left( \frac{9}{0.9382} \right) = \cosh^{-1} 9.592 = 2.9513
\]

\[
E_1 = 0.9382 \cosh 5.9026 = 0.9382 \times 8.9216 \text{ GeV}
\]

Hence the conventional synchrotron must produce 171.6 GeV protons to be equivalent to an intersecting beam machine of 9 GeV per beam.

4) What is the total energy in the c.m. system of a collision between a 6.0 GeV electron and a proton at rest?

Answer:

Let \( \rho \) be the ratio between proton and electron mass:

\[
\rho = \frac{m_p}{m_e}
\]
The parameter of motion of the electron in the lab. system is \( a_1 \), that of the c.m. system \( a_c \). In the c.m. system the momenta of electron and proton are equal. Hence

\[
\frac{m_e c \sinh (a_1 - a_c)}{\rho m_e c \sinh a_c} = \frac{m_p c \sinh a_1}{\rho m_p c \sinh a_1} = \frac{\sinh a_1 \cosh a_1 - \cosh a_1 \sinh a_1}{\cosh a_1 + \rho} = \frac{\sinh a_1}{\cosh a_1 + \rho}
\]

The collision energy in the c.m. system is

\[
E_{\text{coll}} = m_e c^2 \left[ \frac{\cosh (a_1 - a_c) + \rho \cosh a_c}{\cosh a_1 + \rho} \right]
\]

Numerical evaluation:

\[
\rho = \frac{m_p}{m_e} = \frac{238.2}{0.511} = 1836
\]

\[
\cosh a_1 \approx \frac{1}{2} e^{a_1} = \frac{6.0 \times 10^9}{5.11 \times 10^5} = 11760 \approx \sinh a_1
\]

\[
a_1 = \log 23520 = 10.0655
\]

\[
\tanh a_c = \frac{11760}{13596} = 0.86495
\]

\[
a_c = 1.3117; \quad \cosh a_c = 1.9926
\]

\[
a_1 = a_c = 8.7538
\]

\[
\cosh (a_1 - a_c) \approx \frac{1}{2} e^{8.7538} = \frac{1}{2} \times 10^3 \times e^{1.0450} = 3164
\]

\[
E_{\text{coll}} = 0.511 \left[ 3164 + 1836 \times 1.9926 \right] \text{ Mev} = 3.49 \text{ Gev.}
\]

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