STABILITY DIAGRAM FOR LINACS WITH ALTERNATING GRADIENT FOCUSING LENSES.
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1. Introduction

In this report a stability diagram will be given for an AG focused linac. To obtain this diagram the following assumptions have been made:

i) The focusing system is periodic with period L.

ii) The defocusing force due to the accelerating wave changes negligibly over one period of the free oscillations. This means among other things that we assume the period of the phase oscillations to be long compared with the period of the free oscillations, and that the energy gain per period of the free oscillations is small.

iii) All lenses are identical, but every second one is turned 90° with respect to the other ones.

The first two assumptions are made in order to enable us to apply the ordinary stability condition for a Hill equation. It is believed that this approach gives a reasonably good accuracy, and can be used for the first steps in designing a linac with AG focusing. However, a final design must be computed through numerically. It looks as if the results obtained here are conservative, as all changes with energy in the linac parameters tend to improve the stability. This, however, is not true when the coupling between phase oscillations and free oscillations is considered. This may cause resonance effects and therefore a build-up of oscillation amplitudes. This may not be dangerous for as short an accelerator as the one we are planning, but a careful examination of this problem will be needed. It will, however, not be treated in this report.

The stability diagram will here be given for two extreme cases: the very simple case of point lenses, and the case of no field-free sections. It will then be seen that if the proper parameters are chosen to represent the linac and the lens system the two diagrams are so similar that the difference between them can be neglected. As actual cases will lie between these two extremes, it can be con-
cluded that the diagram we have found can be used for any practical lens arrangement.

2. Point Lenses.

By a point lens is meant an infinitely thin four-pole lens that changes the derivative of a particle orbit by an amount ±dx, where x is the displacement of the orbit from the axis. The upper sign is for a defocusing lens (in the z-direction), and the lower sign is for a focusing lens. In a linear lens δ is independent of x.

The relation between a point just in front of a lens and just after it is

\[
\begin{pmatrix}
x(\pm 0)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
+3 & 1
\end{pmatrix} \begin{pmatrix}
x(\pm 0)
\end{pmatrix}
\]

Between the lenses the equation of the particle orbit is

\[
\frac{d^2 x}{dz^2} = k z = 0
\]  

where k is considered to be constant (if the assumption it is satisfied). This constant depends on the linear parameters, such as accelerating field, phase velocity, phase angle, frequency etc. It can be shown that

\[
k \cdot \frac{E}{m c^2} = \beta^{-3} \alpha \ln \varphi
\]  

where E is the amplitude of the accelerating field, \( \lambda_0 \) the free-space wavelength, \( \varphi \) the phase angle, measured from the peak of the wave, and \( \beta \) is the particle velocity over the velocity of light.

The solution of (2) and the derivative of the solution can be written as
\[
\begin{pmatrix}
    x_x \\
    \cdot \\
    x'
\end{pmatrix} =
\begin{pmatrix}
    \cosh[k'] z & \frac{1}{k'} \sinh[k'] z \\
    \frac{1}{k'} \sinh[k'] z & \cosh[k] z
\end{pmatrix}
\begin{pmatrix}
    x_x' \\
    \cdot \\
    x'
\end{pmatrix} \tag{4}
\]

where \(x_x\) and \(x'_x\) are the initial values.

The transfer matrix over one period \(L\) is consequently
\[
T = \begin{pmatrix}
    C & S/[k'] \\
    k'S & C
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    C & S/[k'] \\
    k'S & C
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix} \tag{5}
\]

where
\[
C = \cosh[k'] L/2, \quad S = \sinh[k'] L/2 \tag{6}
\]

Half of the trace of \(T\) is
\[
\cos \mu = C^2 + S^2 = \frac{1}{2} (\delta L/2)^2 \left( \frac{S}{k' L/2} \right)^2 \tag{7}
\]

and the stability condition is, as we know, that
\[
-1 < \cos \mu < 1 \tag{8}
\]

However, we get a more convenient expression if we rearrange (7) in the following way:

Let the defocusing forces in (7) tend to zero. We then get \(\cos \mu\) for what can conveniently be called the "empty" system, i.e., the same system with the RF switched off, but still assuming the particles to have the proper momentum in each point. The corresponding \(\mu\) we give the index \(0\), and get
\[
\cos \mu_0 = 1 - \frac{1}{2} (\delta L/2)^2 \tag{9}
\]

This is then a parameter characterizing the lens system only, and containing no linac parameter apart from the particle momentum.

From this equation we then substitute in (7) for \((\delta L/2)\) and obtain
\[
\cos \mu = \cos^2 \theta + \sin^2 \theta \left(1 - \cos \mu_0 \right) \left(\frac{\sin \theta}{\sqrt{k L}}\right)^2
\]

(10)

The two limits of the stability region are given by the two curves one obtains by putting \(\cos \mu = \pm 1\) in (10).

The limiting curve \(\cos \mu = 1\) \((\mu = 0)\) is given by

\[1 - \cos \mu_0 = 2k(L/2)^2\]

(11)

and the other limiting curve \(\cos \mu = -1\) \((\mu = \pi)\) is given by

\[1 - \cos \mu_0 = 2k(L/2)^2 \cos^2 \left(\sqrt{\frac{k}{L}} \frac{1}{2}\right)\]

(12)

In Fig. 1 these two limiting curves have been drawn as dotted curves, and the region between the curves is the stable region. The parameter \((1 - \cos \mu_0)\), characterizing the lens system itself, is along the horizontal axis, and \(k(L/2)^2\) is along the vertical axis.

Before we discuss this diagram in more detail we shall consider another example:

3. Focusing System with no Field Free Sections.

The other extreme case is a focusing system with no field free sections. This can be worked through in the same way as above, but the calculations will not be given here. The result, however, can be presented in exactly the same way as in the case of point lenses, and is shown by the fully drawn curves marked \(\mu = 0\) and \(\mu = \pi\) in Fig. 1.

As noticed, the difference between the results in the two extreme cases is very small and can in fact be neglected compared with other errors introduced, for example by the assumptions on which the analysis is based.

The diagram can therefore be used for any lens arrangement. Of the parameters characterizing the lens system only \(\cos \mu\) enters the discussion of the stability and where to work on the stability diagram. How to obtain the \(\cos \mu\) wanted from these considerations is purely a practical problem.

The defocusing problems are most serious near the input end of the linac, and we shall therefore only consider that part of the accelerator. That means that there are particles performing phase oscillations to the very limits of the phase stable region. That again means that if \( k \) in eq. (2) on Fig.1 for the synchronous particle is \( k_s \), there are particles in the bunch experiencing a radial force corresponding to \( k = -k_s \) (which for these particles acts focusing). We now state that also these particles must be inside the stability diagram, i.e. their working point must be above the \( \mu = \pi \) line. That, however, means that the working point of the synchronous particle must be below the curve one obtains by turning the \( \mu = \pi \) curve about the horizontal axis (the curve A-B).

The maximum \( k \) a particle in a bunch can experience is

\[
 k_{\text{max}} = k_s \sin \varphi' / \sin \varphi_s
\]

where \( \varphi' = 2\varphi_s \) for \( \varphi_s < 45^\circ \) and \( \varphi' = \pi/2 \) for \( \varphi_s > 45^\circ \).

These particles must have their working point below the \( \mu = 0 \) line. This gives a new limiting curve for the working point of the synchronous particle. This curve is obtained by multiplying the ordinates for the \( \mu = 0 \) curve by \( \sin \varphi_s / \sin \varphi' \). Two such curves have been drawn into the diagram in Fig.1 (the curves marked \( \varphi_s = 60^\circ \) and \( \varphi_s = 30^\circ \)).

It is thus noticed that the region for the working point of the synchronous particle is much smaller than the main stability region, and consists only of the triangular shaped area OAB.

At low energies, such as we have near the injection end of the linac, the accelerating field will be limited by the maximum value one can choose for \( k_s \) from the stability diagram. That means that we want to work near the injection end in the top corner of the stability triangle (point A). If this point is chosen as the working point there are particles in the bunch with working points everywhere on the vertical line through this top point, quite out to the stability limits \( \mu = 0 \) and \( \mu = \pi \). Due to the phase oscillations a particle's working point moves
vertically along this line in the stability diagram.

As the energy of the particles increases the working point of the synchronous particles moves, and then in general downwards and to the left on the diagram, so that we get further away from the instability limits, if we do not at the same time increase either the accelerating field or $\mu_0$, or both, to make up for this motion. Especially near the injection it is desirable to do that in order to reach the maximum field as quickly as possible. This cannot be done continuously, but only at the beginning of each accelerating section.

5. Example.

As an example we shall consider the conditions at the injection end of the first section of a helix accelerator for 0.5 - 50 MeV, focused by magnetic four-pole lenses.

We choose the following basic parameters:

- Free space wavelength $\lambda_0 = 3 \text{ m}$
- Synchronous phase (measured from the peak of the wave) $\psi_s = 60^\circ$
- Period of focusing system $L = 0.5 \text{ m}$

The choice of wavelength is mainly governed by helix-diameter considerations. A large phase angle is chosen in order to get a wide trapping region. In choosing the period of the focusing system I assumed that the lenses ought to be at least twice as long as the diameter of the system it is going to surround. This diameter is 8-10 cm, and consequently I have chosen for this example $L = 0.5 \text{ m}$ as a reasonable period.

From the stability diagram we find for point A, which we, according to the considerations made in section 4, choose as the working point

$$k_s (L/2)^2 = 0.53$$
$$1 - \cos \mu_0 = 0.29 \quad \text{or} \quad \cos \mu_0 = 0.29$$
We substitute for \( k(L/2)^2 \) in eq. (3) and find \( E = 308 \text{ kV/m} \), and the corresponding energy gain per unit length is \( E \cos \varphi_B = 154 \text{ keV/m} \).

As noticed we get a rather low accelerating field, but as the energy increases, one can, as already mentioned, increase the field considerably. For instance, if the first section accelerates only up to 1 MeV, we can at the input end of the next section employ almost three times as strong fields as in the first section.

To find an approximate value for the field gradient needed in the four-pole lenses to get the required \( \cos \mu \), of about -0.29, we neglect possible field-free sections and find then that a field gradient of \( \frac{dE}{dx} = 470 \text{ Gs/cm/cm} \) is needed, which should be quite easy to achieve.


In the discussion of which region of the stability diagram can be used for the working point of the synchronous particle, it was assumed that no particle should at any time of its acceleration be allowed to get outside the main stability limits \( \cos \mu = \pm 1 \).

This may, for several reasons, be a stricter requirement than actually needed. Firstly, as the particle is accelerated, the stability conditions improve because its velocity increases, and a particle that at injection is just outside the stability region, may after a while, and before its radial amplitude has increased appreciably, find itself inside the stable region. Secondly, the accelerator is not infinitely long, but, on the contrary, rather short, and even if some particles are part of the time a little outside the stability limits, the amplitude may not have time to grow to a dangerous magnitude.

This, however, is partly counteracted by the fact that the stability conditions are based on the assumption that the amplitude should not grow to infinity, whereas we cannot permit the amplitude to grow outside the size of the vacuum chamber. If that had been considered when stating the stability conditions, a narrower stability region would have been found.
These considerations again show that for detailed analysis of a proton linac numerical computations are needed. But for the first estimates the method described here should give satisfactory results.

From the example given, it can also be concluded that it now seems practically possible to focus a helix type of linac by means of the alternating gradient method. This means that we shall have to consider the helix in comparison with the type using drift tubes for our injection machine. A report containing some very tentative tables comparing these two types will be issued shortly.

Fig. 1. Stability diagram for linacs with alternating-gradient focusing.