A METHOD FOR COMPUTING THE LONGITUDINAL COUPLING IMPEDANCE OF CIRCULAR APERTURES IN A PERIODIC ARRAY OF INFINITE PLANES

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The diffraction of the electromagnetic field created by a charge travelling on the axis of circular apertures in a set of perfectly conducting infinite planes is described by the field travelling with the charge itself and by the radiation from the plates, which has also a travelling wave character. Accordingly we represent all the fields as a superposition of two parts: a part generated by the charge in free space and a part created by the presence of the screens, which together must satisfy the boundary conditions. These are generally of mixed type (on the plate and in the hole) and lead to two integral equations. A general procedure is shown to transform this system into only one Fredholm integral equation of the second kind.

1. INTRODUCTION

Let us consider a particle of charge \( Q \) travelling at velocity \( \vec{v} \) on the axis of circular holes in a set of parallel planes. The geometry is shown in Fig. 1.

We shall use cylindrical coordinates whose \( \vec{z} \) axis passes through the center of the apertures and is perpendicular to the plane of the screens. We shall assume that the charge moves in the positive \( \vec{z} \) direction. The theory of the diffraction of a plane wave by a circular aperture in an infinite screen can easily be found in the literature.\(^1\), \(^2\) The problem is usually analysed by modal expansion methods which give the solution as an
Periodic array of circular apertures in infinite plane screens.

FIGURE 1

infinite sum of eigenfunctions of the wave equation in a particular coordinate system. However, this solution has the shortcoming of being badly convergent, especially for the case of short wavelengths. It is well known that a point charge crossing the hole will excite a continuous spectrum of frequencies, which extends to very high frequencies for ultrarelativistic charges; this general feature of the diffraction-radiation problem makes modal expansion really impracticable for our problem, even for an approximate solution.

The charge moving with uniform velocity in vacuum radiates only because of the optical inhomogeneities present near its path. The radiation is due to the diffraction of the field at the edges of the holes.3

The field created by the charge in the presence of the screen will interact with the charge itself so that, together with the phenomenon of radiation, we should find a decrease of the particle velocity.4, 5 Such a radiation problem is very difficult to solve and therefore a simplifying assumption will be made: we shall suppose that the charge moves at constant velocity during its flight. The constant velocity can be imagined as being maintained by an external source. The result will be a good approximation provided that the velocity of the charge does not change significantly during the interaction with the screen. This assumption can be considered to be realistic when dealing with ultrarelativistic charges.

The fields of an ultrarelativistic charge are essentially confined within an angular region of aperture $\approx 1/\gamma$, where $\gamma$ is the energy of the charge expressed in rest mass units. As long as the charge is far from the hole, it barely perceives the presence of the screen. Its image charges are at a great distance so that there is only very weak interaction with them;6, 7 moreover, they move at constant velocity towards the center of the hole. In this situation, which persists up to quite small distances of the charge from the hole, little radiation is expected. Only when the edge of the hole is seen by the charge within the narrow $(1/\gamma)$ cone of its field, will the image charges experience a sudden change of their motion since they are released and start moving radially back to infinity: this process lasts for the time of passage of the charge through the hole and it is the main reason for radiation. The more relativistic is the charge, the shorter is the radiation time and the wider is the spectrum of radiation.

The problem will be treated as a boundary-value problem for Maxwell’s equations:
we have the radiation condition at infinity, the condition on the tangential component of the electric field on the screen, and the Meixner (edge) condition for the discontinuity at the edge of the holes. The last condition will ensure the uniqueness of the solution for our problem.

This diffraction problem is described by the field \((\vec{E}_0, \vec{H}_0)\) travelling with the charge itself and by the radiation \((\vec{E}, \vec{H})\) from the plates, which has a travelling wave character. Accordingly, we can represent all the fields and/or potentials as the superposition of two terms: a term generated by the charge in free space and a term created by the presence of the screens, which together must satisfy the boundary conditions; these are generally of mixed type (on the screen and on the hole) and lead to two integral equations. We thus write

\[
\vec{E}_t = \vec{E}_0 + \vec{E},
\]

\[
\vec{H}_t = \vec{H}_0 + \vec{H}.
\]

It is worth noting that, owing to the symmetry of the problem, the induced currents on the screens are directed radially and the only components of the field are \(E_r, E_z, H_\varphi\). Moreover, since the induced currents are orthogonal to the edge, the Meixner or edge condition requires that the components of the electric field orthogonal to the edge diverge as \(d^{-1/2}\), where \(d\) is the distance from the edge.

Our approach to the solution of this problem consists in reducing it to a system of dual integral equations, involving Bessel functions for the surface current density on the screen (the same thing occurs, for example, in the solution of axially-symmetric boundary-value problems in elasticity and electrostatics). These systems have been studied by many authors; a quite complete summary of all the methods of solution can be found in Sneddon. The basic idea we used is to reformulate the problem as a Fredholm integral equation of the second kind with a continuous kernel, for a certain auxiliary function; the analytical solution of this new equation is, in general, not so easy to obtain. A numerical treatment can give satisfactory results, at least for not too high frequencies. The way to reformulate the problem is not unique, as demonstrated in several papers by Lebedev and Skal’skaya. This variety of methods can be used to find a fast converging integral equation, i.e. an equation where the free term is already a good approximation of the complete solution; this is not an easy task. Lebedev-Skal’skaya’s works are also very useful in order to understand the aim of these transformations; they contain very nice manipulations of some integral relations.

It should be borne in mind that the above boundary-value problem, at least for the case of a single screen, has a rigorous solution which is obtainable by separation of variables in oblate spheroidal coordinates. This solution is given by a series of spheroidal wave functions; but it is suitable for numerical calculations only at low frequencies.

It is also worth emphasizing that the Wiener–Hopf techniques cannot be used for our problem.

It will be assumed in the following that the order of integration in repeated integrals, and the orders of differentiation and integration, can be reversed as necessary without explicit justification.
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2. FIELD OF A UNIFORMLY MOVING CHARGE

Let us consider a cylindrical coordinate system \((r, \varphi, z)\). The charge is located on the \(z\)-axis and moves with velocity \(\vec{v}\) in the positive direction. It can be shown that the expressions of the fields without screens\(^{20}\) are in the \(\omega\)-domain:

\[
H_{0\varphi}(r, z; \omega) = \frac{Q|\omega|}{2\pi v\gamma} K_1 \left( \frac{|\omega|r}{v\gamma} \right) e^{-j\omega/v},
\]
\[
E_{0r}(r, z; \omega) = \frac{Q|\omega|\mu}{2\pi \beta^2 \gamma} K_1 \left( \frac{|\omega|r}{v\gamma} \right) e^{-j\omega/v},
\]
\[
E_{0z}(r, z; \omega) = \frac{jQ\omega\mu}{2\pi (\beta\gamma)^2} K_0 \left( \frac{|\omega|r}{v\gamma} \right) e^{-j\omega/v},
\]

where \(j\) is the imaginary unit, \(\mu\) is the vacuum permittivity, \(\gamma = 1/\sqrt{1 - \beta^2}\), \(\beta = v/c\), and \(c\) the velocity of light. The asymptotic behavior of the modified Bessel functions of the second kind is given by

\[
K_n(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + O\left( x^{-1} \right) \right],
\]

so it is easy to verify that the fields associated with the charge moving in free space are strongly attenuated radially.

3. METHOD OF DETERMINING THE INDUCED CURRENTS

The method given hereafter considers an unknown distribution of induced currents on the plates as the source of radiated fields. The component of the electric field along the surface of the plates is set equal to the negative component of the electric field due to the charge: we obtain an integral equation where the unknown function is the space time transformed current.

Let us consider the current distribution on the \(n\)-th plate:

\[
j^n_r(r, z; t) = j^n_r(r; t) \delta(z - nL),
\]

where \(L\) is the distance between the plates. Because of the source travelling at velocity \(\vec{v}\), the current on the \(n\)-th plate will have the same configuration as that on the 0-th plate.

\*The conventions for the Fourier transforms used in this work are
a) time-dependent function

\[
G(\omega) = \int_{-\infty}^{+\infty} g(t)e^{-j\omega t} dt, \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\omega)e^{j\omega t} d\omega;
\]
b) space-dependent function

\[
G(k_z) = \int_{-\infty}^{+\infty} g(z)e^{jk_z z} dz, \quad g(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(k_z)e^{-jk_z z} dk_z.
\]
at a time retarded by $nL/v$:

\[ j_r^n(r; t + nL/v) = j_r^0(r; t) . \]

Making use of the shift theorem for the Fourier transform, we get:

\[ J_r^n(r, z; \omega) = e^{-j\omega nL/v} J_r^0(r; \omega) \delta(z - nL) , \tag{4} \]

where the quantity $J_r^0(r; \omega)$, i.e. the Fourier transform of the current on the plate $n = 0$, becomes the relevant unknown of the problem. For each frequency component of current, we can find the expression of the Hertz potential $\Pi(r, z; \omega)$ by means of the equation\(^2\)

\[ \nabla^2 \Pi + k^2 \Pi = -\frac{j}{j\omega\varepsilon} j, \]

where $k = \omega/c$ is the wave number. Firstly we solve for a $\delta$-function source located at the point $r_0, z_0$ which yields the Green’s function $G(r, r_0, z_0; \omega) = G(r, r_0, z_0; \omega) \hat{r}$ as solution of the equation

\[ \nabla^2 G + k^2 G = \frac{j}{j\omega\varepsilon} \frac{\delta(r - r_0)\delta(z - z_0)}{2\pi r_0} \hat{r}, \]

which, in cylindrical coordinates, becomes

\[ \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} - \frac{1}{r^2} G + \frac{\partial^2 G}{\partial z^2} + k^2 G = \frac{j}{j\omega\varepsilon} \frac{\delta(r - r_0)\delta(z - z_0)}{2\pi r_0} . \]

We can now transform this partial differential equation into an ordinary algebraic equation by means of the double (Hankel–Fourier) integral transformation\(^2\)

\[ G(k_r, k_z; r_0, z_0; \omega) = \int_0^\infty r J_1(rk_r) \left[ \int_{-\infty}^{+\infty} G(r, z; r_0, z_0; \omega) e^{jzk_z} dz \right] dr , \tag{5} \]

which, from the properties of the $\delta$-functions, enables us to write

\[ G(k_r, k_z; r_0, z_0; \omega) = \frac{j}{2\pi \omega \varepsilon} \frac{J_1(r_0 k_r) e^{jzk_z}}{k^2 - k_z^2 - k_r^2} . \]

We can now integrate Green’s function multiplied by the current $\vec{J}(r_0, z_0; \omega)$ over the whole space $V_0$, even if the actual current flows on the plates only for $r > a$ (hole radius). Eventually we shall impose the condition that the current vanishes in the hole, which is a condition for the Hankel transform of the current. We obtain

\[ \Pi_r(r, z; \omega) = \frac{j}{4\pi^2 \omega \varepsilon} \sum_{n=-\infty}^{+\infty} \int_{V_0} J_r^n(r, z_0; \omega) \left\{ \int_{0}^{\infty} k_r J_1(rk_r) J_1(r_0 k_r) \right. \left[ \int_{-\infty}^{+\infty} \frac{e^{jzk_z(z_0 - z)}}{k^2 - k_z^2 - k_r^2} dk_z \right] dk_r \right\} dV_0 , \]
where \( J^0_r(r_0, z_0; \omega) \) is given by Eq. (4) and \( dV_0 = r_0 dr_0 dz_0 d\varphi_0 \). Performing the integration over \( z_0 \) and \( \varphi_0 \), we get:

\[
\Pi_r(r, z; \omega) = \frac{j}{2\pi\omega \varepsilon} \sum_{n=-\infty}^{+\infty} e^{-j\omega nL/v} \int_0^\infty k_r J_1(k_r r) \left\{ \int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{k^2 - k_z^2} \right. \\
\left. \int_0^\infty r_0 J_1(r_0 k_r) J_r^0(r_0; \omega) dr_0 \right\} dk_z.
\]

The integral over \( r_0 \) is simply the Hankel transform of the current \( J^0_r(r_0; \omega) \):

\[
F(u) = \int_0^\infty r_0 J_1(ur_0) J^0_r(r_0; \omega) dr_0
\]

where \( u \) is the radial wavenumber \( k_r \). The inverse transform reads

\[
J^0_r(r_0; \omega) = \int_0^\infty u F(u) J_1(ur_0) du.
\]

From now on we choose the transform \( F(u) \) as the unknown of the problem. Equation (6) becomes

\[
\Pi_r(r, z; \omega) = \frac{j}{2\pi\omega \varepsilon} \sum_{n=-\infty}^{+\infty} e^{-j\omega nL/v} \\
\int_0^\infty u F(u) J_1(ur) \left[ \int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{k^2 - u^2 - k_z^2} dk_z \right] du.
\]

The integration in square brackets over \( k_z \) may be performed * by means of the residue theorem; in fact, putting \( U = \sqrt{k^2 - u^2} \), the integrand function exhibits two simple poles at \( k_z = \pm U \). It is found that

\[
\int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{U^2 - k_z^2} dk_z = j\pi \frac{e^{-jU|z-nL|}}{U}.
\]

Finally expression (8) can be rewritten as

\[
\Pi_r(r, z; \omega) = -\frac{1}{2\omega \varepsilon} \int_0^\infty \frac{u F(u) J_1(ur)}{U} \left[ \sum_{n=-\infty}^{+\infty} e^{-j(U|z-nL|+\omega nL/v)} \right] du.
\]

Because the fields \( \vec{E} \) and \( \vec{H} \) are related to the Hertz potential \( \vec{\Pi} \) by

\[
\vec{E} = k^2 \vec{\Pi} + \nabla(\nabla \cdot \vec{\Pi}) \\
\vec{H} = j\omega \varepsilon \nabla \times \vec{\Pi}
\]

*In the evaluation of the residues it is necessary to take into account a small imaginary part for \( k = \omega \sqrt{\varepsilon\mu}, \)

where \( \varepsilon\mu \) is considered to be complex, so that \( \text{Im}(k) < 0 \). For the two poles at \( k_z = \pm U \), we then have \( \text{Im}(U) < 0 \) or \( U = \sqrt{k^2 - u^2} = -j\sqrt{u^2 - k^2} \) when \( u > \text{Re}(k) \). By taking \( \text{Im}(U) < 0 \), we implicitly satisfy the radiation condition at infinity.
introducing the notation

\[ S(u, z) = \sum_{n=-\infty}^{+\infty} e^{-j(U|z-nL|+\omega n L/v)} \] (10)

we have the following expressions for the fields *:

\[
\begin{align*}
E_r(r, z) &= -\frac{\zeta_0}{2k} \int_0^\infty u F(u) \sqrt{k^2 - u^2} S(u, z) J_1(ur) du \\
E_z(r, z) &= -\frac{\zeta_0}{2k} \int_0^\infty \frac{u^2 F(u)}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} J_0(ur) du \\
H_\phi(r, z) &= -\frac{j}{2} \int_0^\infty \frac{u F(u)}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} J_1(ur) du
\end{align*}
\] (11) (12) (13)

(\zeta_0 = 120\pi \Omega \) is the free space impedance). Let us remember that \( \sqrt{k^2 - u^2} = -j\sqrt{u^2 - k^2} \). The series that defines the function \( S(u, z) \) can be summed in closed form; we obtain for \( |z| \leq L \):

\[
S(u, z) = j \frac{\sin[(L - |z|)\sqrt{k^2 - u^2}] + e^{-j\omega \text{sgn}(z)L/v} \sin(|z|\sqrt{k^2 - u^2})}{\cos(L\sqrt{k^2 - u^2}) - \cos(\omega L/v)}.
\]

If \( z \) is not in the range \( |z| \leq L \), then the function \( S(u, z) \) can be computed by means of the relation

\[
S(u, z + L) = S(u, z)e^{-j\omega L/v}.
\]

It should be noted for future use that

\[
S(u, 0) = \frac{j \sin (L\sqrt{k^2 - u^2})}{\cos (L\sqrt{k^2 - u^2}) - \cos (\omega L/v)}
= \frac{\sinh (L\sqrt{u^2 - k^2})}{\cosh (L\sqrt{u^2 - k^2}) - \cos (\omega L/v)}.
\] (14)

It is not difficult to compute the derivative along \( z \) of the function \( S(u, z) \) for \( |z| < L \):

\[
\frac{1}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} = -j \text{sgn}(z) \frac{\cos [(L - |z|)\sqrt{k^2 - u^2}] - e^{-j\omega \text{sgn}(z)L/v} \cos (z\sqrt{k^2 - u^2})}{\cos (L\sqrt{k^2 - u^2}) - \cos(\omega L/v)},
\]

*Remembering that \( \Pi = \hat{r} \Pi_r \), we can write

\[
\begin{align*}
E_r &= \frac{\partial^2 \Pi_r}{\partial r^2} + \frac{1}{r} \frac{\partial \Pi_r}{\partial r} + \left( k^2 - \frac{1}{r^2} \right) \Pi_r, \\
E_z &= \frac{1}{r} \frac{\partial^2 (r \Pi_r)}{\partial z \partial r}, \\
H_\phi &= j\omega \varepsilon \frac{\partial \Pi_r}{\partial z}.
\end{align*}
\]
and its value for \(z = 0\):
\[
\left[ \frac{1}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} \right]_{z=0} = -j \text{sgn}(z) + \frac{\sin(\omega L/v)}{\cos \left( L\sqrt{k^2 - u^2} \right) - \cos(\omega L/v)}.
\]
(15)

From Eq. (14) we see that
\[
\lim_{u \to \infty} S(u, 0) = 1,
\]
and, because \(\text{Im}\sqrt{k^2 - u^2} < 0\), we also have
\[
\lim_{L \to \infty} S(u, 0) = 1 \quad \text{for any } u.
\]

Similarly, the last term of Eq. (15) vanishes when \(L \to \infty\). Therefore, as expected when \(L \to \infty\), Eqs. (11) to (13) reduce to the formulae given in the Appendix for the case of a single screen.

As already mentioned, the solution can be found as a superposition of the solution of the inhomogeneous equations in free space and a solution of the homogeneous equations, chosen in such a way as to fulfill the boundary conditions on the plates. Accordingly the two conditions which are to be satisfied are
\[
J^0_r(r; \omega) = 0, \quad 0 \leq r \leq a, \quad \text{(16)}
\]
\[
E_{0r}(r, z = 0; \omega) + E_r(r, z = 0; \omega) = 0, \quad r > a. \quad \text{(17)}
\]

Therefore the two conditions that allow us to find the expression of the current transform \(F(u)\) can easily be obtained as a system of dual integral equations (extensively studied in Ref. 22) from Eqs. (2), (16), and (17)
\[
\int_0^\infty u F(u) J_1(ur) du = 0, \quad 0 \leq r \leq a \quad \text{(18)}
\]
\[
\int_0^\infty u F(u) \sqrt{u^2 - k^2} S(u, 0) J_1(ur) du = C \frac{j k^2}{\pi \beta^2 \gamma} K_1 \left( \frac{\omega r}{v \gamma} \right), \quad r > a \quad \text{(19)}
\]

where \(\omega > 0\). In the Appendix are given the formulae for the case of a single screen.

This system of integral equations can be transformed into one equation with a singular kernel. In fact if we introduce the unitary step function \(\mathcal{U}(x)\), the previous system becomes formally
\[
\int_0^\infty u F(u) J_1(ur) du = \frac{1}{C} \left\{ Q \frac{j k^2}{\pi \beta^2 \gamma} K_1 \left( \frac{\omega r}{v \gamma} \right) + \int_0^\infty u F(u) \left[ C - \sqrt{u^2 - k^2} S(u, 0) \right] J_1(ur) du \right\} \mathcal{U}(r-a), \quad \text{(20)}
\]

where \(C\) is any constant independent of \(u\), different from 0 and \(\infty\). Equation (20) can be interpreted as a Hankel transform of \(F(u)\); so, making use of the inversion formula (7) and of the fundamental result
\[
\int_0^\infty r J_1(ur) J_1(vr) dr = \frac{\delta(u-v)}{u} = \frac{\delta(u-v)}{v},
\]
after some algebraic manipulations we obtain

\[ \varphi(v) = B(v) - a^2 \int_0^\infty u \varphi(u) \left[ \frac{C}{\sqrt{u^2 - k^2 S(u,0)}} - 1 \right] N(v,u) du, \]  

(21)

where we put

\[ \varphi(u) = F(u) \sqrt{u^2 - k^2 S(u,0)}, \]  

(22)

\[ B(u) = \frac{Qjk^2}{\pi \beta^2} \int_a^\infty r K_1(\kappa r) J_1(ur) dr \]

\[ = \frac{u}{\kappa u^2 + \kappa^2} \left[ \kappa a K_1(\kappa a) J_0(ua) + (\kappa a)^2 K_0(\kappa a) \frac{J_1(ua)}{ua} \right], \]  

(23)

\[ N(v,u) = \frac{1}{a^2} \int_0^a r J_1(ur) J_1(vr) dr = N(u,v) \]

\[ = \frac{uv}{2} \frac{J_0(va) J_2(ua) - J_0(ua) J_2(va)}{u^2 - v^2}. \]  

(24)

\( S(u,0) \) is given by Eq. (14) and \( \kappa = k/(\beta \gamma) \). It appears that \( N(u,v) \) is largest on the diagonal \( u = v \), where

\[ N(u,u) = \frac{1}{2} \left[ J_1^2(au) - J_0(au) J_2(au) \right], \]

with

\[ \frac{dN(u,u)}{d(au)} = \frac{1}{au} J_0(au) J_2(au). \]

The integral Eq. (21) represents a Fredholm equation of the second kind with an asymmetrical and meromorphic kernel, where \( v \geq 0 \); the left-hand side is the Hankel transform of \( E_r(r,0) \), whilst in the right-hand side \( B(v) \) is the contribution from the screen and the remaining integral is the contribution from the hole.

4. GENERAL TRANSFORMATIONS

The aim of this section is to propose two general transformations useful for the solution of the dual integral Eqs. (18) and (19); in other words, making use of an auxiliary function, we can represent the system of dual integral equations with only one Fredholm integral equation of the second kind, which is easier to solve and/or to treat numerically.

We begin by reducing the dual integral Eqs. (18) and (19) to a form suitable for our purpose. Our object is to replace the Bessel functions appearing in Eqs. (18) and (19) by sine functions. We can rearrange Eq. (18) by means of an Abel’s transformation, using the following representation of the Bessel function:

\[ J_1(ur) = \frac{2}{\pi r} \int_0^r \frac{x \sin(ux)}{\sqrt{r^2 - x^2}} dx, \]

Eq. (18) becomes

\[ \int_0^\infty u F(u) \sin(ur) du = 0, \quad 0 \leq r \leq a. \]
Equation (19) can be simplified by means of the two integrals
\[
\int_r^\infty \frac{J_1(ux)}{\sqrt{x^2 - r^2}} \, dx = \frac{\sin ur}{ur}, \quad \int_r^\infty \frac{K_1(ux)}{\sqrt{x^2 - r^2}} \, dx = \frac{\pi e^{-ur}}{2ur}, \quad u > 0.
\]
Thus the previous system of dual integral equations can be rewritten as
\[
\int_0^\infty u F(u) \sin(ur) \, du = 0, \quad 0 \leq r \leq a, \quad (25)
\]
\[
\int_0^\infty F(u) \sqrt{u^2 - k^2} S(u, 0) \sin(ur) \, du = \frac{Qj\kappa}{2\beta} e^{-\kappa r}, \quad r > a. \quad (26)
\]
The second step is to introduce an auxiliary function; we shall discuss two kinds of such functions. We have to find an *ad hoc* representation of the unknown function \( F(u) \), so that one of the two equations of the system is automatically satisfied. By taking the inverse sine transform of (26), it is apparent that any function given as
\[
F(u) \sqrt{u^2 - k^2} S(u, 0) = Qj\kappa \frac{u}{u^2 + \kappa^2} \left[ \int_0^a p(x) \sin(ux) \, dx \right], \quad (27)
\]
where \( p(x) \) is some unknown auxiliary function, continuous together with its first derivative in the closed interval \([0, a]\), automatically satisfies\(^{27}\) Eq. (26); this supposes that \( \text{Im} Jk^2 - u^2 < 0 \). Then Eq. (25) becomes
\[
\int_0^\infty \frac{u}{\sqrt{u^2 - k^2} S(u, 0)} \left[ \frac{u}{u^2 + \kappa^2} - \int_0^a p(x) \sin(ux) \, dx \right] \sin(ur) \, du = 0 \quad (28)
\]
\[
(0 \leq r \leq a)
\]
where
\[
\frac{1}{S(u, 0)} = \frac{\cos (L\sqrt{k^2 - u^2}) - \cos (\omega L/v)}{j \sin (L\sqrt{k^2 - u^2})} = \frac{\cosh (L\sqrt{u^2 - k^2}) - \cos (\omega L/v)}{\sinh (L\sqrt{u^2 - k^2})}. \quad (29)
\]
With this formulation, the function \( F(u) \) has been replaced by the left hand side of Eq. (27), that is the function \( \varphi(u) \) of Eq. (22), as the unknown of our problem.

It has already been shown\(^{28}\) that Eq. (28), which seems to be a Fredholm integral equation of the first kind for \( p(r) \), can be transformed into a Fredholm equation of the second kind because its kernel reveals the presence of a \( \delta \)-function. Thus, after some algebraic manipulations, we can finally write
\[
p(r) = T(r) + \frac{1}{2} \int_0^a [G(|x - r|) - G(x + r)] p(x) \, dx, \quad 0 \leq r \leq a, \quad (30)
\]
where the kernel and the free term of this equation are respectively
\[
G(y) = \frac{2}{\pi} \int_0^\infty \left[ 1 - \frac{u}{\sqrt{u^2 - k^2} S(u, 0)} \right] \cos(uy) \, du \quad (31)
\]
In Eqs. (31) and (32), we must again assume that $\text{Im}(k) < 0$. It can be shown that with the expression (27) for $F(u)$, the edge condition for $E_r$ at $r = a$ is automatically satisfied.

Another possible auxiliary function $f(t)$ which we can try is defined by

$$u F(u) = \frac{jk}{\pi \beta} \int_a^\infty f(t) \sin(ut) dt.$$  \hfill (33)

Equation (25) is now automatically satisfied, whilst Eq. (26) becomes

$$\int_a^\infty \left[ \int_0^\infty \frac{\sqrt{u^2 - k^2} S(u, 0)}{u} \sin(ut) \sin(ut) du \right] f(t) dt = \frac{\pi}{2} e^{-\kappa r}, \quad r > a,$$

which seems to be a Fredholm equation of the first kind, but which can be easily reduced to one of the second kind, by making use of the relation\(^22\)

$$\int_0^\infty \sin(ut) \sin(ut) du = \frac{\pi}{2} [\delta(t - r) - \delta(t + r)].$$

So we get

$$f(r) = e^{-\kappa r} + \frac{1}{2} \int_a^\infty [L(|t - r|) - L(t + r)] f(t) dt, \quad r > a,$$  \hfill (34)

where the kernel is given by

$$L(y) = \frac{2}{\pi} \int_0^\infty \left[ 1 - \frac{\sqrt{u^2 - k^2} S(u, 0)}{u} \right] [\cos(uy) - \cos(ub)] du.$$  \hfill (35)

The $\cos(ub)$ term has been added in order to cancel the pole of the integrand at $u = 0$; $b$ can be any length $\geq 0$ without affecting Eq. (34).

Let us notice that in both Fredholm integral Eqs. (30) and (34) for the auxiliary functions, the kernel is symmetrical and continuous.

5. REMARK

It is worth comparing the Hertz potential expression (9) with that given\(^2,29-32\) for the case of a charged rod moving near a grating of infinite half-planes (see Fig. 2)

$$\Pi_y(y, z; \omega) = \frac{j}{2\omega \varepsilon} \int_{-\infty}^{+\infty} \frac{F(u) e^{-juy}}{\sqrt{k^2 - u^2}} \cdot \left\{ \sin(\sqrt{k^2 - u^2}(z - L)) - e^{-j\omega L/v} \sin(\sqrt{k^2 - u^2} z) \right\} du,$$
for \(0 \leq z \leq L\). In this case we have an \(x\)-independent expression relative to current flowing in the \(\hat{y}\) direction. As can be seen, the expressions are quite similar; this looks reasonable indeed, and by a further insight into the method used we can say that they are completely equivalent. In fact, in deriving the expression of the potentials for both cases one supposes that the current flows on the whole plane, but eventually one imposes the currents to be zero in the hole \((r < a)\) or in the half-plane \((y < 0)\). The only difference is that in our case there is a cylindrical symmetry, i.e. currents are \(\varphi\)-independent and \(\hat{r}\) directed, whilst for the case of the rod of Fig. 2 there is a rectangular symmetry, i.e. currents are \(x\)-independent and \(\hat{y}\) directed.

6. LONGITUDINAL COUPLING IMPEDANCE

The longitudinal coupling impedance is defined by \(^\text{33-35}\)

\[
Z_{\parallel}(k) = -\frac{1}{Q} \int_{-\infty}^{+\infty} E_z(r = 0, z; k)e^{jz/k/\beta} \, dz,
\]

where \(E_z\) is the radiated field. Because in our case we have a periodic structure, the quantity

\[
E_z(r = 0, z)e^{jz/k/\beta}
\]

is periodic in \(z\), with period \(L\). Therefore the integral (36) taken from \(-NL\) to \(+NL\) reads

\[
Z_{\parallel}(k) = -\frac{1}{Q} \int_{-NL}^{+NL} E_z(r = 0, z; k)e^{jz/k/\beta} \, dz
\]

\[
= -\frac{2N}{Q} \int_{0}^{L} E_z(r = 0, z; k)e^{jz/k/\beta} \, dz.
\]
The total impedance $Z$ is proportional to the number of cell, $2N$; we thus redefine an impedance per cell $Z$ as

$$Z_{II}(k) = - \frac{1}{Q} \int_0^L \frac{1}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} e^{jk/\beta} dz$$

(38)

where the integral is now taken over a single cell. Using Eq. (15)

$$\int_0^L \frac{1}{\sqrt{k^2 - u^2}} \frac{\partial S(u, z)}{\partial z} e^{jk/\beta} dz = \frac{2k}{\beta(u^2 + \kappa^2)}.$$ 

(39)

From Eqs. (12) and (39), we finally obtain

$$\frac{2\pi}{\zeta_0} Z_{II}(k) = \frac{2\pi}{Q\beta} \int_0^\infty \frac{u^2}{u^2 + \kappa^2} F(u) du,$$

(40)

where $\zeta_0 = 120\pi \Omega$ is the characteristic impedance of free space. It is interesting to note that this expression is formally identical to the one given in the Appendix for the case of a single screen.

In the general expression (40) the impedance is a function of the unknown $F(u)$; we can rewrite it in terms of the auxiliary functions $p(x)$ and $f(t)$. Making use of Eqs. (27) and (33), we get

$$\frac{2\pi}{\zeta_0} Z_{II}(k) = \pi \frac{jk}{\beta^2} \left[ \frac{2}{\pi} \int_0^\infty \frac{u^3}{(u^2 + \kappa^2)^2} \sqrt{u^2 - k^2} S(u, 0) du - \int_0^a p(x) T(x) dx \right]$$

(41)

or

$$\frac{2\pi}{\zeta_0} Z_{II}(k) = \pi \frac{jk}{\beta^2} \int_a^\infty f(t) e^{-\kappa t} dt.$$

(42)

In spite of its simple form, Eq. (42) is only useful if one can solve the integral Eq. (34) for $f(t)$.

An approximate expression for the coupling impedance can be found from the integral Eq. (21). If it is possible to approximate the right-hand side of this equation with the free term only, we can write

$$F(u)\sqrt{u^2 - k^2} S(u, 0) \approx B(u),$$

(43)

which means that we are neglecting the contribution of the radiated electric field in the aperture. Combining Eq. (43) with the relation (40), we get with (29)

$$\frac{2\pi}{\zeta_0} Z_{II}(k) \approx \frac{2}{\beta^2} \int_0^\infty \frac{u^3}{(u^2 + \kappa^2)^2} \left[ \kappa a K_1(\kappa a) J_0(au) + (\kappa a)^2 K_0(\kappa a) \frac{J_1(au)}{au} \right] \cosh \left( L \sqrt{u^2 - k^2} \right) - \cos(\omega L/v) \sqrt{u^2 - k^2} \sinh \left( L \sqrt{u^2 - k^2} \right) du.$$

(44)

It is interesting to note that the integrand of the last expression has a finite number of simple poles on the real axis, located at

$$u^2 = k^2 - \left( \frac{n\pi}{L} \right)^2,$$

where $n$ is any integer less than $kL/\pi$. 
7. CONCLUSIONS

We have considered a point charge travelling on the axis of a periodic array of circular holes in perfectly conducting infinite planes. The Hankel transform of the current induced in the planes is determined by a set of dual integral equations, whose solution is expressed in terms of an auxiliary function which verifies a Fredholm integral equation of the second kind, and which can thus be computed by numerical methods. The longitudinal coupling impedance per period (or cell) of the array is then easily computed by quadrature from the auxiliary function.

The numerical work will hopefully be performed in the near future; not only will it provide numerical values for the coupling impedance per cell as a function of frequency, but it will at the same time ascertain the range of validity of the approximate expression (44) which is proposed here.

REFERENCES

34. L. Palumbo and V. G. Vaccaro, CERN 87-03 (1987).
APPENDIX

In this appendix we give some formulae for the radiated fields and for the induced current in the case of a single plane. Because in this case, from Eq. (10), for \( L \to \infty \) we have

\[
S(u, z) = e^{-j|z|U} = e^{-|z|\sqrt{u^2-k^2}},
\]

the electromagnetic fields due to the radiation from the plate as a function of the unknown \( F(u) \), that is the Hankel transform of the current \( J_r(r, k) \) induced on the metallic plate by the particle, are given in the \( \omega \)-domain by

\[
E_r(r, z) = \frac{j\zeta_0}{2k} \int_0^{\infty} uF(u)\sqrt{u^2-k^2}e^{-|z|\sqrt{u^2-k^2}} J_1(ur)du
\]

\[
E_z(r, z) = \frac{j\zeta_0}{2k} \text{sgn}(z) \int_0^{\infty} u^2F(u)e^{-|z|\sqrt{u^2-k^2}} J_0(ur)du
\]

\[
H_\varphi(r, z) = -\frac{\text{sgn}(z)}{2} \int_0^{\infty} uF(u)e^{-|z|\sqrt{u^2-k^2}} J_1(ur)du
\]

where \( k = \omega/c \) and Re\((\sqrt{u^2-k^2}) > 0 \). The fundamental system of integral Eqs. (18) and (19) which determines the unknown \( F(u) \) reads

\[
\int_0^{\infty} uF(u)J_1(ur)du = 0, \quad 0 \leq r \leq a,
\]

\[
\int_0^{\infty} uF(u)\sqrt{u^2-k^2} J_1(ur)du = Q \frac{jk^2}{\pi \beta^2 \gamma} K_1\left(\frac{\omega r}{v \gamma}\right), \quad r > a.
\]

Also in this case the system yields one Fredholm equation of the second kind with an asymmetrical and singular kernel

\[
\sqrt{v^2-k^2} F(v) = B(v) - a^2 \int_0^{\infty} uF(u)\sqrt{u^2-k^2} \left[ \frac{C}{\sqrt{u^2-k^2}} - 1 \right] N(v, u)du,
\]

where \( k > 0 \) and the functions \( B(v) \) and \( N(v, u) \) are the same as before. If we take \( C = jk \) and let \( k \to \infty \), we obtain formally

\[
\sqrt{u^2-k^2} F(u) \approx B(u),
\]

which is the basis of an approximate expression\(^{28} \) for the real part of \( Z_{||}(k) \) when \( ka \to \infty \). The validity of such an approximation has been discussed by Bekefi.\(^{36} \)

The functions \( G(y) \) and \( L(y) \), respectively defined in Eqs. (31) and (35), have been computed in\(^{10} \) for the case \( S(u, 0) = 1 \), and the result is \( (k > 0) \)

\[
G(y) = k \left[ J_1(ky) - jH_1(ky) + \frac{j2}{\pi} \right],
\]

\[
L(y) = k \left\{ J_{i1}(kb) - J_{i1}(ky) - j \left[ H_{i1}(kb) - H_{i1}(ky) \right] \right\}
\]

where \( J_1(x) \) and \( H_1(x) \) are the ordinary Bessel and Struve functions,\(^{25} \) whilst \( J_{i1}(x) \) and \( H_{i1}(x) \) are the integral functions of Bessel and Struve, defined by the formulae

\[
J_{i1}(x) = \int_0^x \frac{J_1(z)}{z} dz, \quad H_{i1}(x) = \int_0^x \frac{H_1(z)}{z} dz.
\]
It should be noted that the equation for $L(y)$ is simplest when taking $b = 0$. Finally the longitudinal coupling impedance becomes

$$\frac{2\pi}{\zeta_0} Z_{\parallel}(k) = \frac{2\pi}{Q\beta} \int_0^\infty \frac{u^2}{u^2 + \kappa^2} F(u) du,$$

as in Eq. (40).