Characterization and simulation of a pillbox cavity

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Abstract- A cylindrical pillbox cavity is characterized and its principal parameters, such as the cut-off frequencies of the modes, the quality factor and R/Q figure, are calculated. Additionally, these measurements are compared with an analytical analysis and with a simulation with the software CST Studio to determine how precise they are.

I. INTRODUCTION

The CERN Accelerator School (CAS) provides a series of courses on particle accelerators and colliders. One of the principal elements in an accelerator are the cavities. For that reason, is of utmost importance to know how to characterize one and the principal parameters that define it.

Nowadays methods for measuring these parameters are automatized and done within minutes or even seconds. With manual measurements, students can better understand theory concepts, and observe effects and problems that do not appear in mechanized procedures.

In this report, several manual measurements of cavity principal figures of merit are presented, and results are given and compared with software simulations.

II. ANALYTICAL ANALYSIS

The first approximation of the cut-off frequencies of the transverse electrical and magnetic modes is obtained based on the dimensions (see Fig. 1) of the cavity under study and using eq. (1) and eq. (2), where \( a \) is the radius and \( h \) the longitude of the cavity.

\[
\begin{align*}
 f_{TE_{nm}} &= \frac{c}{2\pi} \sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{nL}{h}\right)^2} \\
 f_{TM_{nm}} &= \frac{c}{2\pi} \sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{nL}{h}\right)^2}
\end{align*}
\]

In our case these two lowest modes are the TE\(_{111}\) and the TM\(_{010}\). Theoretically, their cut-off frequencies are 766 MHz and 776 MHz, but the exact values may vary due to inaccuracies of the dimensions and geometric variations (like the beam ports). The mode that is of interest is the TM\(_{010}\) mode, as it is the one which has an E-field along the z-axis that allows to accelerate the charged particles of the beam.

Next important figure is the \( Q_0 \) factor, which is defined by the losses of the cavity compared to the stored energy, and can be calculated by measuring the bandwidth relative to its central frequency. Analytically, it can be obtained as shown in eq. (3), where the skin depth, \( \delta \), is defined in eq. (4).

\[
Q_0 = \frac{a}{\delta} \left[1 + \frac{a}{h}\right]^{-1}
\]

\[
\delta = \sqrt{\frac{2}{\omega_0 \sigma \mu}}
\]

For the cavity under test, considering a conductivity of stainless steel of \( 1.3 \cdot 10^6 \) S/m and a permeability of \( 4\pi \cdot 10^{-7} \) H/m, the analytical \( Q_0 \) factor is equal to 6321.8.

The last important parameter is the R/Q value. It describes how efficient the cavity transfers its stored energy to the beam [3], and it is defined as ratio between the shunt impedance, \( R \), and the \( Q_0 \) factor, as defined in eq. (5), being \( V_{acc} \) the accelerating voltage and \( U \) the stored energy.

\[
\frac{R}{Q} = \frac{V_{acc}^2}{2\omega_0 U} = \frac{V_{acc}^2}{2\omega_0 U}
\]

Analytically, for cylindrical cavities and the mode TM\(_{010}\), the value of R/Q can be determined using the formula in eq. (6). For the given dimensions of our cavity \( R/Q = 10.4 \) Ω.

\[
\frac{R}{Q} = 128 \cdot \sin^2 \left(\frac{1.2024 h}{a}\right)
\]

III. SET-UP AND MEASUREMENTS

A. Cut-off frequencies

To measure the parameters and figures of merit of the cavity, a vector network analyser (VNA) is used. First, to locate the TM\(_{010}\) mode, a magnetic loop is attached to one of the measurement ports of the cavity. The S\(_{11}\) measurement is shown in Fig. 2. A further investigation needs to be performed to determine which mode corresponds to each resonance.
There are two simple ways of doing this: analyzing the reflected signal \( (S_{11}) \), or the transmitted signal \( (S_{21}) \).

- **Reflected signal**
  If an electric probe is introduced longitudinally in one of the beam ports along the symmetry axis of the cavity, the \( S_{11} \) measurement of the first resonance stays unaltered while the second one shifts slightly in frequency. This means, the second resonance couples to the electric probe, thus it has a longitudinal electric field component (TM mode), while the first one doesn’t (TE mode).

- **Transmitted signal**
  On the other hand, a measurement with two magnetic probes can be performed to measure the transmitted signal. In this case, the first mode disappears, while the 2\textsuperscript{nd} TM mode preserves as its magnetic field couples to the two loop probes.

\[
\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} \tag{7}
\]

A critical coupling measurement is performed using the Smith chart as display format. Considering the cavity equivalent circuit as a parallel RLC circuit, at direct current (frequency zero) it acts like a short (due to the inductance), and at resonance as a match. By rotation of the magnetic probe the coupling changes, and the match point for critical coupling at the resonant frequency can be tuned, as shown in Fig. 3.

Once in critical coupling, the bandwidth can be calculated by taking the frequencies crossing the 0.5 circle in the Smith chart, at the points with -3 dB amplitude in the logarithmic format or at the maximum and minimum of the imaginary part. After the characterization of the resonance frequency and the bandwidth, the loaded \( Q \) can be calculated, and thus, the unloaded \( Q \) is twice that value, which is close to the analytical calculation. The results are shown in Table I.

\[
\begin{array}{l|l}
  f_0 & 774.085 MHz \\
  f_{-3dB.1} & 773.964 MHz \\
  f_{-3dB.2} & 774.214 MHz \\
  \text{BW} & 250.5 kHz \\
  Q_L & 3090.2 \\
  Q_0 & 6180.3 \\
\end{array}
\]

Table I. Values for the \( Q_0 \) calculation.

C. R/Q

As already mentioned, the R/Q parameter defines how much accelerating voltage can be achieved, and depends only on the geometry of the cavity. This voltage is given by the integrated electric field, eq. (8), here the \( E_z \)-field component along the \( z \) symmetry axis of the cavity.

\[
V_{\text{acc}} = \left| \int E(z) \cos \left( \frac{\omega z}{\beta c} \right) dz \right| \tag{8}
\]

To measure the electric field along the propagation axis, \( z \), a perturbation method is used. When a small object is introduced in a cavity, the resonance frequency varies and the electric field can be calculated as is described from Slater’s theorem in eq. (9):

\[
\frac{f - f_0}{f_0} = \frac{1}{D} \left( -K_1\varepsilon_0 E_z^2 - K_2\varepsilon_0 E_z^2 - K_3\mu_0 H_z^2 - K_4\mu_0 H_z^2 \right) \tag{9}
\]

\[
E(z) = \sqrt{\frac{f(z) - f_0}{f_0} \frac{-1}{K_1\varepsilon_0}} \tag{10}
\]

Where \( K_1, K_2, K_3 \) and \( K_4 \) are coefficients proportional to the electric and magnetic polarizability of the object. For our analysis, an elongated object is preferred, such that the dominant coupling field is the longitudinal electric field component. For that reason, \( K_1 \) is larger than the other coefficients, which can be neglected, see eq. (10). The coefficient of interest depends on the geometry of the perturbing object, in this case a needle, and can be evaluated following eq. (11), where the electric field is normalized by the root of the stored energy, \( U \).
\[ K_1 = \frac{\pi}{3} \frac{l}{a} \left[ \sinh^{-1} \left( \frac{2}{3\pi} \frac{l}{a} \right) \right]^{-1} \] (11)

It should be noted that here \( l \) refers to half the length of the object. The needle used for our measurements had a length of 10 mm \( (l = 5 \text{ mm}) \) and a radius \( a = 0.59 \text{ mm} \). With these dimensions follows: \( K_1 = 9.65 \times 10^{-8} \).

To measure the frequency shift, the object is pulled through the cavity by utilizing a thin, non-metallic wire. The set-up schematics is shown in Fig. 4(a), and the practical set-up in Fig. 4(b). Additionally, metal flanges are used to close in the cavity ports, which present a better-defined boundary.

Two different methods are used to measure the R/Q value with the bead: directly or with the phase shift.

- **Direct measurement using a S11 reflection coefficient**
  Using one magnetic probe, the resonant frequency is measured without the perturbing object. Then, the object is pulled through the cavity, and the frequency shift is monitored in steps of 1 cm. Because of the symmetry of the cavity, it is sufficient to probe only half of the cavity. The results are shown in Fig. 5(blue). Integrating the electrical field, a value of 15.5 for R/Q is obtained.

- **Phase shift measurement based on the S21 transmission coefficient**
  The second method is indirect, as the frequency shift is obtained through the phase of a transmitted signal at the resonance frequency. Therefore, two magnetic probes are used. It is of utmost importance to precisely excite the resonance frequency, as all following measurement steps depend on it. Additionally, both probes should be tuned for a very weak coupling, around -0.5 dB peak amplitude in S11 and S22.

  Once the exact value of the TM_{010} resonance frequency is determined, the center frequency of the VNA is set to \( f_0 \), and a zero-span frequency (0 Hz) is configured to excite and analyze only at that particular frequency. Similar to the direct method, the needle is pull through the cavity, in steps of 1 cm, and the phase shift in S21 is measured. The results are shown in Fig. 5(red); in this case an R/Q of 16.5 is obtained, similar to the previous method.

  It has been observed that phase drifts over time, most likely due to temperature variations, and it will alter the measurement result. To minimize this influence, the bead-pull phase measurement has to be performed quickly. A simple help consists in the attachment of a ruler, such that the centimetre steps can be determined faster.

**IV. NUMERICAL SIMULATIONS**

**A. CST simulation**

A numerical simulation with CST is performed in order to compare it to the real measurements. A model like the one showed in Fig. 1 is used, and the fields and cut-off frequencies for the three lowest modes are obtained. The two first correspond to the same mode with different polarization, and are the TE_{111}, as it has no electric field along the z axis. On the contrary, the third mode (TM_{010}) has it.
The TM\textsubscript{010} electric field along the z axis in the cavity is calculated with CST, as shown in Fig. 7. Additionally, it is compared with the two electric fields obtained with the measurements. As it can be observed, the results are similar, with an error of ~3% at the peak. From these electric fields, the R/Q values are obtained. In the Table II there is a summary with the results and the ones obtained directly with CST.

<table>
<thead>
<tr>
<th>Method</th>
<th>Q\textsubscript{0} value</th>
<th>R/Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency shift</td>
<td>6180.3</td>
<td>15.54</td>
</tr>
<tr>
<td>Phase shift</td>
<td>6485.9</td>
<td>12.74</td>
</tr>
<tr>
<td>CST</td>
<td>16.55</td>
<td></td>
</tr>
<tr>
<td>CST electric field</td>
<td>14.61</td>
<td></td>
</tr>
</tbody>
</table>

Table II. Results with different methods.

B. Circuit simulation

Last, a circuit simulation is performed with Qucs software. The equivalent circuit of the cavity, as mention before, is a parallel RLC circuit (Fig. 8). A transformer is added to consider the coupling of the probe to the cavity. The values of the components can be calculated using the eq.(12) to eq.(15).

\begin{align*}
R &= \frac{R}{Q} \cdot Q \\
\omega_0 &= \frac{1}{\sqrt{LC}} \\
Q &= \frac{R}{\omega_0 L} = R \cdot \omega_0 C \\
n &= \sqrt{\frac{R_{gr}}{R}}
\end{align*}

Considering the values for R/Q and Q previously measured with the frequency shift method, the equivalent RLC circuit has the values shown in Table III.

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>95.79 kΩ</td>
</tr>
<tr>
<td>L</td>
<td>3.817 nH</td>
</tr>
<tr>
<td>C</td>
<td>13.27 pF</td>
</tr>
<tr>
<td>n</td>
<td>0.0228</td>
</tr>
</tbody>
</table>

Table III. Values for the equivalent circuit.

When simulating this circuit in Qucs, the resonance appears in the expected frequency, at 774 MHz, as shown in Fig. 9.
V. CONCLUSIONS

The principal parameters of the pillbox cavity have been obtained, along with an analytical analysis and simulations.

The analytical analysis is a first approximation to the actual results, but depends highly on the dimensions of the cavity and geometry, which cannot always be measured with high accuracy.

On the other hand, the CST simulation was quite close to the actual measurement, particularly to the frequency shift method. Even though, it could be refined with the addition of the measurement ports and probes.

The differences between the two methods of measurement were most probably due to the phase shift of the VNA. This could be avoided with an automatized system, but as mentioned before, it is not the point of these measurements.

In conclusion, the analysis of a simple cavity, performing different measurements, calculations and simulations, is a very good approach to the basic concepts of accelerating cavities, electromagnetic fields and real set-ups.

REFERENCES