Analytical model of impedance in elliptical beam pipes

Arthur Pesah, BE-ABP-HSC
Supervisors: Benoît Salvant, Nicolo Biancacci, Mauro Migliorati
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Abstract

Beam instabilities are among the main limitations in building higher intensity accelerators. Having a good impedance model for every accelerators is necessary in order to build components that minimize the probability of instabilities caused by the interaction beam-environment and to understand what piece to change in case of intensity increasing. Most of accelerator components have their impedance simulated with finite elements method (using softwares like CST Studio), but simple components such as circular or flat pipes are modeled analytically, with a decreasing computation time and an increasing precision compared to their simulated model. Elliptical beam pipes, while being a simple component present in some accelerators, still misses a good analytical model working for the hole range of velocities and frequencies.

In this report, we present a general framework to study the impedance of elliptical pipes analytically. We developed a model for both longitudinal and transverse impedance, first in the case of a perfectly conduction pipe, then taking resistivity into account. We compared our results in the limit cases of a round pipe and a flat pipe with existing models for those two components, and showed that they are identical for longitudinal and quadrupolar impedance, but slightly different for dipolar impedance.

1 Introduction

When going to the main restaurant at CERN, one can see a screen with the current state of the LHC, updated in real time: energy of the beam, intensity, luminosity, etc. Many physicists going for a lunch or a beer at the restaurant are looking carefully at this screen, hoping to see at the top the mention PROTON PHYSICS: STABLE BEAMS, which means that the protons are oscillating in the accelerator with a controlled amplitude and that physicists are allowed to perform experiments with the beam. However, it happens sometime to lose the beam, and there can be multiple reasons for that, such as a malfunction with any of the ten thousands of equipments present in the 27 km of the LHC. But the main causes of beam losses are instabilities: the amplitude of the oscillations suddenly increases dangerously and the operators have to dump the beam. Instabilities can also have many causes, one of them being called impedance instabilities: the electromagnetic field created by the beam interacts with the walls of the accelerator, perturbing the protons behind with an unwanted Lorentz force. Those effects are particularly important at high intensity and are one of the main limiting factors when trying to build higher intensity accelerators. In order to model impedance effects, one can either use simulation softwares (like CST Studio), necessary for complex components, or develop an analytical model in order to have a lower computation time. Good analytical models have been developed for circular and flat beam pipes, but current models for elliptical pipes only use big approximations, such as low frequency or high velocity ($\beta = 1$). The goal of my project at CERN this summer was to contribute to the development of an analytical model of impedance for those elliptical chambers and to develop reusable Python codes to compare our results with simulations.
2 Previous works on ellipses

2.1 Why to study ellipses?

Before starting our overview of the current state of science on elliptical impedance, we first need to answer a legitimate question: why do we need to develop an analytical model for impedance in elliptical pipes?

The first answer is because there are elliptical chambers in some accelerators. It’s not directly the case of the LHC, but the Proton Synchrotron (PS), central part of its injector chain, has some elliptical pipes, for instance in its section 43 (cf figure 2).

The impedances of PS elliptical beam pipes are currently modeled by two methods. The first one is with CST Studio, a software that can simulate wake fields and impedances of any 3D structure using finite elements method. The problem with CST is that it is much slower than analytical models (it can take between few hours and a day while analytical models takes only few seconds). It can be fine for only one structure, but when combining all the structures to build the general model of impedance of an accelerator, it is always to good to reduce computation time.

The second method is to consider elliptical pipes either as round pipes or flat pipes (depending on the eccentricity of our ellipse). Both are well known analytical models that can be computed easily with codes like Impedance Wake 2D (IW2D). The problem here is that there can be a huge difference in terms of impedance between round pipes, flat pipes and elliptical pipes, mostly for quadrupolar impedance as we will show later.

Having an analytical model for ellipses would allow a generalization of codes like IW2D, reducing the computation cost or the precision of current models and facilitating implementation of elliptical
chambers in future accelerators.

2.2 Previous analytical models
Most of the previous works on elliptical impedance either use some kind of approximations regarding frequency, conductivity or velocity, or truncate series to their first term. This last approximation can be found in [7] where the author derives the longitudinal impedance of an elliptic pipe using the same methods, but taking only 0th order Mathieu functions into account. Vaccaro and Palumbo [5] also derived an expression of the elliptical impedance, using a different formalism and approximations on frequency and conductivity: \((\frac{k_b}{\gamma})^2 \ll 1\) and \(\sigma > > \omega \epsilon\). Finally, in the 1990s, many attempts were done to obtain a general expression of impedance for any structure (see [6], [9], [8]) and ellipses were often used as an example of application. The main limitation is that when applying the general formula to ellipses, it is still required to compute the electromagnetic field on the boundaries, and the calculations are usually done using approximations like the ones above, or ultra-relativistic limit \((\beta = 1)\). Those approximations have shown limitations in experiments, for instance for components with low conductivity carrying a low-beta beam (see introduction of [4]). In this work, we will derive a precise formula working for the whole range of velocities, frequencies and conductivities.

3 Elliptical impedance for a perfectly conducting wall
We will start our study of elliptical impedance by considering the case of a perfectly conducting boundary. In that case, there can be two sources of impedance at low \(\beta\): direct space-charge impedance, which corresponds to a particle in free-space and indirect space-charge impedance, which takes the boundary into account.

3.1 Maxwell equations inside an ellipse
3.1.1 Elliptical coordinates
In order to simplify the expression of the boundary conditions, we will do all our calculations in elliptical coordinates. If we consider an elliptical boundary with a semi-major axis \(a\) and semi-minor axis \(b\), with a focal \(F = \sqrt{a^2 - b^2}\), elliptical coordinates are defined by the two parameters \(\mu\) and \(\phi\) so that:

\[
\begin{align*}
x &= F \cosh(\mu) \cos(\phi) \\
y &= F \sinh(\mu) \sin(\phi)
\end{align*}
\]

Figure 3: Elliptical coordinates
As you can see in figure 3, \( \mu \) represents the size of the ellipse, while \( \phi \) represents the angular position on the ellipse. The boundary can now be expressed by the simple equation \( \mu = \mu_0 \) with \( \mu_0 \) such as \( \frac{E_z}{G} = \frac{1}{\cosh(\mu_0)} \).

### 3.1.2 Longitudinal electric field equations for a charge in the center

To derive an expression of the impedance, we first have to compute the electromagnetic field created by a charge moving in the center of the pipe with a normalized velocity \( \beta < 1 \). We will consider only the longitudinal electric field, as it is the only one present in the expression of the impedance (both longitudinal and transverse, as we will show later). As shown in [2], Maxwell’s equations for the direct field \( E_z^d \) and the indirect field \( E_z^i \) become:

\[
\nabla_t E_z^d + k_t^2 E_z^d = -G\delta(0)
\]
\[
\nabla_t E_z^i + k_t^2 E_z^i = 0
\]

with \( k_t^2 = \left( \frac{\varepsilon}{\varepsilon_0} \right)^2 - \left( \frac{\varepsilon}{\varepsilon_0} \right)^2 < 0 \) and \( G = \frac{jZ_0 Qk_0}{2\pi \beta^2 \gamma} \).

It looks very similar to the classical waveguide equation, but the fact that \( k_t^2 \) is negative (since the velocity of the charge is imposed) makes it harder.

In order to solve it in elliptical coordinates, we can perform a separation of variables. For instance, if we write the indirect field as \( E_z^i = U(\mu)V(\phi) \), we obtain the following equations:

\[
\frac{d^2 U}{d\mu^2} - (C + 2q\cosh(2\mu))U = 0
\]
\[
\frac{d^2 V}{d\phi^2} + (C + 2q\sinh(2\phi))V = 0
\]

with \( q = \frac{k_t^2 F^2}{4} \) and \( C \) a separation constant. Those equations are known as Mathieu equations, and can be solved using Mathieu functions.

### 3.1.3 Solutions in terms of Mathieu function

Mathieu functions are the equivalent of Bessel functions for ellipses. They can be written either as a series of \( \cosh \) and \( \sinh \), or as a series of Bessel functions. There are many types of Mathieu functions, which are classified in table 3.1.3

<table>
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<tr>
<th></th>
<th>Even</th>
<th>Odd</th>
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<tr>
<td>First kind</td>
<td>Ordinary</td>
<td>Modified</td>
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<td>( J_n )</td>
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Table 1: Categorization of Mathieu functions

<table>
<thead>
<tr>
<th></th>
<th>Ordinary</th>
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<tbody>
<tr>
<td>First kind</td>
<td>( J_n )</td>
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<td>Second kind</td>
<td>( Y_n )</td>
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Table 2: Categorization of Bessel functions

As we will prove later, there is a strong equivalence between modified Mathieu and Bessel functions. The separation between even and odd Mathieu functions will correspond to the difference between the dipolar and the quadrupolar cases (leading charge in the center or shifted).

Using the boundary condition \( E_z(\mu_0) = 0 \), we can prove that the electric field is expressed in terms of even modified Mathieu functions:

\[
E_z = 2\pi G \sum_{l=0}^{\infty} \frac{A_{2l}^l}{v_{2l}^l} ce_{2l}(\frac{\pi}{2} - \phi, q) \left[ F_{2l}(\mu, q) - \frac{F_{2l}(\mu_0, q)}{C_{2l}(\mu_0, -q)} Ce_{2l}(\mu, q) \right]
\]

(4)
where $A_n^2l$ and $p_{2l}$ are coefficients in the expansion of Mathieu functions.

It can be compared to the round case, where we can find the equivalence stated above:

$$E_z = G \left[ K_0 \left( \frac{k_0 r}{\beta \gamma} \right) - \frac{K_0 \left( \frac{k_0 a}{\beta \gamma} \right)}{I_0 \left( \frac{k_0 a}{\beta \gamma} \right)} I_0 \left( \frac{k_0 r}{\beta \gamma} \right) \right]$$

### 3.2 Longitudinal impedance

We can now easily derive an expression of the longitudinal impedance, starting from the definition:

$$Z_{||} = -\frac{1}{Q} \int_{-\infty}^{\infty} E_z (\mu = 0, \phi = \frac{\pi}{2}) \, dz$$

We can now consider the derivative of $Z$ along the $z$ axis, in order to have an expression independent of the length of the cavity.

$$\frac{dZ_{||}}{dz} = -\frac{E_z (\mu = 0, \phi = \frac{\pi}{2})}{Q}$$

Thus, using the previous expression of the electric field, we get:

$$\frac{dZ_{||}}{dz} = 2\pi G \sum_{l=0}^{\infty} A_n^2l \frac{F_{ek2l}(\mu, -q)}{C_{ek2l}(\mu, -q)} C_{ek2l}(\mu, -q)$$

We wrote a Python code to plot this impedance in function of the frequency, for different ellipses, and to compare it to the round and flat cases given by IW2D:

![Figure 4: Longitudinal impedance for different $q_r = \frac{a-b}{a+b}$, using $\beta = 0.9$ and $b = 35 \text{ mm}$](image)

We can see that the curves correspond almost precisely between our model and IW2D for the cases $q_r \sim 0$ and $q_r \sim 1$.

### 3.3 Quadrupolar impedance

The derivation of the quadrupolar impedance is quite the same, using its slightly different definition:

$$Z_{\perp y} = \frac{j}{Q_{\Delta y}} \int_{-\infty}^{\infty} E_y (\mu = \Delta \mu, \phi = \frac{\pi}{2}) + v B_z (\mu = \Delta \mu, \phi = \frac{\pi}{2}) \, dz$$
Using Maxwell equations, we can show that only the dependence in $E_z$ remains:

$$Z_{\perp, y} = -\frac{1}{\Omega \Delta y k_{\perp}} \int_{-\infty}^{\infty} \frac{\partial E_z}{\partial y} \, dz$$

Converting this expression in elliptical coordinates, we obtain:

$$\frac{dZ_{\perp, y}}{dz} = -\frac{1}{\Omega k_{\perp} F^2 \sinh(\Delta \mu) \cosh(\Delta \mu)} \frac{\partial E_z}{\partial \mu}$$

### 3.4 Dipolar impedance

Dipolar impedance is slightly more complicated, because we need to derive a new electromagnetic field, generated by a charge shifted from the center. Here is the expression we obtained:

$$E_z^{(dip)} = -\frac{\pi G_{\perp y} d}{\beta \gamma} \sum_{m=0}^{\infty} (-1)^m \frac{A_{2m+1}^2}{p_{2m+1}^2} \sin (2m+1) \phi \cos(q) \frac{G e k_{2m+1}(\mu_0, -q) S c_{2m+1}(\mu, -q)}{S c_{2m+1}(\mu_0, -q)}$$

From this longitudinal electric field, we can obtain the transverse impedance along $x$ or $y$ using the same derivation than for the quadrupolar impedance. For instance, on the $y$ axis, we have:

$$\frac{dZ_{\perp, y}^{(dip)}}{dz} = -\frac{1}{\Omega k_{\perp} F^2} \frac{\partial E_z}{\partial \mu}$$

$$\frac{dZ_{\perp, y}^{(dip)}}{dz} = \frac{\pi G}{F} \sum_{m=0}^{\infty} (-1)^m \frac{A_{2m+1}^2}{p_{2m+1}^2} \sin (2m+1) \phi \cos(q) \frac{G e k_{2m+1}(\mu_0, -q) S c_{2m+1}(\mu = 0, -q)}{S c_{2m+1}(\mu_0, -q)}$$

Here are the graphs representing the impedances (along $y$ and $x$) in function of the frequency:

![Graphs representing dipolar impedance](image)

Figure 5: Dipolar impedance in $y$ plane (left) and $x$ plane (right) for different $q_r = \frac{a-b}{a+b}$, using $\beta = 0.9$ and $b = 35$ mm.

We can notice a slight difference between our model and IW2D for the flat case, which doesn’t disappear with an increase of $q_r$. The cause of this difference is still under investigation.

### 4 Elliptical impedance for a resistive wall

The next step to improve our model is to take into account boundaries with finite conductivity.

#### 4.1 Electromagnetic field

As before, we first have to derive from Maxwell’s equations an expression of the longitudinal electric field for a resistive boundary. Using $E_z^c = Z_{\perp} H_{\phi}^{tot}$ (where $Z_{\perp}$ is the surface impedance [1]) and computing $H_{\phi}^{tot}$ with Maxwell equations (using $E_z^{PEC}$), we get:

$$E_z^c(\phi, \mu, -q) = \frac{\pi G d Z_{\perp}}{2 F Z_0 \gamma} \sum_{p=0}^{\infty} (-1)^p C_{2p+1} \sin (2p+1) \phi \cos(q) S c_{2p+1}(\mu, -q)$$
with:

\[ C_{2p+1} = \frac{1}{Sc_{2p+1}(\mu_0, -q)} \sum_{m=0}^{\infty} A_{2m+1}^1 W_{2m+1} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} A_{2p+1}^1 A_{2m+1}^1 \text{Proj}_{r,t}(\mu_0) \]

\[ W_{2m+1} = -Gek_{2m+1}(0, -q) Se'_{2m+1}(0, -q) = \frac{\mu'_{2m+1} c_{2m+1}(0, q)}{\pi \sqrt{q} A_{2m+1}^1} Se'_{2m+1}(0, -q) \]

4.2 Dipolar impedance

4.2.1 The notion of form factor

In order to compare directly the elliptical impedance with the circular impedance, we will use what is often called form factor in the literature (see [3] for instance) and defined as follow in the case of an ellipse:

\[ FF = \frac{Z_{\perp,\text{ellipse}}}{Z_{\perp,\text{cylinder}}} \]

The goal being to compare our expression with known cases, we will only compute it at low frequency, since expressions of form factors at low frequency can be find in the literature [3].

The transverse impedance of the elliptical pipe is given by:

\[ Z_{\perp,\text{ellipse}}' = -\frac{1}{Qk_z F} \frac{\partial E_z}{\partial \mu} \]

where the longitudinal electric field has been given above.

The impedance of a round pipe can be derived from [2] and [1] for low frequency:

\[ Z_{\perp,\text{circular}}' = \frac{Z_z}{\pi k_0 d^3} \]

4.2.2 Numerical results

The electric field for a resistive wall being much more complicated than for a perfectly conductive one, we haven’t obtained the results expected by the end of my internship. But our current state of research can still be presented.

The goal was to plot the form factor at low frequency for different \( q_r \) (like previously). When \( q_r \to 0 \), it should be equal to 1 (the elliptical impedance becomes a circular one), and for other \( q_r \), we should find the graph represented in figure 6 (computed in [3]):

![Figure 6: Form factor obtained after correcting the coefficient (source: [3])](image-url)
The first problem was that the form factor was not equal to 1 in the case of low \( q_r \). To check if it was a numerical or calculation problem, I did the asymptotic development of the form factor for \( q_r \rightarrow 0 \) and found the exact missing coefficient (see last section of Appendix A for the calculation). So there must be a factor missing in the expression of the electric field or somewhere else, that we haven’t found yet.

A second problem is that, even after correcting the factor issue, we obtained the graph represented in figure 7:

![Form Factor Obtained After Correcting the Coefficient](image)

**Figure 7:** Form factor obtained after correcting the coefficient

That fact that there is such a difference between figures 6 and 7 must be the sign of an error, either numerical or in the calculations.

The next steps to find the origin of the error were to do other asymptotic developments in order to check the calculations (for \( q_r \rightarrow 1 \) or \( \beta = 1 \) for instance), or to compare the electric field we calculated with a simulated electric field (using CST Studio).

## 5 Experimental study of instabilities

In order to get a better overview of what an instability really is, I have been proposed to study instabilities happening in real time in the LHC. Machine Developments (MD) are time slots allocated to accelerator physicists for testing their models of the machine and helping to improve the accelerator. The impedance team has done several MD during the summer, and I have been participating to two of them. During the first one, several instabilities occurred, and my job was to fit an exponential to the amplitude evolution curve and to extract its growth rate. For that, I created a Python program which takes two timestamps, a beam number and an axis, and display some interesting curves for the study of instabilities, including the amplitude and its exponential fitting (cf figure 8). I used PyTimber, a library which allows the user to get most of the variables of the beam and the LHC in real time.
Figure 8: Code for quick instabilities analysis, that produce the growth rate of an instability given the time interval during which the instability happened. The top-left figure corresponds the transverse position of the beam in function of time. On the right is its Fourier transform. On the bottom is the amplitude of the main frequency in the function of time, with the exponential fitting.

To perform the fitting, the program clean the data using a Gaussian filter, determine the the end of the instability with a condition on the second derivative (its cancellation), and use the function `curve_fit` of the library Scipy to perform the fitting on the cleaned data.

Acknowledgement

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Conclusion

My work during this summer internship at CERN brought new contributions to the elliptical impedance model that we are currently developing. My main contributions were to compute the different impedance expressions given the electromagnetic field calculated by the team and to develop a Python code implementing Mathieu functions and displaying the impedance for different kind of ellipses, in order to compare the model to the round and flat cases. It worked well for the perfectly conducting pipe, in spite of a slight difference between our model for $q_r$ close to 1 and the flat pipe model that we still need to explain. For the resistive wall, there seems to be a problem either in the expression of the dipolar electromagnetic field or in the code (probably both), leading to a wrong form factor. Those two issues are currently under investigations, by benchmarking the different expression with simulations on CST Studio and by doing more asymptotic analyses.
Another minor contribution was to develop a Python script which can get in few seconds the growth rate of an instability between two times. This code might be used during future MD in order to get quick informations about instabilities happening in real-time or to analyze the data afterward.

On a personal note, this project allowed me to learn a lot about the fascinating field of accelerator physics. I am not sure to do research in that field in the future, but I am glad to have been able to open the black box that accelerators constituted to me before. This project also helped me to progress in my mathematical understanding of special functions and to be aware of the numerical problems that can occur when trying to compute them numerically inside series. This summer at CERN was one of the best in my life and I am grateful to the impedance team for contributing to it.

References


