SYNCHROTRON MOTION IN THE PRESENCE OF SPACE CHARGE

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INTRODUCTION

While the synchrotron motion with negligibly small space charge forces is well known\(^1\), this is less true for high intensity beams, i.e. when the space charge forces are no longer small with respect to the external (RF) focusing force. We study here the static case (solution of Poisson's equation followed by a relativistic correction) for four-particle distributions:

i) constant density in longitudinal phase space\(^2\text{-}^4\),

ii) parabolic distribution in geometric space\(^5\text{,}^6\),

iii) \(\cos^2\) distribution in geometric space\(^7\), and

iv) constant volume density inside an ellipsoid in geometric space.

The results are given in terms of bucket (or bunch) areas, bunching factors and synchrotron frequencies as a function of the number of circulating particles.

To keep the mathematics transparent and simple, and to facilitate comparison with previous calculations\(^8\), we work directly with the equations of synchrotron motion rather than with Hamiltonians. Simplified expressions are often used for the space charge fields. The underlying assumptions and the resulting restrictions are as follows.

For a sinusoidal RF voltage, with frequency errors and space charge fields neglected, the equations of synchrotron motion are written as\(^1\)

\[
2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_s} \right) = eV_{RF} \left\{ \begin{array}{ll}
\sin \phi - \sin \phi_0 & \text{below transition energy} \\
-\left[ \sin(\phi - 2\phi_0) + \sin \phi_0 \right] & \text{above transition energy}^*)
\end{array} \right.
\]

\[
\frac{d\phi}{dt} = \frac{h_0 \omega_s \eta \Delta E}{\beta^2 \gamma E_0}
\]

where \(\omega_s\) is the angular particle revolution frequency, \(\phi\) is counted from the zero-crossing of the RF voltage, and the other symbols have the usual meanings (given for example in Ref. 1).

\(*)\) Above the transition energy this particular formulation is chosen to ensure the computational convenience of having \(\phi\) always in the interval \(-\pi \leq \phi \leq \pi\) (for stable motion), and to maintain the value of \(\phi_0\).
If only $\gamma$ is time-dependent, the two equations in (1.1) are transformed, by means of elementary operations, into

$$
\frac{2\pi R^2 E_o}{h c^2} \frac{d}{dt} \left( \frac{\gamma^3}{\gamma^2 - 1} \frac{d\phi}{dt} \right) = eV_{RF} \begin{cases}
\sin \phi - \sin \phi_o, & \text{below transition} \\
-\left[ \sin(\phi - 2\phi_o) - \sin \phi_o \right], & \text{above transition}
\end{cases}
$$

(1.2)

Let $s$ be the longitudinal space coordinate $^*$. If the space charge forces are not negligible, the right-hand side of Eq. (1.2) is modified by the addition of the term $^{2-6}$.

$$
2\pi \text{Re } E_s
$$

where $E_s$ is the longitudinal component of the electric field (in the laboratory frame) produced by the charge of the beam, in the presence of all surrounding structures.

In general $E_s$ is a function of both the longitudinal and the transverse space coordinates, i.e.

$$
E_s = E_s(s, x, z).
$$

Hence, in the presence of space charge the longitudinal motion is generally coupled to the transverse motion. In the present approach this coupling, and any extra contribution to the radial displacement

$$
\Delta R = \frac{\alpha R}{h \omega_s} \frac{d\phi}{dt}
$$

(1.3)

will be neglected, i.e. we assume the existence of a space charge potential

$$
V_{sc} = 2\pi R E_s(s).
$$

(1.4)

Properties of the transverse motion will enter Eq. (1.2) only via the momentum compaction factor $\alpha$ and Eq. (1.4) via the transverse beam dimensions.

Since the calculation of $E_s$ in the presence of surrounding structures is quite involved, a simplified approach will be used as a first step. The surrounding structures will be limited to a uniform lossless vacuum chamber, coaxial with the beam. For simplicity it will be further assumed that the vacuum chamber and the beam cross-section are circular, the beam radius being

$$
a_b[m] = \sqrt{\frac{\epsilon_v}{\epsilon_H} \frac{\epsilon_H}{\epsilon_v}}
$$

(1.5)

$^*$) Coordinate system: $x$ horizontal (transverse), $z$ vertical, $s$ longitudinal coordinate.
where \( \varepsilon_H, \varepsilon_V \) are the emittances (in units of rad m) and \( \bar{\beta}_H, \bar{\beta}_V \) the mean values of the amplitude functions \( \beta_H(s), \beta_V(s) \) (in metres).

To determine the field distribution, a given line charge distribution \( \lambda(s) \) per unit length of \( s \) will be assumed explicitly or implicitly, and in general the corresponding static potential will be calculated by means of the coaxial condenser formula [which does not take into account the effects at the ends of a bunch\(^*\)]

\[
V(s) = \frac{\lambda(s) ds}{C(s) ds},
\]

where \( C(s) \) is the capacitance per unit length of \( s \) between a beam of radius \( a_b(s) \) and the vacuum chamber of radius \( a_v' \). The expression \( C(s) = C(a_b, a_v') \) for a coaxial circular cylinder is recalled in Appendix I. The longitudinal component of the electric field \( E_{so} \), on the axis of the beam, is then taken to be

\[
E_{so} = \frac{\partial V(s)}{\partial s}.
\]

A relativistic correction is applied to Eq. (1.7) by replacing all lengths \( l \), measured along the \( s \)-axis, by \( \gamma l \), and the charge density \( \lambda(s) \) by \( \gamma^{-1} \lambda(s) \) \(^{**}\). Often this amounts\(^2\) to writing

\[
E_s = \gamma^{-2} E_{so}.
\]

Since \( E_{so} \) derives from a Coulomb potential (pure repulsion between particles), it cannot contribute to a change of bunch energy. The term \( 2\pi \Re E_s \) should thus have no effect on the value of the synchronous phase angle \( \phi_0 \).

Taking into account the accelerator geometry, the relation between the longitudinal space coordinate \( s \) and the phase coordinate \( \phi \) in Eq. (1.2) is taken to be

\[
s = \frac{R}{\hbar} (\phi - \phi_0).
\]

\(^*\) These effects are considered rigorously in the case of an ellipsoidal charge distribution (in Section 6).

\(^{**}\) This is to transform the line densities measured in the laboratory frame to the particle rest frame in which the electrostatic fields \( E_{so} \) are computed.
After substituting $\phi$ for $s$ in Eq. (1.4), the synchrotron equation with space charge becomes, with Eq. (1.2)

$$
\frac{2\pi R^2 E_0}{c^2} \frac{d}{dt} \left( \frac{\gamma^3}{\alpha\gamma^2 - 1} \frac{d\phi}{dt} \right) = eV_{RF} \left\{ \begin{array}{ll}
\sin \phi - \sin \phi_o & , \alpha\gamma^2 < 1 \\
-\sin(\phi - 2\phi_o) - \sin \phi_o & , \alpha\gamma^2 > 1
\end{array} \right\} + 2\pi R eE_s (\phi - \phi_o). \quad (1.9)
$$

Having thus added the space charge term with the proper phase, let us now consider the integration of Eq. (1.9).

Because of Eqs. (1.7) and (1.8) we have

$$
\int E_s (\phi - \phi_o) d\phi = \frac{h}{R} V_s (s), \quad (1.10)
$$

and the left-hand side represents a potential function, say $V(\phi)$. $V(\phi)$ is a fixed function of $\phi$. [The relation (1.10) is not valid when Eq. (1.3) is taken into account.]

If the variation of $\gamma$ with $t$ is neglected, Eq. (1.9) can be integrated with respect to $\phi$

$$
\frac{2\pi R^2 E_0}{c^2} \frac{\gamma^3}{\alpha\gamma^2 - 1} \left( \frac{d\phi}{dt} \right)^2 + C_0 = -eV_{RF} \left\{ \begin{array}{ll}
\cos \phi + \phi \sin \phi_o & , \alpha\gamma^2 < 1 \\
-\cos(\phi - 2\phi_o) - \phi \sin \phi_o & , \alpha\gamma^2 > 1
\end{array} \right\} + 2\pi R eV(\phi), \quad (1.11)
$$

where $C_0$ is an integration constant. This is the basic equation used in this paper and we shall now discuss it and the implications of its use.

The right-hand side of Eq. (1.11) is a fixed function of $\phi$, describing the potential well in which the accelerated charges are trapped. This potential well is by definition the same for all charges, i.e. for all trajectories $\phi(t) = y = y(\phi)$ resulting from Eq. (1.11), whether they are near to or far from the charge boundary curve.

In principle, it is possible to choose an infinite variety of functions for $\lambda(s)$, but in practice this choice is limited by the effort necessary to evaluate Eq. (1.11). The distributions mentioned earlier were chosen for the following reasons:

i) constant density in $(\phi, \dot{\phi})$ space leads to a $\lambda(s)$ of the same shape as the $(\phi, \dot{\phi})$ trajectories, [i.e. $\lambda(\phi) = \text{const.} \dot{\phi}(\phi)$; it is thus stationary in Kapchinski's sense in the presence of a uniform vacuum chamber].
ii) a parabolic $\lambda(s)$ leads to an $E_s$ linear in $s$,

iii) a $\cos^2$ distribution leads to the absence of discontinuity of $E_s$ at
the ends of the bunch,

iv) the field component $E_{so}$ produced by an ellipsoid of constant volume
density can be calculated without using the capacitance formula (1.6).

The case (iv) can thus be used to check the accuracy of Eq. (1.7). Furthermore, in the absence of a vacuum chamber, (iv) is the only known distribution yielding an $E_{so}$ not containing transverse coordinates. In other words, without the vacuum chamber, (iv) constitutes the only known case where the longitudinal and transverse space charge forces are independent of each other, i.e. space charge fields do not couple transverse and longitudinal motions and hence Eq. (1.4) is rigorously correct.

The comparison between the different distributions is made as follows. In all cases this comparison is based on the same operating parameters $\phi_0$, $V_{RF}$, $N$. For a full bucket the implicitly defined bunch length is used, i.e. the length resulting from the solution of Eq. (1.11) for each distribution. For a partially filled bucket the bunch length for negligible space charge is assumed to be the same for all distributions. The latter assumption is obviously not quite correct but computationally convenient.

In many practical cases one wishes to evaluate the effect of space charge after some adiabatic damping has taken place. It is then necessary to solve Eq. (1.9), taking into account the fact that $E_s(\phi - \phi_0)$ is a function of $\gamma$. Such a solution is much more involved than the first integral (1.11), obtained with the assumption that $\gamma$ is constant. If the law of adiabatic damping used to calculate the damped bunch lengths [which determines the constant $C_0$ in Eq. (1.11)] is not consistent with the distribution giving rise to $V(\phi)$, there will be disagreement between the bunch width and height obtained from Eq. (1.11) and that obtained from the law of adiabatic damping (or from a direct measurement, even if the space charge model for a fixed $\gamma$ is correct).

It is worthwhile to note that the approach based on Eqs. (1.6), (1.7)
and (1.11) breaks down when the vacuum chamber contains an irregularity
such as a gap. For such a case the relativistically corrected solution $E_{so}$
of Poisson's equation should be replaced by a solution $E_s$ of Maxwell's equations.

2. CONSTANT DENSITY IN $(\phi, \dot{\phi})$ SPACE

Assume that $\sigma = \text{const.}$ is the surface charge density in $(\phi, \dot{\phi})$ space and that the transverse beam cross-section is circular and constant. From Eq. (1.8):

$$ds = \frac{R}{h} d\phi, \quad \frac{ds}{dt} = \frac{R}{h} \frac{d\phi}{dt} = \frac{R}{h} \dot{\phi}(t).$$

(2.1)

Consider $\dot{\phi}(t) > 0$ and $\dot{\phi}(t) \to 0^+$, then

$$\lambda(\phi) = 2\sigma \dot{\phi}(t), \quad \lambda(s) = \frac{2\sigma h}{R} \dot{\phi}(t).$$

(2.2)

From the assumptions above and Eq. (1.7)

$$E_{so} = \frac{1}{C_b} \frac{\partial \lambda(s)}{\partial s} = \frac{h}{C_b} \frac{d\lambda(\phi)}{dt} \frac{dt}{ds} = \frac{2\sigma h^2}{C_b R^2} \ddot{\phi}(t),$$

(2.3)

where (in the MKSA system, used throughout this paper, cp. Appendix 1)

$$C_b = \frac{4\pi \varepsilon_o}{g_o}, \quad g_o = 1 + 2\ln \frac{a_v}{a_b}.$$  

(2.4)

Replacing $R$ by $\gamma R$, and noting that $\sigma$ contains no length element along the $s$-axis,

$$E_s = \frac{\sigma g_o h^2}{2\pi \varepsilon_o R^2 \gamma^2} \ddot{\phi}(t).$$

(2.5)

Ignoring for the moment what happens above transition, Eq. (1.9) becomes

$$2\pi \frac{R^2 E_o}{\hbar c^2} \frac{\gamma^3}{\alpha \gamma^2 - 1} \ddot{\phi}(t) = eV \left[ \sin \phi(t) - \sin \phi_o \right] + \frac{\sigma g_o h^2 e}{\varepsilon_o R \gamma^2} \dddot{\phi}(t)$$

(2.6)

To reduce the order of formula (2.6) by one unit, we write for brevity

$$A = 2\pi \frac{R^2 E_o}{\hbar c^2} \frac{\gamma^3}{\alpha \gamma^2 - 1}, \quad B = \frac{\sigma g_o h^2 e}{\varepsilon_o R \gamma^2},$$

and

$$y = \frac{d\phi}{dt}, \quad \frac{d^2\phi}{dt^2} = y \frac{dy}{d\phi},$$

(2.7)

giving

$$(Ay - B) \frac{dy}{d\phi} = e V_{RF} (\sin \phi - \sin \phi_o), \quad y > 0 \text{ and } y \to 0^+. $$

(2.8)
An integration with respect to $\phi$ yields

$$\frac{1}{2} A y^2 - B y = C_2 - e V_{\text{RF}} (\cos \phi + \phi \sin \phi_0), \quad y > 0 \text{ and } y \to 0^+,$$  \hspace{1cm} (2.9)

which coincides with Eq. (1.11), provided $V(\phi) = B y$. Symmetry requires invariance of Eq. (2.9) for $y < 0$, and Eq. (2.9) can be written in the more general form

$$\frac{1}{2} A y^2 - B |y| = C_2 - e V_{\text{RF}} (\cos \phi + \phi \sin \phi_0).$$  \hspace{1cm} (2.10)

The abscissa of Eq. (2.8) defining the separatrix is at $\phi_2 = \pi - \phi_0$ (as in the absence of space charge), giving

$$C_2 = e V_{\text{RF}} (\cos \phi_2 + \phi_2 \sin \phi_0),$$

and the equation of the separatrix becomes

$$y^2 - \frac{2B}{A} y = -\frac{2e V_{\text{RF}}}{A} \left[ \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0 \right], \quad y > 0 \text{ and } y \to 0^+$$  \hspace{1cm} (2.11)

Noting that

$$B = -\sigma_{\text{E}} e^2 h^3 c^2 1 - \alpha^2 \quad \frac{e V}{A} = \frac{\Omega_0^2}{\cos \phi_0}$$

where

$$\Omega_0 = \sqrt{\frac{h e V_{\text{RF}} c^2 1 - \alpha^2}{2 \pi R^2 E_0 \gamma^3}}$$

is the (angular) small-amplitude synchrotron frequency in the absence of space charge, the explicit equation of the separatrix is

$$y(\phi) = \frac{B}{A} + \sqrt{\frac{B}{A}} + \frac{2 \Omega_0^2}{\cos \phi_0} \left[ \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0 \right], \quad y > 0 \text{ and } y \to 0^+.$$  \hspace{1cm} (2.14)

The bucket half-height being defined as

$$y_{\text{max}} = \max_{-\pi < \phi < +\pi} y(\phi),$$

one readily finds

$$y_{\text{max}} = y(\phi_0).$$  \hspace{1cm} (2.15)
Let $\phi_1$ be a root of
\[ \cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2)\sin \phi_0 = 0, \]  
(2.17)
then the area enclosed by the separatrix (bucket area) is
\[ \alpha_0 = 2 \int_{\phi_1}^{\phi_2} y(\phi) \, d\phi. \]  
(2.18)
If the separatrix is also the charge envelope (fully filled bucket), the total number of accelerated particles becomes
\[ N = \frac{\sigma}{e} h \alpha_0. \]  
(2.19)

We now consider the problem of a partially filled bucket. Let $\phi_{2e}$ have a given value in the interval $\phi_0 < \phi_{2e} < \phi_2$, and let $\phi_{1e}$ be a root of
\[ \cos \phi_{1e} - \cos \phi_{2e} + (\phi_{1e} - \phi_{2e})\sin \phi_0 = 0. \]  
(2.20)
If the charge fills only the interval $(\phi_{1e}, \phi_{2e}) < (\phi_1, \phi_2)$, the charge envelope curve is
\[ y_e(\phi) = \frac{B}{A} + \sqrt{\left(\frac{B}{A}\right)^2 + \frac{2\Omega^2}{\cos \phi_0} \left[ \cos \phi - \cos \phi_{2e} + (\phi - \phi_{2e})\sin \phi_0 \right]} \quad, \quad y_e > 0 \]  
(2.21)
for $y_e \to 0^+$. The number of accelerated particles becomes
\[ N = 2 \frac{\sigma}{e} h \int_{\phi_{1e}}^{\phi_{2e}} y_e(\phi) \, d\phi. \]  
(2.22)

One of the assumptions is that the longitudinal dependence of the space charge force is the same for all particles. In other words, in the present case of partially filled buckets, it is necessary to move the space charge term $-B|y_e|$ [originating from Eq. (2.10)] to the right-hand side and to redetermine the constant $C_2$. Consider an abscissa $\phi_{2i}$, $\phi_0 < \phi_{2i} \leq \phi_{2e}$. The trajectory $y_i(\phi)$, located inside the charge envelope curve (2.21), will thus be described by ($y_i > 0$)
\[ y_i(\phi) = \sqrt{\frac{2\Omega^2}{\cos \phi_0} \left[ \cos \phi - \cos \phi_i + (\phi - \phi_i)\sin \phi_0 + \frac{2B}{A} \left( y_e(\phi) - y_e(\phi_{2i}) \right) \right]} \]  
(2.23)
Let \( \phi_{1i} \leq \phi_{1e} < \phi_0 \), be a root of

\[
\cos \phi_{1i} - \cos \phi_{2i} + (\phi_{1i} - \phi_{2i}) \sin \phi_0 + \frac{2B}{A} \left[ y_e(\phi_{1i}) - y_e(\phi_{2i}) \right] = 0 .
\tag{2.24}
\]

The oscillation period on the trajectory (2.23) is

\[
T_i = 2 \int_{\phi_{1i}}^{\phi_{2i}} \frac{d\phi}{y_i(\phi)} ,
\tag{2.25}
\]

giving the synchrotron frequency \( f_{1i} = T_i^{-1} \).

Since the integrals (2.18), (2.22) and (2.25) cannot be evaluated in closed form, it is necessary to have recourse to numerical calculations. These calculations are facilitated when Eq. (2.14) is normalized in some way. In order to have consistency with some results already available, three different normalizations will be described.

i) Let

\[
y_n = y \sqrt{\frac{\cos \phi_0}{2 \Omega_o^2}} , \quad S_c = \frac{B}{A} \sqrt{\frac{\cos \phi_0}{2 \Omega_o^2}} > 0 .
\tag{2.26}
\]

With this choice there is agreement for \( S_c = 0 \) with the CPS and PSB tables\(^9\).

The normalized expressions are (\( y_n > 0, y_n \to 0^+ \))

\[
y_n^2 + 2S_c y_n = \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0
\]

\[
y_n(\phi) = -S_c + \sqrt{S_c^2 + \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0}
\tag{2.14a}
\]

\[
\alpha = \frac{16}{\sqrt{2}} \frac{2\Omega_o^2}{\cos \phi_0} \alpha(S_c y_n)
\tag{2.18a}
\]

where

\[
\alpha(S_c y_n) = \frac{\sqrt{2}}{8} \int_{\phi_1}^{\phi_2} y_n(\phi) \, d\phi
\tag{2.27}
\]

\[
y_n(\phi) = -S_c + \sqrt{S_c^2 + \cos \phi - \cos \phi_{2e} + (\phi - \phi_{2e}) \sin \phi_0}
\tag{2.21a}
\]
\[ y_{ni}(\phi) = \sqrt{\cos \phi - \cos \phi_{21} + (\phi - \phi_{21}) \sin \phi_o - 2s_c \left[ y_{ne}(\phi) - y_{ne}(\phi_{21}) \right] } \]  
(2.23a)

\[ \cos \phi_{1i} - \cos \phi_{21} + (\phi_{1i} - \phi_{21}) \sin \phi_o - 2s_c \left[ y_{ne}(\phi_{1i}) - y_{ne}(\phi_{21}) \right] = 0 \]  
(2.24a)

\[ T_{ni} = 2 \int_{\phi_{1i}}^{\phi_{2i}} \frac{d\phi}{y_{ni}(\phi)} \quad \text{and} \quad f_{si} = \frac{2}{\int_{\phi_{1i}}^{\phi_{2i}} \frac{d\phi}{y_{ni}(\phi)} \cos \phi_o} \]  
(2.25a)

Because

\[ \sigma = - \left( \frac{\sigma A}{B} \frac{2\Omega^2_0}{\cos \phi} \right)_{sc} = \left( \frac{2\pi R^3 \varepsilon_o \varepsilon_o}{g_o e h^3 c^5} \frac{\gamma^5}{\left( 1 - \alpha \gamma^2 \right)^2 \cos \phi_o} \right)_{sc} \]  
(2.28)

\[ N = \frac{32 \varepsilon_o \varepsilon_o}{\sqrt{2}} \frac{RV_{RF}}{g_o e h} \frac{\gamma^2}{s_c \alpha(S_{cn})} \]  
(2.19a)

and for a partially filled bucket,

\[ N = \frac{32 \varepsilon_o \varepsilon_o}{\sqrt{2}} \frac{RV_{RF}}{g_o e h} \frac{\gamma^2}{s_c \alpha(S_{cne})} \]  
(2.22a)

where

\[ \alpha(S_{cne}) = \sqrt{2} \int_{\phi_{1e}}^{\phi_{2e}} y_{ne}(\phi) d\phi \]  
(2.29)

Finally with Eq. (2.2)

\[ \lambda(\phi) = \begin{cases} 
\left( \frac{4V_{RF} R \gamma^2 \varepsilon_o}{g_o h^2} \right)_{sc} y_{ne}(\phi), & \phi_1 \leq \phi_1e \leq \phi \leq \phi_2 \\
0, & -\pi \leq \phi \leq \phi_1e, \phi_2e \leq \phi \leq \phi + \pi 
\end{cases} \]  
(2.30)

From Eqs. (2.2) and (1.8)

\[ \lambda(s) = \frac{h}{R} \lambda(\phi) \ast, \quad \phi = \phi_o + \frac{h}{R} s \]  
(2.31)

\[ \ast \] Note that Eqs. (2.3) and (2.31) are not self-consistent electrostatically. Let \( \mathbf{E} = \mathbf{E}(E_{s0}, E_{y0}, E_{o0}) \). \( E_{o0} \equiv 0 \) because of the assumed symmetry. Substituting Eqs. (2.3) and (2.31) into Poisson's equation

\[ \text{div} \mathbf{E} = \frac{\partial \xi}{\partial s} + \frac{1}{r} \frac{\partial (r \mathbf{E}_o)}{\partial r} = \frac{\lambda(s)}{\pi a_b^2 \varepsilon_o} \]

yields \( E_{s0} \neq 0 \). Solving for \( E_{s0} \), it is easily verified that curl \( \mathbf{E} \neq 0 \). The lack of electrostatic self-consistency is a consequence of the assumption \( E_{s0} = \text{ct.} \text{.f.} r = \sqrt{x^2 + z^2} \leq a_b \).
From the preceding formulae the bunching factor \( B \), defined by

\[
B = \frac{N_e}{2\pi R \lambda(s)_{\text{max}}},
\]

(2.32)
can be calculated by combining Eqs. (2.26), (2.22a) and (2.30) as follows:

\[
B = \frac{2\sqrt{2}}{\pi} \frac{\alpha(S_c)_{\text{ne}}}{y_{\text{ne max}}},
\]

(2.33)
where

\[
y_{\text{ne max}} = y_{\text{ne}}(\phi_o),
\]

(2.34)
and \( \alpha(S_c)_{\text{ne}} \) are to be evaluated at the given intensity \( N \). Equation (2.33) remains valid for fully filled buckets because \( y_{\text{ne}}(\phi) \to y_n(\phi) \) as \( \phi_2 \to \phi \).

ii) In Ref. 2 there are graphs of the bucket area \( A(\lambda, \Gamma) \) (where \( \Gamma = \sin \phi_0 \)) as a function of a space charge parameter \( \lambda \) proportional to \( \sigma \). These graphs are based on the normalization

\[
\frac{1}{2} y_{[2]}^2 + \lambda_{[2]} y_{[2]} = \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0, \quad y_{[2]} > 0.
\]

(2.11b)
Thus

\[
y_{[2]} = \sqrt{2} y_n, \quad \lambda_{[2]} = 2 S_c, \quad A(\lambda, \Gamma) = \frac{16}{\sqrt{2}} \alpha(S_c)_{\text{ne}},
\]

(2.35)
and

\[
N = \frac{e V \gamma^2}{4\pi h g_o E_o \rho_p} \lambda_{[2]} A(\lambda, \Gamma),
\]

(2.19b)
where \( eV, E_o \) and \( R, \rho_p \) should be in the same units, respectively \( \rho_p \) designates the classical particle (proton) radius.

iii) In Ref. 3 there are graphs of the bucket area \( I_A(\Gamma, \lambda) \) based on the normalization

\[
\frac{1}{2} y_{[3]}^2 + \sqrt{2} \lambda_{[3]} y_{[3]} = \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0, \quad y_{[3]} > 0.
\]

(2.11c)
Although \( y_{[3]} = \sqrt{2} y_n \) and \( \lambda_{[3]} = \sqrt{2} S_c \), the graphs of Fig. 7 in Ref. 3 correspond to

\[
I_A(\Gamma, \lambda) = 4 \sqrt{2} \alpha(S_c)_{\text{ne}}, \quad \lambda = S_c
\]

(2.19c)
We now consider the situation above transition. As was already stated in Ref. 2, below transition space charge effects at the end of the bunch (i.e. for $\phi = \phi_1 e$ and $\phi = \phi_2 e$) are not accurately treated. For $\sigma \neq 0$ the separatrix (2.14) has zero slope at $\phi = \phi_2$ (see Fig. 1a), and the charge envelope curves (2.21) have a discontinuity of slope at $\phi = \phi_1 e$ and $\phi = \phi_2 e$, i.e.

$$\lim_{y_e \to 0^+} \frac{dy_e}{d\phi} \neq \lim_{y_e \to 0^-} \frac{dy_e}{d\phi}.$$  \hspace{1cm} (2.36)

This discontinuity of slope is a consequence of the singularity of the space charge force (2.5) as $\dot{\phi}(t) \to 0$.

![Diagram showing transition between below and above conditions](image)

**Fig. 1:** RF "bucket" in the presence of space charge with constant density in $y_n-\phi$ space.

In order to avoid complete re-examination and rewriting of the equations already developed, we have used the following prescriptions to ensure that the formulae (2.6) to (2.35) remain mathematically valid and computationally straightforward above transition

$$1 - \alpha y^2 \to |1 - \alpha y^2|$$ \hspace{1cm} (2.37)

$$S_c \to -S_c, \quad \phi_2 = \pi - \phi_0 \to \phi_2 = -\pi + 3\phi_0^*),$$ \hspace{1cm} (2.38)

$$\cos \phi - \cos \phi_2 + (\phi - \phi_2)\sin \phi_0 + \cos(\phi - 2\phi_0) - \cos(\phi_2 - 2\phi_0) - (\phi - \phi_2)\sin \phi_0$$

*) Taken together with the convention adopted for Eq. (1.1), this corresponds to a reflection of the bucket by the plane $\phi = \phi_0$. 


The normalized separatrix, for example, becomes

\[ y_n^2 - 2S_c y_n = \cos(\phi - 2\phi_o) - \cos(\phi_2 - 2\phi_o) - (\phi - \phi_2)\sin \phi_o \quad (2.39) \]

implicitly, and

\[ y_n(\phi) = + S_c + \frac{\sqrt{2S_c^2 + \cos(\phi - 2\phi_o) - \cos(\phi_2 - 2\phi_o) - (\phi - \phi_2)\sin \phi_o}}{2S_c} \quad (2.40) \]

explicitly \((S_c > 0, y_n \geq 0)\). In contrast to the case below transition, it follows from Eq. (2.39) that for \(\phi = \phi_1\) and \(\phi = \phi_2\), where the right-hand side of Eq. (2.39) vanishes, the ordinate \(y_n\) can take two distinct values: \(y_n = 0\) and \(y_n = +2S_c\). The charge envelope curve (2.40) contains thus a discontinuity (jump from \(y_n = 0\) to \(y_n = 2S_c\)) at \(\phi = \phi_2\) (Fig. 1b). This discontinuity is obviously also a consequence of the singularity of the space charge force (2.5) as \(\dot{\phi}(t) \to 0\). The jump in amplitude being less acceptable for the purpose in hand than a jump in slope, the solution (2.39) is deemed to fail (cf. Section Vb of Ref. 2).

The edge effect leading to a discontinuity of \(dy_e/d\phi\) below transition and of \(y_e(\phi)\) above transition can be removed by introducing finite range space charge forces\(^ {10}\) or, more simply, by replacing the space charge term \(2S_c y_n(\phi)\) in Eqs. (2.11a) and (2.39) by

\[ 2S_c f(\phi) y_n(\phi) \quad , \quad (2.41) \]

where \(f(\phi)\) is an arbitrary continuously differentiable function verifying the conditions

\[ f(\phi_o) = 1 \quad , \quad f'(\phi_o) = 0 \quad , \quad f(\phi_2) = f(\phi_3) = 0 \quad . \quad (2.42) \]

A particular example is

\[ f(\phi) = 2 - x^m + m_1 x^2 - (1 + m_2) x^{m_2}, \]

\[
\begin{align*}
x &= \frac{\phi - \phi_o}{\phi_2 - \phi_o}, \quad \phi_0 < \phi < \phi_2, \quad \phi_0 < \phi < \phi_1, \quad \phi_1 < \phi < \phi_o, \quad \text{below transition,} \\
x &= \frac{\phi - \phi_o}{\phi_o - \phi_1}, \quad \phi_1 < \phi < \phi_o, \quad \phi_0 < \phi < \phi_1, \quad \text{above transition,} 
\end{align*}
\]

with a suitable choice of the constants \(m, m_1, m_2\) (for example \(m = 10\), \(m_1 = m_2 = 0\)). The preceding formulae are unchanged, subject to \(S_c \to S_c f(\phi),\)
except that Eq. (2.22a) should be replaced by

$$N = \frac{S_c}{30\pi} \frac{RV}{g_0} \frac{\gamma^2}{eH_c} \int_{\phi_1}^{\phi_2} f(\phi) y_n(\phi) d\phi.$$  \hspace{1cm} (2.44)

3. PARABOLIC DISTRIBUTION

To take into account possible asymmetric bunches, two half-parabolae are used to describe the charge distribution. Assume that

$$\lambda(s) = \frac{3Ne}{2h(s_2 - s_1)} \begin{cases} 1 - \frac{s^2}{s_2^2}, & 0 \leq s \leq s_2 \\ 1 - \frac{s^2}{s_1^2}, & s_1 \leq s \leq 0 \end{cases}$$  \hspace{1cm} (3.1)

where, in accordance with Eq. (1.8)

$$s_2 = \frac{R}{h} (\phi_2 - \phi_0), \quad s_1 = \frac{R}{h} (\phi_1 - \phi_0),$$  \hspace{1cm} (3.2)

and that the beam has a constant circular cross-section $\pi a^2$. Equation (3.1) implies

$$\int_{s_1}^{s_2} \lambda(\xi) d\xi = \frac{Ne}{h}.$$  \hspace{1cm} (3.3)

From Eqs. (1.6), (1.7), (3.1) and remembering that the relativistic correction requires $s \rightarrow \gamma s$, $s_1 \rightarrow \gamma s_1$, $s_2 \rightarrow \gamma s_2$.

$$E_s = - \frac{3Ne g_0}{4\pi \varepsilon_0 \gamma^2 h(s_2 - s_1)} \begin{cases} \frac{s}{s_2^2}, & 0 \leq s \leq s_2 \\ \frac{s}{s_1^2}, & s_1 \leq s \leq 0 \end{cases},$$  \hspace{1cm} (3.4)

or, with Eq. (3.2)

$$E_s = - \frac{3Ne g_0 h}{4\pi \varepsilon_0 \gamma^2 R^2(\phi_2 - \phi_1)(\phi_1 - \phi_o)^2} \begin{cases} (\phi - \phi_o), & \phi_o \leq \phi \leq \phi_2 \\ (\phi_2 - \phi_o)^2 (\phi - \phi_o), & \phi_1 \leq \phi \leq \phi_o \end{cases}$$  \hspace{1cm} (3.5)
Letting

\[ S_c = \frac{3 \, \text{Ne} \, g_0 \, h}{2 \, \varepsilon_0 \, \gamma^2 \, R \, V_{\text{RF}} (\phi_2 - \phi_1)(\phi_2 - \phi_0)^2} \]  

(3.6)

and neglecting adiabatic damping, the synchrotron equation below transition becomes with Eqs. (1.9) and (2.13),

\[ \ddot{\phi} = -\frac{\Omega_o^2}{\cos \phi_0} \left\{ \sin \phi - \sin \phi_0 - \left[ \frac{S_c (\phi - \phi_0)}{S_c (\phi_2 - \phi_0)} \right] \right\} \]  

(3.7)

The abscissa \( \phi_2 \) of the saddle point is a root of

\[ \sin \phi_2 - \sin \phi_0 - S_c (\phi_2 - \phi_0) = 0, \quad \phi_0 < \phi_2 < \pi - \phi_0, \]  

(3.8)

which is not independent of space charge. The first integral of Eq. (3.7) is

\[ y^2(\phi) = \dot{\phi}^2(\phi) = \frac{2\Omega_o^2}{\cos \phi_0} \left[ \cos \phi + \phi \sin \phi_0 + \left( \frac{1}{2} S_c (\phi - \phi_0)^2 + C_+ \right. \right. \frac{\left. \phi_0 < \phi < \phi_2 \right] } \]  

(3.9)

\[ y(\phi_2) = 0 \quad \text{and} \quad y(\phi_0^+) = y(\phi_0^-) \quad \text{imply} \quad C_+ = C_- = -\cos \phi_2 - \phi_2 \sin \phi_0 - \frac{1}{2} S_c (\phi_2 - \phi_0)^2. \]

(3.10)

The minimal abscissa \( \phi_1 \) of the separation is a root of

\[ \cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_0 = 0, \]  

(3.11)

and it also depends on \( S_c \) (through \( \phi_2 \)).

With the normalization

\[ y_n(\phi) = \sqrt{\frac{\cos \phi_0}{2\Omega_o^2}} y(\phi) \]  

(3.12)

used previously [cf. Eqs. (2.26)], the equation of the separatrix becomes

\[ y_n(\phi) = \left[ \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0 + \frac{1}{2} S_c \left\{ \left( \phi - \phi_0 \right)^2 - \left( \phi_2 - \phi_0 \right)^2, \phi_0 < \phi < \phi_2 \right\} \right. \]  

\[ \left. \left[ (\phi_2 - \phi_0)^2 (\phi_1 - \phi_0)^2 - (\phi - \phi_0)^2 - (\phi_2 - \phi_0)^2, \phi_0 < \phi < \phi_1 \right] \right\}^{1/2} \]  

(3.9a)
The bucket height, area and the number of accelerated particles are therefore

\[ y_{n \text{ max}} = y_n(\phi_0), \]  

(3.13)

\[ \alpha(S_c)_{n} = \frac{\sqrt{2}}{8} \int_{\phi_1}^{\phi_2} y_n(\phi) d\phi, \]  

(3.14)

\[ N = \frac{2e_0 \gamma^2 R V_{\text{RF}}(\phi_2 - \phi_1)(\phi_2 - \phi_0)^2}{3 h g_0 e S_c}. \]  

(3.15)

Let \( \phi_{2e} < \phi_2 \) be known if the charge fills only a part of the bucket area. The Eqs. (3.9a) and (3.11) to (3.15) remain unchanged, except that \( \phi_{1e}, \phi_{2e} \) and \( y_{ne}(\phi) \) take the place of \( \phi_1, \phi_2 \) and \( y_n(\phi) \).

Suppose that the charge envelope curve \( y_{ne}(\phi) \) is known. The trajectory passing through a fixed abscissa \( \phi_{2i}, \phi_0 < \phi_{2i} \leq \phi_{2e} \) is given by

\[ y_{ni}(\phi) = \left[ \cos \phi - \cos \phi_{2i} + (\phi - \phi_{2i})\sin \phi_0 + \frac{1}{2} S_c \left\{ \frac{(\phi - \phi_0)^2 - (\phi_{2i} - \phi_0)^2}{\phi_{1i} - \phi_0}, \phi_0 < \phi < \phi_{2i} \right\} \right]^{\frac{1}{2}}, \]  

(3.16)

where \( \phi_{1i} \) is a root of

\[ \cos \phi_{1i} - \cos \phi_{2i} + (\phi_{1i} - \phi_{2i})\sin \phi_0 + \frac{1}{2} S_c \left( \frac{\phi_{2i} - \phi_0}{\phi_{1i} - \phi_0} \right)^2 (\phi_{1i} - \phi_0)^2 - \frac{1}{2} S_c (\phi_{2i} - \phi_0)^2 = 0. \]  

(3.17)

The synchrotron period on \( y_{ni}(\phi) \) is

\[ T_{ni} = 2 \int_{\phi_{1i}}^{\phi_{2i}} \frac{d\phi}{y_{ni}(\phi)}. \]  

(3.18)

Substituting Eq. (3.1) into the definition of the bunching factor Eq. (2.32) yields

\[ B = \frac{\phi_2 - \phi_1}{3 \pi}. \]  

(3.19)
Above transition Eq. (3.8) is replaced by
\[ + \sin(\phi_2 - 2\phi_o) + \sin \phi_o + S_c (\phi_2 - \phi_o) = 0, \quad -\pi + 3\phi_o \leq \phi_2 < \phi_o \] (3.20)

The other pertinent formulae become:

**\( \phi_1 \)**

Root of
\[ \cos(\phi_2 - 2\phi_o) - \cos(\phi_1 - 2\phi_o) + (\phi_1 - \phi_2)\sin \phi_o = 0 \] (3.21)

**Separatrix**

\[ y_n(\phi) = \left[ \cos(\phi_2 - 2\phi_o) - \cos(\phi_2 - 2\phi_o) - (\phi_2 - \phi_1)\sin \phi_o - \frac{1}{2} S_c \left\{ (\phi_2 - \phi_1)^2 - (\phi_2 - \phi_o)^2, \phi_2 \leq \phi < \phi_o \right\} \right] \]

\[ \left[ (\phi_2 - \phi_o)^2 (\phi_1 - \phi_o)^{-2} (\phi - \phi_o)^2 - (\phi_2 - \phi_o)^2, \phi_o \leq \phi \leq \phi_1 \right] \frac{1}{y_n(\phi)} \] (3.22)

**Bucket area**

\[ \alpha(S_n) = \frac{\sqrt{2}}{8} \int_{\phi_2}^{\phi_1} \frac{d\phi}{y_n(\phi)} \] (3.23)

**Interior trajectory**

\[ y_{ni}(\phi) = \left[ -\cos(\phi_{2i} - 2\phi_o) + \cos(\phi - 2\phi_o) - (\phi_2 - \phi_{2i})\sin \phi_o - \frac{1}{2} S_c \left\{ (\phi_2 - \phi_1)^2 - (\phi_{2i} - \phi_o)^2, \phi_{2i} \leq \phi \leq \phi_o \right\} \right] \]

\[ \left[ (\phi_{2e} - \phi_o)^2 (\phi_{1e} - \phi_o)^{-2} (\phi - \phi_o)^2 - (\phi_{2i} - \phi_o)^2, \phi_o \leq \phi \leq \phi_{1i} \right] \frac{1}{y_{ni}(\phi)} \] (3.24)

where
\[ \cos(\phi_{2i} - 2\phi_o) - \cos(\phi_{1i} - 2\phi_o) + (\phi_{1i} - \phi_{2i})\sin \phi_o + \frac{1}{2} S_c \left( \phi_{2e} - \phi_o \right)^2 - \frac{1}{2} S_c (\phi_{2i} - \phi_o)^2 = 0. \] (3.25)

**Synchrotron period**

\[ T_{ni} = 2 \int_{\phi_{2i}}^{\phi_{1i}} \frac{d\phi}{y_{ni}(\phi)} \] (3.26)

The parabolic distribution applies equally well below or above transition. It is thought, however, that it is fairly realistic only for low intensity
beams\textsuperscript{11}). At least in the case of transversal motion it leads to rather high thresholds\textsuperscript{12}).

As in the case of the distribution $\sigma = \text{Const.}$ in $(\phi - \phi)$ space, the Eqs. (3.1) and (3.4) are inconsistent electrostatically.

4. $\cos^2$ DISTRIBUTION

In the same general fashion as in Section 3 assume that

$$\lambda(s) = \frac{2Ne}{h(s_2 - s_1)} \begin{cases} 
\cos^2 \frac{\pi s}{2s_2}, & 0 \leq s \leq s_2 \\
\cos^2 \frac{\pi s}{2s_3}, & s_1 \leq s \leq 0 
\end{cases}, \quad (4.1)$$

where $s_1$, $s_2$ are given by Eq. (3.2), and assume further that the beam has a constant circular cross-section $\pi a^2$. Equation (4.1) implies

$$\int_{s_1}^{s_2} \lambda(\zeta) d\zeta = \frac{Ne}{h}. \quad (4.2)$$

From Eqs. (1.6), (1.7) and (4.1) and applying the relativistic correction $s \rightarrow \gamma s$, $s_1 \rightarrow \gamma s_1$, $s_2 \rightarrow \gamma s_2$

$$E_s = -\frac{Ne g_0}{4\varepsilon \gamma^2 h(s_2 - s_3)} \begin{cases} 
\frac{1}{s_2} \sin \frac{\pi s}{s_2}, & 0 \leq s \leq s_2 \\
\frac{1}{s_3} \sin \frac{\pi s}{s_3}, & s_3 \leq s \leq 0 
\end{cases}. \quad (4.3)$$

Eliminating $s$, $s_1$, $s_2$

$$E_s = -\frac{Ne g_0 h}{4\varepsilon \gamma^2 R^2(\phi_2 - \phi_1)} \begin{cases} 
\frac{1}{\phi_2 - \phi_0} \sin \frac{\pi (\phi - \phi_0)}{\phi_2 - \phi_0}, & \phi_0 \leq \phi \leq \phi_2 \\
\frac{1}{\phi_1 - \phi_0} \sin \frac{\pi (\phi - \phi_0)}{\phi_1 - \phi_0}, & \phi_1 \leq \phi \leq \phi_0 
\end{cases}. \quad (4.4)$$

In order to have the same reference conditions as in the case of the parabolic distribution it is necessary to let

$$S_c = \frac{\pi^2 Ne g_0 h}{2\varepsilon \gamma^2 R \sqrt{\frac{V_{RF}(\phi_2 - \phi_1)(\phi_2 - \phi_0)^2}}}. \quad (4.5)$$
Neglecting adiabatic damping, the synchrotron equation below transition becomes
\begin{equation}
\dot{\phi} = - \frac{\Omega_0^2}{\cos \phi_0} \left[ \sin \phi - \sin \phi_0 - \left( \begin{array}{c}
\frac{\phi_2 - \phi_0}{\pi} \sin \frac{\phi - \phi_0}{\phi_2 - \phi_0}, \phi_0 < \phi < \phi_2 \\
\frac{\phi_2 - \phi_0}{\phi_1 - \phi_0} \sin \frac{\phi - \phi_0}{\phi_1 - \phi_0}, \phi_1 < \phi < \phi_0
\end{array} \right) \right]. \tag{4.6}
\end{equation}

The abscissa \( \phi_2 \) of the saddle point is a root of
\begin{equation}
\sin \phi_2 - \sin \phi_0 = 0. \tag{4.7}
\end{equation}

It is thus independent of space charge and equal to \( \phi_2 = \pi - \phi_0 \). The first integral of Eq. (4.6) is
\begin{equation}
y^2(\phi) = \dot{\phi}^2(\phi) = \frac{2\Omega_0^2}{\cos \phi_0} \left[ \cos \phi + \phi \sin \phi_0 - \left( \begin{array}{c}
S_c \left( \frac{\phi_2 - \phi_0}{\pi} \right)^2 \cos \frac{\phi - \phi_0}{\phi_2 - \phi_0} + C_+, \phi_0 < \phi < \phi_2 \\
S_c \left( \frac{\phi_2 - \phi_0}{\phi_1 - \phi_0} \right)^2 \cos \frac{\phi - \phi_0}{\phi_1 - \phi_0} + C_-, \phi_1 < \phi < \phi_0
\end{array} \right) \right]. \tag{4.8}
\end{equation}

\( y(\phi_2) = 0 \) implies
\begin{equation}
C_+ = \cos \phi_2 + \phi_2 \sin \phi_0 + S_c \left( \frac{\phi_2 - \phi_0}{\pi} \right)^2. \tag{4.9}
\end{equation}

Continuity of \( y(\phi) \) at \( \phi = \phi_0 \) implies
\begin{equation}
C_- = C_+. \tag{4.10}
\end{equation}

The minimal abscissa \( \phi_1 \) of the separatrix is a root of
\begin{equation}
\cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_0 = 0, \tag{4.11}
\end{equation}

and it is also independent of \( S_c \). With the normalization (3.12), the equation of the separatrix becomes
\begin{equation}
y_n(\phi) = \left[ \cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_0 - S_c \left( \frac{\phi_2 - \phi_0}{\pi} \right)^2 \right] \left[ 1 + \cos \pi \phi \phi_2 - \phi_0 \right]^{1/2}, \tag{4.12}
\end{equation}

\( \phi_0 < \phi < \phi_2 \); \( \left[ \cos \pi \phi \phi_1 - \phi_0 + 1, \phi_1 < \phi < \phi_0 \right] \right) \right]}
The number of accelerated particles, the bucket height and area are

\[
N = \frac{2e_o \gamma^2 R V_{RF} (\phi_2 - \phi_1)(\phi_2 - \phi_o)^2}{\pi^2 g_o e h} \quad S_c \quad (4.13)
\]

\[
y_{n \text{ max}} = y_n(\phi_o) \quad (4.14)
\]

\[
\alpha(S_c)_n = \frac{\sqrt{2}}{8} \int_{\phi_1}^{\phi_2} y_n(\phi) d\phi. \quad (4.15)
\]

For \( \phi_o = 0 \), the last expression simplifies into the explicit form

\[
\alpha(S_c)_n = \sqrt{1 - S_c} = \sqrt{1 - \frac{g_o e h N}{4\pi e_o R \gamma^2 V_{RF}}} \quad (4.15a)
\]

used in Ref. 7.

If the charge fills only a part of the bucket area and \( \phi_{2e} \) is known
the preceding equations remain unchanged, except that \( \phi_{1e}, \phi_{2e}, y_{ne}(\phi) \)
take the place of \( \phi_1, \phi_2, y_n(\phi) \).

Once the charge envelope curve \( y_{ne}(\phi) \) is known, the trajectory passing
through the fixed abscissa \( \phi_{2i}, \phi_0 < \phi_{2i} < \phi_{2e} \), is given by

\[
y_{ni}(\phi) = \left[ \cos \phi - \cos \phi_{2i} + (\phi - \phi_{2i}) \sin \phi_0 + S_c \left( \frac{\phi_{2e} - \phi_{2i}}{\pi} \right)^2 \right] \left[ \cos \pi \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} - \cos \pi \frac{\phi - \phi_0}{\phi_{2e} - \phi_0} \right],
\]

\[
\phi_0 < \phi < \phi_{2i} \right]; \left[ \cos \pi \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} - \cos \pi \frac{\phi - \phi_0}{\phi_{2e} - \phi_0}, \phi_{1i} < \phi < \phi_o \right] \right]^{1/2},
\]  

(4.16)

where \( \phi_{1i} \) is a root of

\[
\cos \phi_{1i} = \cos \phi_{2i} + (\phi_{1i} - \phi_{2i}) \sin \phi_0 +
\]

\[
+ S_c \left( \frac{\phi_{2e} - \phi_{2i}}{\pi} \right)^2 \left[ \cos \pi \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} - \cos \pi \frac{\phi - \phi_0}{\phi_{2e} - \phi_0} \right] = 0 \quad (4.17)
\]

The synchrotron period on \( y_{ni}(\phi) \) is

\[
T_{ni} = 2 \int_{\phi_{1i}}^{\phi_{2i}} \frac{d\phi}{y_{ni}(\phi)}. \quad (4.18)
\]
Substituting Eq. (4.1) into Eq. (2.32) yields for the bunching factor

\[
B = \frac{\phi_2 - \phi_1}{4 \pi} \tag{4.19}
\]

Above transition the saddle point abscissa \( \phi_2 = \pi - \phi_0 \) is replaced by \( \phi_2 = -\pi + 3\phi_0 \). The equation of the separatrix becomes

\[
y_n(\phi) = \left[ \cos(\phi - 2\phi_0) - \cos(\phi_2 - 2\phi_0) - (\phi_2 - \phi_0)\sin \phi_0 + \right.
\]
\[
\left. S_c \left( \frac{\phi_2 - \phi_0}{\pi} \right)^2 \right] \left[ 1 + \cos \frac{\phi - \phi_0}{\phi_2 - \phi_0} \right], \quad \phi_2 \leq \phi \leq \phi_0 \right] \left[ \cos \frac{\phi - \phi_0}{\phi_1 - \phi_0} + 1, \quad \phi_0 \leq \phi \leq \phi_1 \right] \right]^{1/2}
\]

where \( \phi_1 \) is a root of

\[
\cos(\phi_1 - 2\phi_0) - \cos(\phi_2 - 2\phi_0) - (\phi_1 - \phi_2)\sin \phi_0 = 0. \tag{4.21}
\]

The Eqs. (4.13) to (4.15) remain unchanged, except that \( \phi_2 \rightarrow \phi_1, \phi_1 \rightarrow \phi_2 \). For a given \( \phi_{2i} \) and a known \( y_{ne}(\phi) \), the equation of an interior trajectory is

\[
y_{ni}(\phi) = \left[ \cos(\phi - 2\phi_0) - \cos(\phi_{2i} - 2\phi_0) - (\phi - \phi_{2i})\sin \phi_0 + \right.
\]
\[
\left. S_c \left( \frac{\phi_{2e} - \phi_0}{\pi} \right)^2 \right] \left[ \cos \frac{\phi - \phi_0}{\phi_{2e} - \phi_0} - \cos \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} \right], \quad \phi_{2i} \leq \phi \leq \phi_0 \right] \left[ \cos \frac{\phi - \phi_0}{\phi_{1e} - \phi_0} - \cos \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} \right], \quad \phi_{1e} \leq \phi \leq \phi_0 \right] \right]^{1/2}
\]

where \( \phi_{1i} \) is a root of

\[
\cos(\phi_{1i} - 2\phi_0) - \cos(\phi_{2i} - 2\phi_0) - (\phi_{1i} - \phi_{2i})\sin \phi_0 + \right.
\]
\[
\left. S_c \left( \frac{\phi_{2e} - \phi_0}{\pi} \right)^2 \right] \left[ \cos \frac{\phi_{1i} - \phi_0}{\phi_{1e} - \phi_0} - \cos \frac{\phi_{2i} - \phi_0}{\phi_{2e} - \phi_0} \right] = 0. \tag{4.23}
\]

The \( \cos^2 \) distribution applies equally well below or above transition. As in the previous two cases, it is however inconsistent electrostatically.
5. **ELLIPSOIDAL DISTRIBUTION** (capacitance formula)

Let

\[
\frac{x^2}{a^2} + \frac{z^2}{b^2} + \frac{s^2}{c_0^2} = 1
\]  
(5.1)

be the surface of an ellipsoid, filled with a charge of constant volume density \( \rho \), and let one such ellipsoid contain the charge

\[
\frac{Ne}{h} = \frac{4}{3} \pi ab c_0 \rho
\]  
(5.2)

A slice of thickness \( ds \) contains the charge

\[
dq = \pi x z \rho \, ds
\]  
(5.3)

The linear charge density is therefore

\[
\lambda(s) = \frac{dq}{ds} = \pi \rho x y
\]  
(5.4)

For simplicity, assume that the ellipsoid (5.1) is one of revolution:

\[
a = b, \quad r^2 = x^2 + z^2 = a^2 \left( 1 - \frac{s^2}{c_0^2} \right)
\]  
(5.5)

Since the cross-section is now \( \pi r^2 \), Eq. (5.4) becomes

\[
\lambda(s) = \pi \rho a^2 \left( 1 - \frac{s^2}{c_0^2} \right)
\]  
(5.6)

i.e. the line-charge distribution is parabolic. The beam radius is, however, not constant, but equal to \( r \).

If the capacitance formula (1.6) still applies

\[
V(s) = \frac{1}{4 \pi \varepsilon_0} \lambda(s) \left( 1 + 2 \ln \frac{a}{r} \right)
\]  
(5.7)

Hence,

\[
E_{so} = \frac{\partial V(s)}{\partial s} = \frac{\rho a^2}{2 \varepsilon_0 c_0^2} \left[ (1 - g_o)s + s \ln(1 - \frac{s^2}{c_0^2}) \right]
\]  
(5.8)

where

\[
g_o = 1 + 2 \ln \frac{a}{r}.
\]
The expression (5.8) is singular for \( s = c_0 \). The relativistic correction requires \( c_0 + \gamma c_0, s + \gamma s, \rho + \gamma^{-1} \rho \):

\[
E_s = \frac{\rho a^2}{2c_0 c_0^2 \gamma^2} \left[ (1 - g_o)s + s \ln(1 - \frac{s^2}{c_0^2}) \right] \tag{5.9}
\]

Neglecting adiabatic damping, the synchrotron equation below transition is, with Eqs. (1.9) and (2.12),

\[
\ddot{\phi} = -\frac{\Omega^2}{\cos^2 \phi_o} \left[ \sin \phi - \sin \phi_o - S_c (\phi - \phi_o) + S_c \frac{\phi - \phi_o}{g_o - 1} \ln(1 - \frac{R^2}{h^2 c_0^2 (\phi - \phi_o)^2}) \right] \tag{5.10}
\]

where

\[
S_c = \frac{3N_e R^2 (g_o - 1)}{4c_0 h^2 c_0^3 \gamma^2 V_{RF}} \tag{5.11}
\]

Equation (5.10) integrates into

\[
y^2(\phi) = \dot{\phi}^2(\phi) = \frac{2\Omega^2}{\cos^2 \phi_o} \left[ \cos \phi + \phi \sin \phi_o + \frac{1}{2} S_c (\phi - \phi_o)^2 + S_c \frac{\phi - \phi_o}{g_o - 1} f(\phi - \phi_o) + C_0 \right] \tag{5.12}
\]

where \( C_0 \) is a constant and

\[
f(\phi) = \frac{1}{2} \phi^2 + \frac{h^2 c_0^2}{2R^2} \left( 1 - \frac{R^2}{h^2 c_0^2} (\phi - \phi_o)^2 \right) \ln(1 - \frac{R^2}{h^2 c_0^2} \phi^2) \tag{5.13}
\]

Since by definition the abscissa \( \phi_2 \) of the saddle point is a root of

\[
sin \phi_2 - \sin \phi_o - S_c (\phi_2 - \phi_o) + S_c \frac{\phi_2 - \phi_o}{g_o - 1} \ln(1 - \frac{R^2}{h^2 c_0^2} (\phi_2 - \phi_o)^2) = 0, \phi_o < \phi_2 < \pi \tag{5.14}
\]

\[
C_0 = -\cos \phi_2 - \phi_2 \sin \phi_o - \frac{1}{2} S_c (\phi_2 - \phi_o)^2 - \frac{S_c}{g_o - 1} f(\phi_2 - \phi_o).
\]

The minimal abscissa \( \phi_1, -\pi < \phi_1 < \phi_o \), is a root of

\[
\cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_o + \frac{S_c g_o}{2(g_o - 1)} \left[ (\phi_1 - \phi_o)^2 - (\phi_2 - \phi_o)^2 \right] + \frac{S_c}{2(g_o - 1)} \left[ f_1 (\phi_1 - \phi_o) - f_1 (\phi_2 - \phi_o) \right] = 0, \tag{5.15}
\]
where
\[ f_1(\phi) = \frac{h^2 c_0^2}{R^2} \left( 1 - \frac{R^2 \phi^2}{h^2 c_0^2} \right) \ln \left( 1 - \frac{R^2 \phi^2}{h^2 c_0^2} \right). \]

Equations (5.14) and (5.15) cannot be solved unless the major ellipsoid axis 2c_0 is first expressed as a function of \( \phi_1 \) and \( \phi_2 \). It appears simplest to let
\[ c_0 = \frac{R}{2h} (\phi_2 - \phi_1) \quad (5.16) \]

but then it should be kept in mind that Eqs. (5.16) and (5.6) are incompatible, because \( \phi_0 \) is not located midway between \( \phi_2 \) and \( \phi_3 \), i.e., unless \( \phi_0 = 0 \),
\[ \frac{1}{2} (\phi_2 - \phi_1) \neq \phi_2 - \phi_0 \neq \phi_0 - \phi_1. \]

It was found in fact, that the two Eqs. (5.14) and (5.15) do not admit a real solution for \( \phi_1 \) and \( \phi_2 \) in the required intervals when \( c_0 \) is defined by Eq. (5.16). The lack of existence of a real solution for \( \phi_1 \) and \( \phi_2 \) is a consequence of the singularity of Eq. (5.8).

In order to neutralize this singularity let
\[ c_0 = \frac{R}{h} (\phi_0 - \phi_1) \quad (5.17) \]

and redefine Eqs. (5.10) and (5.12) as follows:
\[ -\frac{\cos \phi \phi_0}{2\Omega_0^2} \phi = \sin \phi - \sin \phi_0 + \left\{ - \frac{S}{c} (\phi - \phi_0) + \frac{S}{c - 1} (\phi - \phi_0) \ln \left( 1 - \frac{\phi - \phi_0}{\phi_0 - \phi_1} \right) \right\}, \quad (5.18) \]
\[ \phi_1 < \phi < 2\phi_0 - \phi_1 \}; \quad [0, 2\phi_0 - \phi_1] \leq \phi \leq \phi_2 \]
\[ \frac{\cos \phi_0}{2 \Omega_0^2} \dot{\phi}^2 = \cos \phi + \phi \sin \phi_0 + \left[ C_+ + \frac{S}{2} (\phi - \phi_0)^2 + \frac{S}{g_0 - 1} f_2(\phi - \phi_0) \right], \]

\[ \phi_1 < \phi < 2\phi_0 - \phi_1 \right] ; \left[ C_-, 2\phi_0 - \phi_1 < \phi < \phi_2 \right] \]

(5.19)

where \( C_- \), \( C_+ \) are constants of integration and

\[ f_2(\phi) = \frac{1}{2} \phi^2 + \frac{1}{2} (\phi_0 - \phi_1)^2 \left[ 1 - \frac{\phi^2}{(\phi_0 - \phi_1)^2} \right] \ln \left[ 1 - \frac{\phi^2}{(\phi_0 - \phi_1)^2} \right] \]

Using \( \dot{\phi}(\phi_1) = 0 \), \( \dot{\phi}(\phi_2) = 0 \) and the continuity of \( \dot{\phi} \) at \( \phi = 2\phi_0 - \phi_1 \), one obtains

\[ C_+ = -\cos \phi_1 - \phi_1 \sin \phi_0 - \frac{S c g_0}{2(g_0 - 1)} (\phi_1 - \phi_0)^2 \]

\[ C_- = -\cos \phi_1 - \phi_1 \sin \phi_0 \]

For a "full" bucket the extremal abscissae \( \phi_2, \phi_1 \) are roots of

\[ \sin \phi_2 - \sin \phi_0 = 0 \]

(5.20)

and

\[ \cos \phi_2 - \cos \phi_1 + (\phi_2 - \phi_1) \sin \phi_0 = 0 \]

(5.21)

respectively. With the normalization (3.12) the equation of the separatrix becomes

\[ y_n^2(\phi) = \cos \phi - \cos \phi_1 + (\phi_2 - \phi_1) \sin \phi_0 + \frac{S c g_0}{2(g_0 - 1)} \left\{ (\phi_2 - \phi_0)^2 - (\phi_1 - \phi_0)^2 \right\} + g_0^{-1} f_3(\phi), \phi_1 < \phi < 2\phi_0 - \phi_1 \right] ; \left[ 0, 2\phi_0 - \phi_1 < \phi < \phi_2 \right] \]

(5.22)
where

\[ f_3(\phi) = (\phi - \phi_1)^2 \left[ 1 - \left( \frac{\phi - \phi_0}{\phi_0 - \phi_1} \right)^2 \right] \ln \left[ 1 - \left( \frac{\phi - \phi_0}{\phi_0 - \phi_1} \right)^2 \right] \]

Similarly to the preceding cases

\[ y_{n \text{ max}} = y_n(\phi_0) \]  \hspace{1cm} (5.23)

\[ (S' c_n) = \frac{\sqrt{2}}{8} \int_{\phi_1}^{\phi_2} y_n(\phi) d\phi \]  \hspace{1cm} (5.24)

and

\[ N = \frac{4R^2 h^2 c \gamma^2 V_{RF}}{3R^2 (g_0 - 1)} S_c \]  \hspace{1cm} (5.25)

For a partially filled bucket, \( \phi_{1e} \leq \phi \leq 2\phi - \phi_{1e} \), Eq. (5.17) is replaced by

\[ c_o = \frac{R}{h} (\phi_0 - \phi_{1e}) \]  \hspace{1cm} (5.26)

and Eqs. (5.18) and (5.19) by

\begin{align*}
&- \frac{\cos \phi}{\Omega_0^2} \phi = \sin \phi - \sin \phi_0 + \\
+ & S_c \begin{cases} 
0, \quad 2\phi_0 - \phi_{1e} \leq \phi \leq \phi_2 \\
-(\phi - \phi_0) + \frac{\phi - \phi_0}{g_0 - 1} \ln \left[ 1 - \left( \frac{\phi - \phi_0}{\phi_0 - \phi_{1e}} \right)^2 \right], \quad \phi_{1e} < \phi < 2\phi_0 - \phi_{1e} \\
0, \quad \phi_1 \leq \phi \leq \phi_{1e} 
\end{cases} 
\end{align*}  \hspace{1cm} (5.27)

\begin{align*}
&\cos \frac{\phi}{\Omega_0^2} \phi^2 = \cos \phi + \phi \sin \phi_0 + \\
+ & \begin{cases} 
C_1, \quad 2\phi_0 - \phi_{1e} \leq \phi \leq \phi_2 \\
C_2 + \frac{S}{2} (\phi - \phi_0)^2 + \frac{S}{g_0 - 1} f_{2e}(\phi), \quad \phi_{1e} \leq \phi \leq 2\phi_0 - \phi_{1e} \\
C_3, \quad \phi_1 \leq \phi \leq \phi_{1e} 
\end{cases} 
\end{align*}  \hspace{1cm} (5.28)
where \( C_1, C_2 \) and \( C_3 \) are integration constants and

\[
f_{2e}(\phi) = \frac{1}{2}(\phi - \phi_0)^2 + \frac{1}{2}(\phi_0 - \phi_{1e})^2 \left[ 1 - \frac{\phi - \phi_0}{\phi_0 - \phi_{1e}} \right]^2 \ln \left[ 1 - \frac{\phi - \phi_0}{\phi_0 - \phi_{1e}} \right] \]

From \( \dot{\phi}(\phi_1) = 0, \dot{\phi}(\phi_2) = 0 \), and the continuity of \( \dot{\phi} \) at \( \phi = \phi_{1e} \) and \( \phi = 2\phi_0 - \phi_{1e} \), it is straightforward to deduce the normalized trajectory tangent to the charge envelope curve at \( \phi_1 \):

\[
y_{ne}^2 = \cos \phi - \cos \phi_1 + (\phi - \phi_1) \sin \phi_0 + \]

\[
\frac{S_c g_o}{2(g_o - 1)} \begin{cases}
0 & , 2 \phi_0 - \phi_{1e} \leq \phi \leq \phi_2 \\
\frac{(\phi - \phi_0)^2 - (\phi_{1e} - \phi_0)^2}{2} + \frac{1}{g_o - 1} f_{3e}(\phi), \phi_{1e} \leq \phi \leq 2 \phi_0 - \phi_{1e} \\
0 & , \phi_1 \leq \phi \leq \phi_{1e}
\end{cases} \quad (5.29)
\]

where

\[
f_{3e}(\phi) = (\phi_0 - \phi_{1e})^2 \left[ 1 - \frac{\phi - \phi_0}{\phi_0 - \phi_{1e}} \right]^2 \ln \left[ 1 - \frac{\phi - \phi_0}{\phi_0 - \phi_{1e}} \right] \]

The Eqs. (5.23), (5.24) and (5.25) remain formally unchanged, except that, whenever necessary, \( \phi_2 \) should be replaced by \( \phi_{2e} \), the root of

\[
\cos \phi_{2e} - \cos \phi_{1e} + (\phi_{2e} - \phi_{1e}) \sin \phi_0 = 0 . \quad (5.30)
\]

The preceding analysis can easily be modified to apply above transition. Since the capacitance formula (1.6) was used to calculate the space charge field \( E_{g0} \), the results are again electrostatically inconsistent.

In order to test the validity of the capacitance formula (1.6), the field \( E_{g0} \) will next be calculated without the use of any simplifying assumptions.
6. ELLIPSOIDAL DISTRIBUTION (solution of Poisson's equation)

6.1 Beam without a vacuum chamber

Because of a multiplicity of symbols, their meanings will be defined at each step, independently from the previous sections. Let the surface of the ellipsoid be given by

\[ \frac{\xi^2}{a^2} + \frac{\phi^2}{b^2} + \frac{\eta^2}{c_0^2} = 1, \quad (6.1) \]

and let \( P(x,z,s) \) be a point inside the ellipsoid at which the potential \( V_P \) and the field \( \mathbf{E}_P \) are to be calculated. Consider \( P \) as the origin of polar coordinates \((r, \phi, \theta)\) with

\[ \xi = x + r \cos \phi \cos \theta, \quad \zeta = z + r \sin \phi, \quad \eta = s + r \cos \phi \sin \theta, \quad dq = r^2 \cos \phi \, d\phi \, d\theta \]

where \( \rho = \text{const.} \) is the volume charge density. The potential at \( P \) is \(^{13}\)

\[ V_P = -\frac{1}{4\pi \varepsilon_0} \int \frac{dq}{r} = W + \frac{\rho}{4\pi \varepsilon_0} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left( \frac{\cos^2 \phi \cos^2 \theta}{a^2} \frac{x^2}{a^2} + \right. \]

\[ \left. + \frac{\sin^2 \phi \, z^2}{b^2} + \frac{\cos^2 \phi \sin^2 \theta \, s^2}{c_0^2} \right) \frac{\cos \phi \, d\phi \, d\theta}{A^2} \]  \( (6.3) \)

where

\[ A = \frac{\cos^2 \phi \cos^2 \theta}{a^2} + \frac{\sin^2 \phi}{b^2} + \frac{\cos^2 \phi \sin^2 \theta}{c_0^2} \]

\[ W = \frac{\rho}{8\pi \varepsilon_0} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos \phi \, d\phi \, d\theta}{A}. \]

\( W \) is the potential at the centre of the ellipsoid. By means of elementary transformations it is possible to express Eq. (6.3) in the form \(^{13}\)
\[ V_p = \frac{\rho \, abe}{4\varepsilon_0} \int_0^\infty \left( 1 - \frac{x^2}{a^2+t} - \frac{z^2}{a^2+t} - \frac{s^2}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}} \right) \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}} \]  

(6.5)

From Eq. (6.5) it follows that the equipotential surfaces \( V = \text{const.} \) can be written

\[ W - V = \frac{\rho \, abe}{4\varepsilon_0} \left( M_x \, s^2 + M_z \, z^2 + M_s \, s^2 \right) \]  

(6.6)

where the coefficients

\[ M_x = \int_0^\infty \frac{dt}{\sqrt{(a^2+t)^3(b^2+t)(c^2+t)}} \], \quad M_z = \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)^3(c^2+t)}} \]  

(6.7)

\[ M_s = \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)^3}} \], \quad M_o = \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}} \]

are the so-called ellipsoid form-factors.

The field \( \vec{E} \) inside the ellipsoid is given by the partial derivatives of Eq. (6.6)

\[ E_x = -\frac{\rho \, abe \, M_x}{2\varepsilon_0 \, x}, \quad E_z = -\frac{\rho \, abe \, M_z}{2\varepsilon_0 \, z}, \]  

(6.8)

\[ E_s = -\frac{\rho \, abe \, M_s}{2\varepsilon_0 \, s}. \]

The charged ellipsoid of constant volume density \( \rho \) has thus the remarkable property that at any interior point \( P(x,z,s) \) the field components \( E_x, E_z \) and \( E_s \) depend solely on the coordinates \( x, z \) and \( s \), respectively.

Suppose now that the point \( P(x,z,s) \) is outside the surface (6.1), and let \( k_o = k_o(x,z,s) \) be the algebraically largest root of the cubic equation

\[ \frac{x^2}{a^2} + \frac{z^2}{b^2 + k_o} + \frac{s^2}{c^2 + k_o} = 1, \]  

(6.9)
then\(^1\)

\[
W - V = \frac{\sigma \alpha \beta \gamma}{4\varepsilon_0} \left[ M_x(k_0) x^2 + M_z(k_0) z^2 + M_s(k_0) s^2 \right]
\]  
\[ \tag{6.10} \]

and

\[
E_x = -\frac{\rho \alpha \beta \gamma M_x(k_0)}{2\varepsilon_0} x, \quad E_z = -\frac{\rho \alpha \beta \gamma M_z(k_0)}{2\varepsilon_0} z
\]
\[
E_s = -\frac{\rho \alpha \beta \gamma M_s(k_0)}{2\varepsilon_0} s
\]  
\[ \tag{6.11} \]

where the new form-factors are

\[
M_x(k_0) = \int_{k_0}^{\infty} \frac{dt}{k_0 \sqrt{(a^2+t)(b^2+t)(c_0^2+t)}}, \quad M_z(k_0) = \int_{k_0}^{\infty} \frac{dt}{k_0 \sqrt{(a^2+t)(b^2+t)(c_0^2+t)}}
\]
\[
M_s(k_0) = \int_{k_0}^{\infty} \frac{dt}{k_0 \sqrt{(a^2+t)(b^2+t)(c_0^2+t)^3}}, \quad M_{o}(k_0) = \int_{k_0}^{\infty} \frac{dt}{k_0 \sqrt{(a^2+t)(b^2+t)(c_0^2+t)}}
\]  
\[ \tag{6.12} \]

Let us now use the field (6.8) in the analysis of a hypothetical synchrotron motion, and assume again that the ellipsoid is one of revolution

\[
a = b = a_b.
\]  
\[ \tag{6.13} \]

The total charge in the ellipsoid being

\[
q = \frac{4}{3} \pi a_b^2 c_0 \rho = \frac{Ne}{h},
\]  
\[ \tag{6.14} \]

\[
E_{so} = -\frac{3Ne R M_s}{8\pi \varepsilon_0 h^2} (\phi - \phi_0).
\]  
\[ \tag{6.15} \]

Using the relativistic correction \(R \to \gamma R, c_0 \to \gamma c_0\),
\[ E_s = -\frac{3Ne \gamma R M}{8\pi \varepsilon_o \hbar^2} (\phi - \phi_o) \quad , \quad M = \int_0^\infty \frac{dt}{(a_b^2+t)(\gamma^2 c_o^2+t)^{3/2}} \quad (6.16) \]

Let \( \Delta = \sqrt{c_1^2 - a_b^2} \), then (with \( \gamma \) absorbed into \( c_1 = \gamma c_0 \))
\[ M = \Delta^{-3} (\ln \frac{c_1 + \Delta}{c_1 - \Delta} - \frac{2\Delta}{c_1}) \quad , \quad (6.17) \]

If
\[ S_c = \frac{3Ne \gamma R M}{4\varepsilon_o \hbar^2 V_{RF}} \quad , \quad (6.18) \]

the synchrotron equation, below transition and without adiabatic damping, becomes
\[ \ddot{\phi} = -\frac{\Omega_o^2}{\cos \phi_o} \left[ \sin \phi - \sin \phi_o - S_c (\phi - \phi_o) \right] \quad (6.19) \]

The abscissa \( \phi_2 \) of the saddle point is a root of
\[ \sin \phi - \sin \phi_o - S_c (\phi_2 - \phi_o) = 0 \quad , \quad \phi_o < \phi_2 < \pi \quad , \quad (6.20) \]

and the normalized separatrix is described by
\[ y_n(\phi) = \sqrt{\cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_o + \frac{S_c}{2} \left[ (\phi - \phi_o)^2 - (\phi_2 - \phi_o)^2 \right]} \quad (6.21) \]

Let \( \phi_1 \) be a root of
\[ \cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_o + \frac{S_c}{2} \left[ (\phi_1 - \phi_o)^2 - (\phi_2 - \phi_o)^2 \right] = 0 \quad , \quad -\pi < \phi_1 < \phi_0 \quad (6.22) \]

The normalized bucket height area and the number of accelerated particles becomes
\[ y_n \text{max} = y_n(\phi_o) \quad (6.23) \]

\[ (S_c)_n = \frac{\sqrt{2}}{8} \int_{\phi_1}^{\phi_2} y_n(\phi) d\phi \quad (6.24) \]

\[ N = \frac{4\varepsilon_o \hbar^2 V_{RF}}{3\varepsilon \gamma R^2 M S_c} \quad . \quad (6.25) \]
Equation (6.25) is not completely defined unless $c_0$ is first expressed as a function of $\phi_1, \phi_2$. Both Eqs. (5.16) and (5.17) are acceptable*).

For a partially filled bucket defined by $\phi_{2e}, \phi_0 < \phi_{2e} < \phi_2$, the abscissa $\phi_{1e}$ is a root of

$$\cos \phi_{1e} - \cos \phi_{2e} + (\phi_{1e} - \phi_{2e}) \sin \phi_0 + \frac{S}{2} \left[ \left( \phi_{1e} - \phi_0 \right)^2 - \left( \phi_{2e} - \phi_0 \right)^2 \right] = 0, \phi_1 < \phi_{1e} < \phi_0 \quad (6.27)$$

The expressions (6.23), (6.24) and (6.25) remain formally valid, provided $\phi_1, \phi_2$ are replaced by $\phi_{1e}, \phi_{2e}$, respectively.

Let us now take into account the field outside the bunch. Absorbing again $\gamma$ into $c_0$, i.e. replacing Eq. (5.17) by

$$c_1 = \frac{\gamma R}{h} (\phi_0 - \phi_1), \quad (6.28)$$

the synchrotron equation becomes

$$- \frac{\cos \phi_0}{\Omega_0^2} \phi = \sin \phi - \sin \phi_0 - S c M_k (\phi - \phi_0), \quad (6.29)$$

*) Equation (6.25) was used recently in the investigation of current limitations in proton linacs\textsuperscript{14)}, since in these accelerators the over-all shielding effect by the drift tubes is estimated to be relatively minor [Eq. given by Eq. (6.16) about 10 to 15\% too high\textsuperscript{15, 16}). With $\lambda = 2\pi R/(h\beta)$ for the equivalent instantaneous wavelength in the linac, the maximum linac current becomes

$$I_{\text{max}} = \frac{\beta c e}{2\pi R} N_{\text{max}} = \frac{16\pi^2 V_{\text{RF max}}}{3e \gamma \beta^2 \lambda^2 c_1 \left( \frac{c}{M} \right)}. \quad (6.26)$$

The values resulting from Eq. (6.26) so far have not been exceeded in practice\textsuperscript{15)} except in one case\textsuperscript{17}) (by about 15\%). For curiosity, we note that Eq. (6.26) would give for the CPS at 50 MeV $N_{\text{max}} = 2.5 \times 10^{12}$ with $\phi_0 = 30^\circ$ and $N_{\text{max}} = 3.6 \times 10^{12}$ with $\phi_0 = 24^\circ$, demonstrating that one relies heavily on the helpful effect of image forces in these accelerators (cf. Table 5.).
where from Eqs. (6.9) and (6.12)

\[
k(\phi) = \frac{s_1^2 - c_1^2 - a_b^2}{2} + \sqrt{\frac{s_1^2 - c_1^2 - a_b^2}{2} + a_b^2(s_1^2 - c_1^2)}
\]

\[
s_1 = \frac{\gamma R}{h} (\phi - \phi_0)
\]

\[
M_k(\phi) = \Delta^{-3} \ln \frac{\sqrt{c_1^2 + k(\phi)} + \Delta}{\sqrt{c_1^2 + k(\phi)} - \Delta} - \frac{2\Delta}{\sqrt{c_1^2 + k(\phi)}}
\]

Inside the bunch, i.e. for \( \phi_1 \leq \phi \leq 2\phi_e - \phi_1 \), \( k(\phi) \leq 0 \) and \( M_k(\phi) = M \).

The saddle point abscissa \( \phi_2 \) is a root of

\[
\sin \phi_2 - \sin \phi_o - S_c \frac{M_k(\phi_2)}{M} (\phi_2 - \phi_o) = 0
\]  

(6.31)

Integrating Eq. (6.29) and normalizing as usual yields the equation of the separatrix

\[
y_n(\phi) = \left\{ \begin{array}{l}
\cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_o + \\
\frac{S}{2M} \left[ M_k(\phi) (\phi - \phi_0)^2 - M_k(\phi_2) (\phi_2 - \phi_0)^2 \right] \end{array} \right\}^{1/2}
\]  

(6.32)

The minimal abscissa \( \phi_1 \) of the separatrix is a root of

\[
\cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_0 + \frac{S}{2} (\phi_1 - \phi_0)^2 - S_c \frac{M_k(\phi_2)}{2M} (\phi_2 - \phi_0)^2 = 0
\]  

(6.33)

For a partially filled bucket \( \phi_{1e} \) is given, and, equating \( \phi_1, \phi_2 \) with \( \phi_{1e}, \phi_{2e} \), Eq. (6.33) has to be solved for \( \phi_{2e} \). For a full bucket both \( \phi_1 \) and \( \phi_2 \) are unknown, except when \( S_c = 0 \), and the Eqs. (6.31) and (6.33) must be solved simultaneously. Numerically this is lengthy but quite straightforward.

The height and area of the charge-envelope curve (6.32) and the number of accelerated particles are still given by the expressions (6.23), (6.24) and (6.25).
Numerical results show that the edge contribution to the self-field is negligible for \( \phi_0 \gtrsim 1^\circ \) but not for \( 4^\circ < \phi_0 \leq 30^\circ \). For example, neglecting it yields about a 10% pessimistic value of \( N \) for \( S_c = 0.1 \). Values of \( \phi_0 > 30^\circ \) were not examined.

6.2 Beam inside a cylindrical vacuum chamber

Let the reference point \( P \), where the potential and the field is to be calculated, lie on a coaxial cylinder of radius \( a_v \). From Eq. (6.9)

\[
k_0(\phi) = \frac{s^2 - c_o^2 - a_b^2 - a_v^2}{2} + \sqrt{\left(\frac{s^2 - c_o^2 - a_b^2 - a_v^2}{2}\right)^2 + a_b^2 s^2 + c_o^2 (a_v^2 - a_b^2)}, \quad (6.34)
\]

and the potential on the cylinder is (in the absence of the vacuum chamber)

\[
V_p(s, a_v) = \frac{3q}{16\pi \varepsilon_0} \left[ M_o(k_o) - M_{ko}(\phi) s^2 - M_{x o}(k_o) a_v^2 \right] = f_v(s), \quad (6.35)
\]

where \( q = Ne/h \) and

\[
M_o(k_o) = \frac{1}{\Delta} \ln \frac{\sqrt{c_o^2 + k_o(\phi)} + \Delta}{\sqrt{c_o^2 + k_o(\phi)} - \Delta}, \quad (6.36)
\]

\[
M_{x o}(k_o) = \frac{1}{2\Delta^3} \ln \frac{\sqrt{c_o^2 + k_o(\phi)} - \Delta}{\sqrt{c_o^2 + k_o(\phi)} + \Delta}.
\]

Consider now the boundary value problem

\[
\Delta V(s, r) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial s^2} = 0, \quad 0 \leq r < a_v, \quad -\infty < s < +\infty
\]

\[
(6.37)
\]

\[
V(s, a_v) = f_v(s)
\]

An approximate solution of Eq. (6.37) was given previously\(^{18}\). To obtain a rigorous solution, assume a solution of Eq. (6.35) in the form
\[ V(s,r) = \int_{-\infty}^{\infty} A(\tau) I_0(\tau r) e^{-i\tau s} d\tau , \]  

(6.38)

where \( A(\tau) \) is an unknown function and \( I_0(y) = J_0(iy) \) is a Bessel function of the first kind. Representing the boundary function \( f_v(s) \) in the form of a Fourier integral

\[ f_v(s) = \int_{-\infty}^{\infty} F_v(\tau)e^{-i\tau s} d\tau , \quad F_v(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_v(s)e^{i\tau s} ds \]  

(6.39)

there results for \( r = a_v \)

\[ \int_{-\infty}^{\infty} A(\tau) I_0(a_v \tau)e^{-i\tau s} d\tau = \int_{-\infty}^{\infty} F_v(\tau)e^{-i\tau s} d\tau , \]

and hence

\[ A(\tau) = \frac{F_v(\tau)}{I_0(a_v \tau)} . \]  

(6.40)

Substituting Eq. (6.40) into Eq. (6.38) and subtracting Eq. (6.38) from Eqs. (6.6) and (6.10) yields the potential at any point inside the vacuum chamber in the presence of the uniformly charged ellipsoid:

\[ V_i(s,r) = \frac{3q}{16\pi\varepsilon_0} (M_o - M_r x - Ms) - 2 \int_{0}^{\infty} \frac{I_0(\tau r)}{I_0(a_v \tau)} F_v(\tau) \cos \tau s \, d\tau \]  

\[ V_e(s,r) = \frac{3q}{16\pi\varepsilon_0} \left[ M_o (k_r) - M_r x (k_r) r^2 - M(k_r) s^2 \right] - 2 \int_{0}^{\infty} \frac{I_0(\tau r)}{I_0(a_v \tau)} F_v(\tau) \cos \tau s \, d\tau \]  

(6.41)

where

\[ k_r(\phi) = \frac{\sqrt{s^2 - c_o^2 - a_b^2 - r^2}}{2} + \sqrt{\frac{s^2 - c_o^2 - a_b^2 - r^2}{2}} \sqrt{a_b^2 s^2 + c_o^2 (r^2 - a_b^2)} \]

\[ M_o (k_r) = M_o (k_o = k_r) , \quad M_o = M_o (k_o = 0) , \]  

\[ M_x (k_r) = M_x (k_o = k_r) , \quad M_x = M_x (k_o = 0) , \]  

(6.42)
and \( V_i(s,r) \), \( V_e(s,r) \) designate the potential inside and outside the ellipsoid, respectively.

The electric field is readily obtained from Eq. (6.41) by differentiation. For example, the longitudinal component of the electric field inside the bunch is

\[
E_s = -\frac{3qM}{8\pi \epsilon_0} s + 2 \int_0^\infty \frac{I_0(\tau \tau)}{I_0(a \tau)} F_v(\tau) s \tau d\tau. \tag{6.43}
\]

It is noteworthy that the second term in Eq. (6.43) possesses a dependence on the transversal coordinate \( r \). The vacuum chamber produces thus a coupling between longitudinal and radial motion.

Consider now the synchrotron motion in the presence of a charged ellipsoid circulating inside a perfectly conducting cylindrical vacuum chamber. For simplicity it will be assumed that \( r = 0 \). In such a case, neglecting adiabatic damping as usual,

\[
2\pi \frac{R^2 E_0}{\hbar c^2} \frac{\gamma^3}{\alpha \gamma^2 - 1} \frac{d^2 \phi}{dt^2} = eV_{RF} (\sin \phi - \sin \phi_0) - \frac{3q\gamma R^2 e}{4\epsilon_0 \hbar} M_k(\phi)(\phi - \phi_0) + \frac{3qRe}{8\epsilon_0} f(\phi - \phi_0) \tag{6.44}
\]

where

\[
f(\phi) = \frac{16\pi \epsilon_0}{3q} 2 \int_0^\infty \frac{\tau F_v(\tau)}{I_0(a \tau)} \sin \frac{\gamma R \phi}{\hbar} \tau d\tau. \tag{6.45}
\]

We examine first the effect of a linearized space charge force, i.e. we approximate \( f(\phi - \phi_0) \) by its first McLaurin term:

\[
\frac{h}{2\gamma R} f(\phi - \phi_0) \propto M_v(\phi - \phi_0) \tag{6.46}
\]

Differentiating Eq. (6.45) with respect to \( \phi \) leads to

\[
M_v = \frac{8\pi \epsilon_0}{3q} 2 \int_0^\infty \frac{\tau^2}{I_0(a \tau)} F_v(\tau) d\tau. \tag{6.47}
\]

The synchrotron Eq. (6.44) simplifies thus into
\[- \frac{\cos \phi_o}{\Omega_o^2} \frac{d^2\phi}{dt^2} = \sin \phi - \sin \phi_o - \frac{3q \gamma R^2}{4V_{RF} c_o \hbar} \left[ M_k(\phi) - M_v \right] (\phi - \phi_o) \quad (6.48)\]

Comparing Eq. (6.48) to Eqs. (6.29) and (6.18) one notices that these equations differ only by the definition of the ellipsoid form factor. Replacing Eq. (6.18) by

\[ S_c = \frac{3Ne \gamma R^2 (M - M_v)}{4 c_o \hbar^2 V_{RF}} \quad (6.49) \]

Equation (6.48) can be written in the form

\[- \frac{\cos \phi_o}{\Omega_o^2} \frac{d^2\phi}{dt^2} = \sin \phi - \sin \phi_o - S_c \frac{M_k(\phi) - M_v}{M - M_v} (\phi - \phi_o) \quad (6.50)\]

which integrates into

\[ y_n(\phi) = \left\{ \begin{array}{l}
\cos \phi - \cos \phi_2 + (\phi - \phi_2) \sin \phi_o + \\
+ \frac{S_c}{2(M - M_v)} \left[ (M_k(\phi) - M_v)(\phi - \phi_o)^2 - (M_k(\phi_2) - M_v)(\phi_2 - \phi_o)^2 \right] \end{array} \right\}^{\frac{1}{2}} \quad (6.51)\]

where \( \phi_2 \), the abscissa of the saddle point, is a root of

\[(M - M_v)(\sin \phi_2 - \sin \phi_o) - S_c \left[ M_k(\phi_2) - M_v \right] (\phi_2 - \phi_0) = 0 \quad (6.52)\]

The minimal abscissa of the separatrix \( \phi_1 \) is a root of

\[(M - M_v)\left[ \cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2) \sin \phi_o \right] + \\
+ \frac{S_c}{2} \left[ (M - M_v)(\phi_1 - \phi_o)^2 - (M_k(\phi_2) - M_v)(\phi_2 - \phi_o)^2 \right] = 0 \quad (6.53)\]

In the case of a full bucket, Eqs. (6.52) and (6.53) must be solved simultaneously, because \( M, M_k(\phi_2) \) and \( M_v \) contain \( \phi_1 \). If the bunch occupies only a part of the bucket, only Eq. (6.53) need be solved. Knowing \( \phi_1, \phi_2 \) (or \( \phi_{1e}, \phi_{2e} \)), \( M \) and \( M_v \) are completely defined and the number of accelerated particles is then

\[ N = \frac{4c_o \hbar^2 V_{RF}}{3Ne R (M - M_v)} S_c \quad (6.54)\]
Comparing the computed values of $N$ by means of Eqs. (6.54) and (5.25) it is found that they agree very closely (within 1%) for the same values of $\phi_1$ (or $\phi_{1e}$). The synchrotron Eq. (5.10), based on the capacitance formula (1.6), is thus a very good approximation of the far more complex Eq. (6.48), at least for the bunch lengths encountered in the PSB and the CPS below transition.

Consider finally the synchrotron Eq. (6.44) containing the full nonlinear space charge field. Using again Eq. (6.49) its first integral is

$$\cos \phi_0 \frac{2 \sin^2 \phi}{2 \sin^2 \phi_0} = \cos \phi + \phi \sin \phi_0 +$$

$$+ \frac{S_c}{c} \left[ M_k(\phi)(\phi - \phi_0)^2 + \left( \frac{\hbar}{\gamma R} \right)^2 f_1(\phi - \phi_0) \right] + C,$$

where $C$ is an integration constant and

$$f_1(\phi) = \frac{16\pi}{3q} c_0 2 \int_0^\infty \frac{F_v(\tau)}{I_0(\alpha_v \tau)} \cos \frac{\gamma R \phi}{h} \tau \, d\tau.$$  \hfill (6.56)

The separatrix passing through the abscissae $\phi_1, \phi_2$ is defined by the roots of the two simultaneous equations

$$(M - M_v)(\sin \phi_2 - \sin \phi_0) - S_c \left[ M_k(\phi_2)(\phi_2 - \phi_0) - \frac{\hbar}{2 \gamma R} f(\phi_2 - \phi_0) \right] = 0, \hfill (6.57)$$

$$(M - M_v)\left[ \cos \phi_1 - \cos \phi_2 + (\phi_1 - \phi_2)\sin \phi_0 \right] +$$

$$+ \frac{S_c}{2} \left[ M_k(\phi_1 - \phi_0)^2 - M_k(\phi_2 - \phi_0)^2 + \left( \frac{\hbar}{\gamma R} \right)^2 \left( f_1(\phi_1 - \phi_0) - f_1(\phi_2 - \phi_0) \right) \right] = 0 \hfill (6.58)$$

In the case of a partially filled bucket only Eq. (6.58) need be solved for $\phi_{1e}$. The normalized separatrix is described by

$$y_n(\phi) = \cos \phi - \cos \phi_2 + (\phi - \phi_2)\sin \phi_0 + \frac{S_c}{2(M - M_v)} \left[ M_k(\phi)(\phi - \phi_0)^2 -$$

$$- M_k(\phi_2)(\phi_2 - \phi_0) + \left( \frac{\hbar}{\gamma R} \right)^2 \left( f_1(\phi - \phi_0) - f_1(\phi_2 - \phi_0) \right) \right].$$ \hfill (6.59)
For Eq. (6.59) one still has the expressions (5.23) and (5.24). Instead of Eq. (5.25) one must of course use Eq. (6.54). A method to compute \( M_v \), \( f(\phi) \) and \( f_1(\phi) \) is described in Appendix 2. The main difference between Eqs. (6.59) and (6.51) lies in the fact that they involve values of \( \phi_1, \phi_2 \) which depend differently on the space charge force.

No systematic study of Eqs. (6.44) and (6.55) was made, because the necessary computer time becomes prohibitive even on the CDC 6600 but, from the dozen or so sample cases examined, it appears that the numerical results are the same as with the capacitance formula (1.6) for values of \( N \) which differ not more than 15\%. Depending on the particular accelerator parameters, the results with Eq. (1.6) were either too pessimistic or too optimistic.

7. NUMERICAL RESULTS

Computer programs are available for computing results for any set of accelerator parameters. Because of their immediate interest, we concern ourselves here with

i) the PSB (at 50 MeV and 800 MeV) and

ii) the CPS (at various energies).

7.1 Bunch area and bunching factor

7.1.1 PSB

Typical curves are shown in Fig. 2 for the bunch area and in Fig. 3 for the bunch height, in both cases as a function of the number of accelerated particles \( N \).

In these graphs and in the tables below, area and height reduction are defined by the ratios

\[
\frac{\alpha(S)N}{\alpha(0)} \text{nom} \quad \text{and} \quad \frac{y_{\text{max}}(N = \text{nom})}{y_{\text{max}}(N = 0)}
\]  

(7.1)

As the ellipsoidal distribution was shown to give essentially the same results as the parabolic one (cf. Section 6), only the results for the first three distributions are plotted.
The main effect of space charge consists in reducing the area enclosed by the charge envelope curve below transition (and increasing it above transition). The amount of reduction (increase) is roughly the same for the four distributions considered, but there exists an essential difference between the $\sigma = \text{const.}$ in $(\phi, \dot{\phi})$ space and $\cos^2$ distributions on the one hand, and the parabolic and ellipsoidal (with edges taken into account) on the other. For the former the width of the charge envelope curve (bunch length) is constant, whereas for the latter it decreases below transition and increases above transition as the number of accelerated particles $N$ increases. In the case of an accelerating bucket below transition there occurs for small $N$ and an ellipsoidal distribution a slight increase of the width of the charge envelope curve resulting from the addition of the assumed symmetrical space charge force to an unsymmetrical RF force. For increased $N$ this anomaly disappears.

For the nominal intensity of $2.5 \times 10^{12}$ protons/ring the computed PSB bunching factors are as follows: $(g_0 \ 50 \text{ MeV} = 2.9, \ g_0 \ 800 \text{ MeV} = 4.3)$

<table>
<thead>
<tr>
<th>$\sigma = \text{const. in } (\phi, \dot{\phi}) \text{ space}$</th>
<th>$50 \text{ MeV, } 12 \text{ kV}$</th>
<th>$800 \text{ MeV, } 12 \text{ kV}$</th>
<th>$800 \text{ MeV, } 3 \text{ kV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_0 = 4.8^\circ$</td>
<td>$0.518$</td>
<td>$0.300$</td>
<td>$0.415$</td>
</tr>
<tr>
<td>$\phi_0 = 4.8^\circ$</td>
<td>$0.528$</td>
<td>$0.272$</td>
<td>$0.411$</td>
</tr>
<tr>
<td>$\phi_0 = 0.1^\circ$</td>
<td>$0.414$</td>
<td>$0.204$</td>
<td>$0.304$</td>
</tr>
</tbody>
</table>

It is seen that in general, the value of $B$ is more model dependent than charge envelope areas and heights. This is due to the rather strong dependence of the central (maximum) charge density on the shape of the charge envelope curve, for a fixed number of accelerated particles, and fixed initial bunch length.
7.1.2 CPS

We turn now to the improved CPS at 800 MeV during the PSB-CPS transfer. In the 20-bunch mode without adiabatic lengthening the CPS parameters are roughly as follows:

\[ N = 10^{13}, \quad \phi_0 = 26^\circ, \quad V = 40 \text{ kV}, \quad a_b = 0.018 \text{ m}, \quad a_v = 0.05 \text{ m}, \quad \Delta \phi = 147^\circ. \]

With these parameters we have the values given in Table 2.

Table 2

<table>
<thead>
<tr>
<th></th>
<th>Area reduction</th>
<th>Height reduction</th>
<th>Bunching factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</td>
<td>0.718</td>
<td>0.768</td>
<td>0.281</td>
</tr>
<tr>
<td>parabolic</td>
<td>0.757</td>
<td>0.774</td>
<td>0.272</td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>0.774</td>
<td>0.683</td>
<td>0.204</td>
</tr>
</tbody>
</table>

In the 20-bunch mode with adiabating bunch lengthening to $220^\circ$, $\phi_0 = 0.1^\circ$, $V = 10$ kV and the other parameters unchanged, we have the values given in Table 3.

Table 3

<table>
<thead>
<tr>
<th></th>
<th>Area reduction</th>
<th>Height reduction</th>
<th>Bunching factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</td>
<td>0.683</td>
<td>0.736</td>
<td>0.421</td>
</tr>
<tr>
<td>parabolic</td>
<td>0.692</td>
<td>0.725</td>
<td>0.406</td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>0.717</td>
<td>0.607</td>
<td>0.304</td>
</tr>
</tbody>
</table>

Let us finally examine the effect of space charge in the present CPS. For convenience, some slightly old parameter values will be used in order
to allow a comparison with the zero-intensity results reported elsewhere\textsuperscript{19}). These particular values are $\phi_0 = 30^\circ$ and $V = 130$ kV, during the whole accelerating cycle. For $N = 10^{12}$ the CPS results are summarized in Table 4.

Table 4

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Bunch length</th>
<th>$\sigma = \text{const. in (} \phi, \dot{\phi} \text{)}$ space</th>
<th>Parabolic</th>
<th>$\cos^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Area reduction</td>
<td>Height reduction</td>
<td>Area reduction</td>
</tr>
<tr>
<td>1.053</td>
<td>Full bucket</td>
<td>0.987</td>
<td>0.978</td>
<td>0.987</td>
</tr>
<tr>
<td>5.6</td>
<td>$33^\circ$</td>
<td>0.902</td>
<td>0.922</td>
<td>0.915</td>
</tr>
<tr>
<td>7</td>
<td>$29^\circ$</td>
<td>1.08</td>
<td>1.07</td>
<td>1.08</td>
</tr>
<tr>
<td></td>
<td>$54^\circ$</td>
<td>1.01</td>
<td>1.01</td>
<td>1.01</td>
</tr>
<tr>
<td>12</td>
<td>$24^\circ$</td>
<td>1.06</td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td>$84^\circ$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Above transition the shorter bunch lengths correspond to an ideal low of adiabatic damping, whereas the longer ones correspond to actual observations\textsuperscript{19}). The distribution $\sigma = \text{const. in (} \phi, \dot{\phi} \text{)}$ space was extended above transition by means of Eq. (2.41) with $m = 10$, $m_1 = m_2 = 0$ in Eq. (2.43). From Table 4 it is apparent that the observed blow-up tends to reduce the area and height difference between $\gamma = 5.6$ and $\gamma = 7$ (which are roughly equidistant from $\gamma$ transition). Below transition the observed bunch shapes have well defined sharp edges\textsuperscript{20}) and they resemble thus the $\sigma = \text{const. in (} \phi, \dot{\phi} \text{)}$ space or the parabolic distribution. Above transition the bunch shape is not completely fixed, but it has rounded edges and resembles more the $\cos^2$ distribution\textsuperscript{21}).

For the present intensity of $N = 2 \times 10^{12}$, if the operating conditions remained constant, the CPS data would be as given in Table 5.
Table 5
CPS values for various energies and $N = 2 \times 10^{12}$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Bunch length</th>
<th>$\sigma = \text{const. in} (\phi, \Phi)$ space</th>
<th>Parabolic</th>
<th>$\cos^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Area reduction</td>
<td>Height reduction</td>
<td>Area reduction</td>
</tr>
<tr>
<td>1.053</td>
<td>Full bucket</td>
<td>0.977</td>
<td>0.983</td>
<td>0.977</td>
</tr>
<tr>
<td>5.6</td>
<td>33$^\circ$</td>
<td>0.81</td>
<td>0.835</td>
<td>0.825</td>
</tr>
<tr>
<td>7</td>
<td>29$^\circ$</td>
<td>1.16</td>
<td>1.13</td>
<td>1.15</td>
</tr>
<tr>
<td></td>
<td>54$^\circ$</td>
<td>1.02</td>
<td>1.02</td>
<td>1.02</td>
</tr>
<tr>
<td>12</td>
<td>24$^\circ$</td>
<td>1.11</td>
<td>1.09</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td>84$^\circ$</td>
<td>1.01</td>
<td>1.01</td>
<td>1.00</td>
</tr>
</tbody>
</table>

From Table 5 it follows that the difference between $\gamma = 5.6$ and $\gamma = 7$ increases considerably for $N = 2 \times 10^{12}$, regardless of the assumed charge distribution.

The calculated CPS bunching factors are given in Table 6.

Table 6
CPS bunching factors

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Bunch length</th>
<th>$\sigma = \text{const. in} (\phi, \Phi)$ space</th>
<th>Parabolic</th>
<th>$\cos^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N = 10^{12}$</td>
<td>$N = 2 \times 10^{12}$</td>
<td>$N = 10^{12}$</td>
<td>$N = 2 \times 10^{12}$</td>
</tr>
<tr>
<td>1.053</td>
<td>full bucket</td>
<td>0.36</td>
<td>0.36</td>
<td>0.35</td>
</tr>
<tr>
<td>5.6</td>
<td>33$^\circ$</td>
<td>0.071</td>
<td>0.069</td>
<td>0.61</td>
</tr>
<tr>
<td>7</td>
<td>29$^\circ$</td>
<td>0.063</td>
<td>0.064</td>
<td>0.054</td>
</tr>
<tr>
<td></td>
<td>54$^\circ$</td>
<td>0.12</td>
<td>0.12</td>
<td>0.10</td>
</tr>
<tr>
<td>12</td>
<td>24$^\circ$</td>
<td>0.052</td>
<td>0.055</td>
<td>0.044</td>
</tr>
<tr>
<td></td>
<td>84$^\circ$</td>
<td>0.18</td>
<td>0.18</td>
<td>0.16</td>
</tr>
</tbody>
</table>
An inspection of Table 6 leads again to the conclusion that the values of the bunching factors discriminate more strongly between classes of charge distribution, i.e. $\sigma = \text{const. in } (\phi, \ddot{\phi})$ space and parabolic against $\cos^2$, than do the values of charge-envelope curve area and height reductions. Bunching factors are, however, little dependent on beam intensity as apparent from Eqs. (2.32), (3.19) and (4.19).

7.2 Synchrotron frequency

For charge distributions with a single maximum at $\phi = \phi_0$ the main effect of space charge consists in lowering the synchrotron frequency below transition and raising it above transition. The change of the small amplitude synchrotron frequency with the number of accelerated particles is roughly proportional to the curvature of the charge density at $\phi = \phi_0$ (bunch centre). These frequency charges are given in terms of the ratio

$$\left[ \frac{T_i(N = N_{\text{nom}})}{T_i(N = 0)} \right]^{-1},$$

(7.3)

normalized by the zero-intensity values (2.13$^{18}$). "Small amplitude" is defined as

$$\phi_{2i} = \phi_0 + 1^0, \ \alpha \gamma^2 < 1 \ \text{and} \ \phi_{2i} = \phi_0 - 1^0, \ \alpha \gamma^2 > 1^0.$$  \hspace{1cm} (7.2)

For the PSB the following values were obtained ($N = 2.5 \times 10^{12}/\text{ring}$):

Table 7

<table>
<thead>
<tr>
<th>Distributions</th>
<th>Energy</th>
<th>50 MeV, 12 kV $\phi_0 = 4.8^0$</th>
<th>800 MeV, 12 kV $\phi_0 = 4.8^0$</th>
<th>800 MeV, 3 kV $\phi_0 = 0.1^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \text{const. in } (\phi, \ddot{\phi})$ space</td>
<td>0.954</td>
<td>0.890</td>
<td>0.796</td>
<td></td>
</tr>
<tr>
<td>Parabolic</td>
<td>0.974</td>
<td>0.815</td>
<td>0.775</td>
<td></td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>0.838</td>
<td>&lt; 0.2</td>
<td>&lt; 0.1</td>
<td></td>
</tr>
</tbody>
</table>

The small-amplitude synchrotron frequency is thus a very sensitive indicator of the charge distribution shape near $\phi = \phi_0$. 
For the improved CPS at 800 MeV and $N = 10^{13}$, the small amplitude synchrotron frequencies are given in Table 8.

<table>
<thead>
<tr>
<th>RF parameters</th>
<th>$\phi_0 = 26^\circ$ 40 kV</th>
<th>$\phi_0 = 0.1^\circ$ 10 kV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</td>
<td>0.862</td>
<td>0.84</td>
</tr>
<tr>
<td>Parabolic</td>
<td>0.816</td>
<td>0.81</td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>&lt; 0.2</td>
<td>&lt; 0.1</td>
</tr>
</tbody>
</table>

Again, the spread between the values for the different distributions is considerable. A similar situation exists for the present CPS ($\phi_0 = 30^\circ$, 130 kV) (see Table 9).

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Bunch length</th>
<th>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</th>
<th>Parabolic</th>
<th>$\cos^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N = 10^{12}$</td>
<td>$N = 2 \times 10^{12}$</td>
<td>$N = 10^{12}$</td>
<td>$N = 2 \times 10^{12}$</td>
</tr>
<tr>
<td>1.053</td>
<td>Full bucket</td>
<td>0.996</td>
<td>0.992</td>
<td>0.994</td>
</tr>
<tr>
<td>5.6</td>
<td>33°</td>
<td>0.96</td>
<td>0.90</td>
<td>0.92</td>
</tr>
<tr>
<td>7</td>
<td>29°</td>
<td>1.03</td>
<td>1.06</td>
<td>1.08</td>
</tr>
<tr>
<td></td>
<td>54°</td>
<td>1.00</td>
<td>1.01</td>
<td>1.01</td>
</tr>
<tr>
<td>12</td>
<td>24°</td>
<td>1.03</td>
<td>1.04</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>84°</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
The blow-up above transition reduces considerably the small-amplitude synchrotron frequency shift, especially for the parabolic and the \( \cos^2 \) distribution.

It was pointed out recently that the "natural" bunch density oscillation, observed by means of a wide band pick-up electrode, is not produced by the central part of the bunch \( (\psi \approx \phi_0) \) as previously assumed \(^{22,23}\), but rather by irregularities on the bunch periphery \(^{24}\), when the latter is inscribed on the phase plane trajectories. This means that the observed frequency of bunch density oscillations should not be compared to a small-amplitude calculated synchrotron frequency, but rather to one of "large" amplitude.

For a rectangular distribution, the "large" amplitude corresponds, of course, to the bunch edges, but the precise definition of the word "large" is not quite clear for the distributions used in this paper. It was suggested that for a Gaussian-like distribution the "large" amplitude to be used is defined by the point of (absolutely) maximum slope \(^{25}\). For example, for the symmetrical \( \cos^2 \) distribution

\[
\lambda(s) = A \cos^2 \frac{\pi}{2} \frac{s}{s_0}, \quad |s| < s_0; \quad \lambda(s) = 0, \quad |s| > s_0, \quad (7.4)
\]

this would be the amplitude \( s_a = \frac{1}{2} s_0 \). For a non-symmetrical \( \cos^2 \) distribution we will use

\[
s_a = \frac{\gamma R}{h} \left[ \phi_0 + \frac{1}{4} (\phi_{2e} - \phi_{1e}) \right], \quad (7.5)
\]

and for distributions having a monotonically varying slope, with a discontinuity at the bunch edges, we will use

\[
s_a = \frac{\gamma R}{h} \left[ \phi_0 + \frac{1}{3} (\phi_{2e} - \phi_{1e}) \right]. \quad (7.6)
\]

Adopting the above definitions of the large amplitude, the PSB synchrotron frequency shifts are given in Table 10.
Table 10
PSB relative large-amplitude synchrotron frequency

<table>
<thead>
<tr>
<th>Distributions</th>
<th>Energy</th>
<th>50 MeV, 12 kV</th>
<th>800 MeV, 12 kV</th>
<th>800 MeV, 3 kV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_0 = 4.8^\circ$</td>
<td>$\phi_0 = 4.8^\circ$</td>
<td>$\phi_0 = 0.1^\circ$</td>
<td></td>
</tr>
<tr>
<td>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</td>
<td>0.934</td>
<td>0.843</td>
<td>0.722</td>
<td></td>
</tr>
<tr>
<td>Parabolic</td>
<td>0.901</td>
<td>0.792</td>
<td>0.697</td>
<td></td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>0.8770</td>
<td>0.558</td>
<td>0.242</td>
<td></td>
</tr>
<tr>
<td>Frequency shift with respect to small amplitudes for $N = 0$</td>
<td>0.820</td>
<td>0.955</td>
<td>0.885</td>
<td></td>
</tr>
</tbody>
</table>

The results are quite similar for the PSB–CPS transfer at 800 MeV and are given in Table 11.

Table 11
CPS relative large-amplitude synchrotron frequency at 800 MeV and $N = 10^{13}$

<table>
<thead>
<tr>
<th>RF parameters</th>
<th>$\phi_0 = 26^\circ, 40$ kV</th>
<th>$\phi_0 = 0.1^\circ, 10$ kV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = \text{const. in } (\phi, \dot{\phi})$ space</td>
<td>0.836</td>
<td>0.778</td>
</tr>
<tr>
<td>Parabolic</td>
<td>0.806</td>
<td>0.745</td>
</tr>
<tr>
<td>$\cos^2$</td>
<td>0.587</td>
<td>0.527</td>
</tr>
<tr>
<td>Frequency shift with respect to small amplitudes for $N = 0$</td>
<td>0.935</td>
<td>0.888</td>
</tr>
</tbody>
</table>

Comparing Table 10 to Table 7 and Table 11 to Table 9 it can be seen that the frequency shift for the $\cos^2$ distribution is considerably smaller at large amplitudes.
<table>
<thead>
<tr>
<th>Y</th>
<th>Bunch length</th>
<th>$\sigma = \text{const. in } (\phi, \phi)$ space</th>
<th>Parabolic $\cos^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.053</td>
<td>Full bucket</td>
<td>0.896 0.994</td>
<td>0.998 0.988</td>
</tr>
<tr>
<td>5.6</td>
<td>33°</td>
<td>0.99 0.957</td>
<td>1.04 1.00</td>
</tr>
<tr>
<td>7</td>
<td>29°</td>
<td>0.99 0.97</td>
<td>1.08 1.01</td>
</tr>
<tr>
<td>12</td>
<td>44°</td>
<td>0.99 0.97</td>
<td>1.05 1.02</td>
</tr>
</tbody>
</table>

*) Frequency shift with respect to small amplitudes for $N = 0$. 

Note: Table 12: CPS relative large-amplitude synchrotron frequency at various energies.
For the present CPS ($\phi_0 = 30^\circ, 130$ kV), the corresponding values are given in Table 12.

As in the case of the small amplitudes, the blow-up at transition reduces the synchrotron frequency shift.

The latest measured CPS values\(^{26}\) are given in Table 13.

**Table 13**

CPS "synchrotron frequencies" for $N = 1.8 \times 10^{12}$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$f_{\text{meas}}$ [Hz]</th>
<th>$f_{\text{calcu}}(N = 0)$ [Hz]</th>
<th>$f_{\text{rel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.518</td>
<td>1120</td>
<td>1240</td>
<td>0.907</td>
</tr>
<tr>
<td>5.740</td>
<td>217.5</td>
<td>239</td>
<td>0.909</td>
</tr>
<tr>
<td>6.963</td>
<td>257.5</td>
<td>284</td>
<td>0.906</td>
</tr>
</tbody>
</table>

Thus, the increase of $f_{\text{rel}}$ expected after transition according to Table 12 has not been observed so far. Unfortunately, a comparison of the values of Table 12 with measured values is affected by the uncertainties of the space charge model (cf. Section 6) and of the meaning\(^*\) of the measured values\(^{22}\). A comparison for reduced RF voltages (on a "flat top") would be even more interesting, because of the relative increase of the space charge forces.

It should also be noted that the values of Table 13 refer to the blow-up bunches. As the increase in $f_{\text{rel}}$ is expected to be substantially larger for the "ideally damped" bunches, it would be interesting to measure the synchrotron frequencies with the Q-jump applied, which strongly reduces bunch blow-up\(^{27}\). In this condition, measurements of synchrotron frequencies above transition (say up to 12 GeV/c or so) would appear to be the most promising tool for studying the charge distribution of the CPS beam. If this distribution were of the $\cos^2$ type, for which there exists a certain likelihood\(^{21}\), excitation and measurement of small-amplitude oscillations would be particularly interesting.

\(^*\) If the observed density oscillations correspond to bunch envelope oscillations (rather than to individual particle oscillations as computed here), the observed frequency shift should be a quarter of the computed shift\(^5,\(^{25}\)).
8. CONCLUSIONS

A comparison of the complete solution of Poisson’s equation for an ellipsoidal charge distribution inside a coaxial circular lossless vacuum chamber [Eqs. (6.44) and (6.55)], with the capacitance formula (1.6), shows that both formulae give the same results for a number of accelerated particles N differing by up to about 15% depending on bunch length and diameter (relative to the chamber diameter). However, as the dependence on N of the quantities considered is not very strong, this result constitutes a justification of the capacitance formula, which was therefore used throughout. Also, as a constant volume density in an ellipsoid gives a parabolic line charge density, numerical results are presented only for the distributions $\sigma = \text{const.}$ in $(\phi, \phi)$ space, and for the parabolic and $\cos^2$ distributions in geometric space.

The reduction (below transition) or increase (above transition) of the bucket (bunch) area and height turns out to be relatively model independent. For the CPS, working at 800 MeV with $V_{RF} = 40$ kV, this reduction reaches roughly 25% for $N = 10^{13}$. At present intensities the changes remain below 10% at all energies. For the PSB, at $2.5 \times 10^{12}$ (per ring), the bucket height reductions come out similarly.

The bunching factor $B$ is more strongly model dependent but weakly intensity dependent. $\sigma = \text{const.}$ in $(\phi, \phi)$ space gives the largest values, a $\cos^2$ distribution the smallest values of $B$ (25% smaller).

The synchrotron frequencies are both model-dependent and intensity-dependent. They should thus provide the most promising tool for studying the charge distribution of a real beam.

Acknowledgements

This work benefited from several discussions with H.G. Hereward. H. Koziol and F. Sacherer made useful comments on parts of the manuscript. The numerical calculations were programmed and carried out by F. Guidici.

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P. Lapostolle
W. Schnell
DERIVATION OF THE CAPACITANCE FORMULA (1.6)

Consider a cylindrical beam of radius a located inside a coaxial, grounded lossless metal cylinder of radius b. Let the beam have a constant volume density \( \rho \). To find the potential \( V(r) \) inside the beam, the boundary-value problem to be solved is

\[
\begin{align*}
\frac{1}{r} \frac{d}{dr} \left( r \frac{dV}{dr} \right) &= 0, \quad a \leq r < b, \\
V(b) &= 0, \\
\left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dV}{dr} \right) = \frac{\rho}{\varepsilon_o} \right), \quad 0 \leq r \leq a, \quad |V(r)| < \infty
\end{align*}
\] (A.1)

The solution of Eqs. (A.1) is of the form

\[
V(r) = C_1 \ln r + C_2 + \frac{\rho}{4\varepsilon_o} r^2.
\] (A.2)

From Eq. (A.2) and the boundary conditions in Eqs. (A.1) it follows that

\[
V(r) = \begin{cases} 
\frac{\rho}{4\varepsilon_o} r^2 - \frac{\rho a^2}{4\varepsilon_o} \left( 1 + 2 \ln \frac{b}{a} \right), & 0 \leq r \leq a \\
\frac{\rho a^2}{2\varepsilon_o} \ln \frac{r}{b}, & a \leq r \leq b
\end{cases}
\] (A.3)

For a point on the beam axis one has thus

\[
V(0) = -\frac{\rho a^2}{4\varepsilon_o} \left( 1 + 2 \ln \frac{b}{a} \right),
\] (A.4)

which coincides with Eq. (1.6) if one sets

\[
\lambda = -\rho \pi a^2.
\]

Modified forms of Eq. (A.4) can be obtained by suitably averaging Eq. (A.3) in the interval \( 0 \leq r \leq a \). For example

\[
\overline{V}_x = \frac{1}{a} \int_0^a V(r) dr = -\frac{\rho a^2}{4\varepsilon_o} \left( \frac{2}{3} + 2 \ln \frac{b}{a} \right)
\] (A.5a)

or

\[
\overline{V}_q = \frac{1}{a^2} \int_0^a V(r) dr = -\frac{\rho a^2}{4\varepsilon_o} \left( \frac{1}{2} + 2 \ln \frac{b}{a} \right)
\] (A.5b)

Alternatively, the results of Eq. (A.4) may be regarded as an "average" appropriate to a non-uniform beam density.
APPENDIX 2.

CALCULATION OF $M_V$, $f(\phi)$ AND $f_1(\phi)$

The integrals

$$F_V(\sigma) = \frac{1}{\pi} \int_0^\infty f_V(s) \cos \sigma s \, ds,$$  \hspace{1cm} (B.1)

$$M_V(r) = k \int_0^\infty \frac{I_0(\sigma \rho)}{I_0(\alpha_V \sigma)} \sigma^2 F_V(\sigma) \, d\sigma, \quad k = \frac{16\pi \varepsilon_0}{3q},$$  \hspace{1cm} (B.2)

$$f(\phi,r) = k \int_0^\infty \frac{I_0(\sigma \rho)}{I_0(\alpha_V \sigma)} \sigma F_V(\sigma) \sin \frac{\gamma R \phi}{h} \sigma \, d\sigma,$$  \hspace{1cm} (B.3)

$$f_1(\phi,r) = k \int_0^\infty \frac{I_0(\sigma \rho)}{I_0(\alpha_V \sigma)} F_V(\sigma) \cos \frac{\gamma R \phi}{h} \sigma \, d\sigma$$  \hspace{1cm} (B.4)

are difficult to calculate because the integrands are oscillating functions possessing a very large number of oscillations inside the interval of integration. Their evaluation can be simplified by taking advantage of the auxiliary functions, valid for $|y| > 0$,

$$g(y,r) = 2 \int_0^\infty \frac{I_0(\sigma \rho)}{I_0(\alpha_V \sigma)} \cos y \sigma \, d\sigma = \frac{2\pi}{\alpha_V} \sum_{n=1}^\infty \frac{J_0(\alpha_V \sigma_n)}{J_1(\alpha_V \sigma_n)} e^{-\frac{\alpha_V}{\alpha_V} |y|}, \quad r < \alpha_V$$  \hspace{1cm} (B.5)

$$g_1(y,r) = 2 \int_0^\infty \frac{I_0(\sigma \rho)}{I_0(\alpha_V \sigma)} \sin y \sigma \, d\sigma = \frac{2\pi}{\alpha_V} \sum_{n=1}^\infty \frac{\alpha_n J_0(\alpha_V \sigma_n)}{J_1(\alpha_V \sigma_n)} e^{-\frac{\alpha_n}{\alpha_V} |y|},$$  \hspace{1cm} (B.6)
where \( J_0(y), J_1(y) \) are Bessel functions of the first kind and \( \alpha_n \) are zeros of \( J_0(\alpha) = 0 \). Using integration by parts, the convolution theorem of Fourier transforms [see, for example, the book by Titchmarsh\(^{28}\)], and the auxiliary functions (B.5) and (B.6), one finds after some manipulation

\[
M_v(r) = \frac{1}{\pi} \int_0^\infty g_1(\xi, r) E_n(\xi) \, d\xi \quad (B.7)
\]

\[
f(\phi, r) = \frac{1}{\pi} \int_0^\infty g(\xi, r)[E_n(\xi - \frac{\gamma R\phi}{h}) - E_n(\xi + \frac{\gamma R\phi}{h})] \, d\xi \quad (B.8)
\]

\[
f_1(\phi, r) = \frac{1}{2\pi} \int_0^\infty g(\xi, r)[V_n(\xi - \frac{\gamma R\phi}{h}) + V_n(\xi + \frac{\gamma R\phi}{h})] \, d\xi \quad (B.9)
\]

where

\[
E_n(s) = M_{v_0}(s) s \quad (B.10)
\]

\[
V_n(s) = M_{x_0}(s) - M_{x_0}(s) a_v^2 - M_{v_0}(s) s^2.
\]

The integrals (B.7), (B.8) and (B.9) have non-oscillatory integrands.

Since the auxiliary functions \( g(y, r), g_1(y, r) \) diminish exponentially for \( x >> 1 \), Laguerre integration can be used. At least seven digit accuracy is obtained for \( r = 0, |\phi| < \pi, \gamma \leq 6, R/h = 5 \) by means of the 32 points Laguerre formula [see for example the book by Krilov and Shulgina\(^{29}\)].

The coefficients \( M_0(k_0), M_x(k_0) \) and \( M_{v_0}(s) \) are defined by Eqs. (6.30), (6.36) and (6.42). The series defining \( g(y, r) \) and \( g_1(y, r) \) converge slowly when \( |y| << 1 \), but in that case the integrals in (B.5) and (B.6) are easily evaluated by means of Laguerre integration. When \( r = 0 \) the expressions (B.7), (B.8) and (B.9) reduce to (B.2), (B.3) and (B.4), respectively.

A more detailed numerical study of the field distribution produced by a uniformly charged ellipsoid was recently presented in an SI seminar\(^{30}\).
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16) I.M. Kapchinsky, Particle dynamics in resonant linear accelerators, (Moscow, 1966).
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26) H.H. Umstätter, Synchrotron frequencies in the PS at $1.8 \times 10^{12}$ pulses beam intensity, comparison with theory, MPS/SR - Note 70-29.
30) J. Trickett, The electrostatic field distribution due to a uniformly charged ellipsoid into a coaxial metallic lossless cylinder, CERN/SI/Int. DL/70-9.
FIG. 2 - PSB: Bunch area enclosed by the normalized charge envelope curve.

\[ \Delta \phi = \phi_2 - \phi_1 \]

\[ a_v = 0.0515 \text{ m} \]
FIG. 3 - PSB: Height of the normalized charge envelope curve \[ (y_{n\text{ max}})_{\phi_s = 0^\circ} = \sqrt{2} \]

\[ \Delta \phi = \phi_2 - \phi_1 \quad a_v = 0.515 \text{ m} \]