TRANSVERSE STABILITY OF A COASTING PSB BEAM

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*) Work started as part of the preparation for PSB running-in, and reported now as further experimental verification appears to be several months away.
1. DISPERSION RELATION COEFFICIENTS (Dipole mode, $\Delta\omega = U + V + jV$)

1.1. Formulae used*) (long wave-length limit, no frequency spread)

We work with a beam of circular cross-section (radius b) circulating in a vacuum chamber of (half) height h **). For the higher mode numbers the chamber wall is rather thick. For the most critical mode $n = 5$, the skin depth (Table 1) is up to an order of magnitude larger than the chamber wall thickness, while staying an order of magnitude smaller than the wall radius.

**Table 1:** Skin depth in PSB vacuum chambers for $n = 5$

<table>
<thead>
<tr>
<th>$Q_{V}$</th>
<th>4.6</th>
<th>4.7</th>
<th>4.8</th>
<th>4.9</th>
<th>4.95</th>
<th>d (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{R-CB}$ (mm)</td>
<td>1.17</td>
<td>1.35</td>
<td>1.66</td>
<td>2.35</td>
<td>3.3</td>
<td>0.4***</td>
</tr>
<tr>
<td>$\delta_{R-CT}$ (mm)</td>
<td>0.35</td>
<td>1.0</td>
<td>1.23</td>
<td>1.75</td>
<td>2.45</td>
<td>1.5</td>
</tr>
</tbody>
</table>

$d =$ wall thickness of vacuum chamber

*) The notation of our formula collection1) and the standard PSB equipment nomenclature2) are used throughout.

**) In case of a chamber with a non-circular cross section, use of $h$ (as adopted here) instead of the "equivalent" radius3a)

\[ r_{eq} \sim \left[ 2\left( \varepsilon_1 - \varepsilon_2 \right) \right]^{-1/3} h \]

leads to values of $V$ which are up to about 20% too large.

***) The effect of the corrugations is neglected throughout, as they change4) the value of $U$ by less than 1%. Besides being due to a favourable geometry, this smallness of the effect results from the fact that the corrugations change mainly the distribution of the magnetic field5) (rather than the electric one), which anyway has a small influence at 50 MeV, the case under discussion here.
Also, on a certain fraction of the PSB circumference the outer medium is ferromagnetic (R-B, R-Q), or dielectric plus resistive or magnetic (R-U, I-KS, R-TR). Thus we need formulae for

(a) Infinite wall thickness

\[
U_{\infty}^* = -U_0 (1 - b^2/h^2)/(2\gamma^2)
\]

(1)

\[
V_{\infty} = U_0 (\beta b/h)^2 \delta/(2h)
\]

(2)

\[
= \frac{N r_{p c}}{2} \delta (\omega_o) \left[ \frac{2\pi h^3 \gamma Q_y (n - Q_y)^{1/2}}{2} \right]^{-1}
\]

(2a)

with

\[
W_0 = N r_{p c} (\pi b^2 Q_y)^{-1}
\]

(3)

and

\[
\delta = \left[ \frac{2\rho}{(\mu_0 \omega_o)} \right]^{1/2}
\]

(4)

as the skin depth, where \( w \approx (n \pm Q_y) \Omega_o \) (\( \Omega_o \) = angular revolution frequency) is the angular frequency of the perturbation of mode number \( n \).

(b) Wall thickness comparable to skin depth (Ref. 7, eq. 6.35)

\[
U = U_{\infty} - U_0 \left(\frac{\beta b}{h}\right)^2 \delta \frac{\sin q}{\cosh q - \cos q}
\]

(5)

\[
V = V_{\infty} \frac{\sinh q + \sin q}{\cosh q - \cos q}
\]

(6)

where \( q = 2d/\delta \).

*) Note that \( U_{\infty} = (Q_o - Q_{o_1}) \Omega_o \) when \( F \) is taken near a half-integral resonance (cp. Ref. 1 p. 26-27). This relation is used when comparing ranges of vertical and horizontal stability (cf. 2.1.3. and 2.2.2.).
(c) Wall thickness small compared to skin depth
(Ref. 8, eq. 2.1.)

\[ |(1+j) \mu_0 / (2h)| \gg d/h \gg \left[ \mu_0 \delta / (2c) \right]^2 \]

\[ U = \frac{W_o}{2} \left( \frac{\beta b}{h} \right)^2 \left[ \frac{D \delta^4 - \delta^2 d(h+d)}{p^2 \delta^4 + d^2(h+d)^2} - \frac{(h/b)^2 - 1}{\gamma^2 - 1} \right] \] (7)

\[ V = \frac{W_o}{2} \left( \frac{\beta b}{h} \right)^2 \frac{\delta^2 d(h+d)}{p^2 \delta^4 + d^2(h+d)^2} \] (8)

where

\[ D = \begin{cases} 
1 - \frac{(h+d)^2}{u^2} \left[ 1 - \frac{(h+d)^2}{u^2} \right]^{-1} & \text{for a perfect conductor at radius } u \\
1 & \text{for air to infinity (outside the vacuum chamber)} \\
\left[ 1 + \frac{(h+d)^2}{w^2} \right]^{-1} & \text{for a perfect magnet at radius } w 
\end{cases} \] (9)

The contributions \( U_i \) from the various elements of fractional length \( c_i = \ell_i / c_o \) are added using

\[ U_{\text{tot.}} = \sum_i c_i U_i \] (10)
In the literature\textsuperscript{9)}, normalized values of $U$ and $V$

are sometimes used:

\begin{equation}
U' = U/|\psi_p| \delta p, \quad V' = V/|\psi_p| \delta p
\end{equation}

where

\begin{equation}
\psi_p^{\pm} = (n \pm Q) d\Omega/dp \pm \Omega_0 2Q/\Delta p
\end{equation}

is the spread in the frequency of the perturbation per momentum

bite and $\delta p$ is half of the momentum spread at the half-height of

the distribution function in momentum, and + and - refer to the

"fast" and "slow" wave, respectively.

1.2. Numerical results

Using Zotter's program\textsuperscript{7}) (which computes directly $U$ and

$V$ for multi-layer circular chambers) and the input data given in

Table 2, total PSB dispersion relation coefficients were worked

out and plotted by computer as a function of the mode number $n$

(Fig. 1) and the value of $\omega_V$ (Fig. 2). In all cases one has

$U > V$, i.e., $\text{Re} \omega = U$ to a very good approximation. Whereas $U$

(which for $U > V$ determines the instability threshold) is largely

independent of $n$ and $\omega$, $V$ (the growth rate, i.e. the reciprocal of

the e-folding time) shows a more complex behaviour. It decreases

with $n$, and in general with $\omega_V$, but for $n = 5$ it increases with $Q_V$.

Thus the shortest e-folding times are to be expected for $n = 5$,

$Q_V \rightarrow 5.0$.

For $n = 5$, $\omega_V = 4.8$ and $N = 2.5 \times 10^{12}$ a few values of

$U$ and $V$ were computed by hand (Table 2), using the approximate

formulae of 1.1. As expected, $U$ (which depends mainly on the beam

radius) changes by less than a factor four from element to element,

while $V$ (which depends on the chamber radius and the electrical

wall properties) varies over a much wider range. Treating the

vacuum chamber in the bending magnets as infinitely thick leads to

a value of $V$ too low by a factor four.
Table 2: Input data for computation of $U$ and $V$ and results obtained from formulae of section 1.1 for $n = 5$, $Q_v = 4.8$

$T = 50$ MeV, $N = 2.5 \times 10^{12}$ (vertical plane)

<table>
<thead>
<tr>
<th>Item</th>
<th>Total length $L$ (m)</th>
<th>$L/C_0$</th>
<th>$h$ (mm)</th>
<th>$d$ (mm)</th>
<th>$\rho$ ($10^{-8} \Omega m$)</th>
<th>$b^*$ (mm)</th>
<th>$\delta_{n=5}$ (mm)</th>
<th>$u$ (mm)</th>
<th>$w$ (mm)</th>
<th>Eq.</th>
<th>$U$ ($10^5$ s$^{-1}$)</th>
<th>$V$ (s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-CB* (magnet chamber)</td>
<td>51.84</td>
<td>0.330</td>
<td>31</td>
<td>0.4</td>
<td>130</td>
<td>13.5</td>
<td>1.66</td>
<td></td>
<td></td>
<td></td>
<td>-4.6</td>
<td>634</td>
</tr>
<tr>
<td>R-CL (s.s. chamber)</td>
<td>35.7</td>
<td>0.227</td>
<td>60</td>
<td>1.5</td>
<td>72</td>
<td>12</td>
<td>1.23</td>
<td></td>
<td></td>
<td></td>
<td>-6.9</td>
<td>65</td>
</tr>
<tr>
<td>R-CT (triplet chamber)</td>
<td>57.76</td>
<td>0.367</td>
<td>60.5</td>
<td>1.5</td>
<td>72</td>
<td>21.5</td>
<td>1.23</td>
<td></td>
<td></td>
<td></td>
<td>-1.9</td>
<td>63</td>
</tr>
<tr>
<td>R-U (PU electr.)</td>
<td>5.5</td>
<td>0.035</td>
<td>70</td>
<td>0.055****</td>
<td>20</td>
<td>21.5</td>
<td>0.65</td>
<td>81</td>
<td></td>
<td>7:8</td>
<td>-2.0</td>
<td>210</td>
</tr>
<tr>
<td>I-KS &amp; R-TR</td>
<td>1.48</td>
<td>0.009</td>
<td>60</td>
<td>0.03****</td>
<td>$10^7$</td>
<td>16.5</td>
<td>325</td>
<td>70</td>
<td>7:8</td>
<td>-3.5</td>
<td>0.16****</td>
<td>925</td>
</tr>
</tbody>
</table>

*) Actual values $b$ (rather than $\sqrt{b(a+b)/2}$) are used for safety ($U$ too high).

**) Effect of corrugations neglected (op Table 1).

***) The effective screening by the (thin) vacuum chamber has been observed previously 10).

*****) (Rough) estimate of average thickness of conducting layer.

*******) This value is probably too low, the assumption of a lossless magnet being too coarse an approximation.

Total PSB values $\Sigma L_i U_i V_i/C_0$ -4 925
In general, the numbers obtained by hand agree satisfactorily with those resulting from the more complete computer calculation. However, one should keep in mind that these formulae cease to be valid above, say, \( n = 100 \), where the wave-length is no longer small compared to the chamber radius. (This is the reason why the curves \( U(Q_V) \) in Fig. 2 coincide for \( 5 < n < 100 \) but are different for \( n > 100 \)).

Having assumed all along that the \( V \) plane is the more dangerous one, it is necessary to work out actual numbers for the \( H \) plane. For computing \( U \) we use the equivalent radii \( \sqrt{a(a+b)}/2 \) where \( a \) and \( b \) are the beam half axes. For \( V \) there does not appear to be a ready formula which corresponds to the geometry of R-CB. To obtain a limiting value we use the formula for a circular chamber with \( R_{-\text{CB}} = 66 \) mm (which gives certainly too low a \( V \) value).

Using \( Q_H = 4.6 \) and the other data from tables 1 and 2, the result is:

\[
U_H \approx 1.5 \times 10^{-5} \text{ s}^{-1} \text{ and } V_H \approx 80 \text{ s}^{-1}
\]

i.e. lower by a factor two and eleven respectively, compared to \( U_V \) and \( V_V \) computed in the same way.

2. **LANDAU DAMPING**

The result of section 1 is that in the absence of Landau damping the nominal PSB beam will, at 50 MeV, experience a vertical dipole instability growing with an e-folding time of the order of one millisecond. As this is of the same order as the time for bunching, one does not expect any dramatic effects. However, considering that (i) some of the approximations used in the computations may be rather crude (though in general we have attempted to be conservative), (ii) some of the equipment not considered here may produce a larger effect than thought (although this may even be beneficial\(^{3b}\)), and (iii) that a bunched beam may also be unstable in the \( Q_V \) range envisaged, it was felt worthwhile to work out in advance some likely values of PSB (zeroth harmonic) sextupole and/or octupole fields. As we are not dealing with Ohm's law, these values should be considered as a general guide rather than a precise forecast!
2.1. Formulae used

We base ourselves on the work of the ISR theory group\(^9,11,12\), introducing however the betatron amplitude distribution of both transverse planes into the dispersion relation. This decreases markedly the instability region (Fig. 3(a) and (c)).

2.1.1. Dispersion relation\(^*)\)

One has:

\[
l = Q_0 \left( U + V + iV \right) \iint f(p)dp \, g^2(b) b^2 db \, h(a) ada \nonumber \]

where \[
\int_0^\infty f(p)dp = 1, \quad \int_0^\infty g(b)db = 1, \quad \text{and} \quad \int_0^\infty h(a)ada = 1
\]

are the normalised distribution functions of the particle momentum, the vertical and the horizontal betatron amplitudes respectively.

Defining \(\delta_a\) and \(\delta_b\) by \(h(\delta_a) = \frac{1}{\delta} h(\delta)\), \(g(\delta_b) = \frac{1}{b} g(b)\), the variables of interest are renormalised as follows:

\[
A = \frac{a^2}{\delta_a^2}, \quad B = \frac{b^2}{\delta_b^2}, \quad P = \frac{P - P_0}{\delta_P}
\]

giving for the distribution functions:

\[
F(P) = f(P_0 + P\delta_P), \quad \int_{-\infty}^\infty F(P)dP = 1, \quad F(\pm 1) = \frac{F(0)}{2}
\]

\[
G(B) = g(\sqrt{B}\, \delta_b) \frac{\delta_b^2}{2}, \quad \int_0^\infty G(B)dB = 1, \quad G(1) = \frac{g(0)}{2}
\]

\[
H(A) = h(\sqrt{A}\, \delta_a) \frac{\delta_a^2}{2}, \quad \int_0^\infty H(A)dA = 1, \quad H(1) = \frac{H(0)}{2}
\]

\(^*)\) All equations are written down for the vertical plane. To apply them to the horizontal plane one should use \(\delta\), \(U\) and \(V\) pertaining to the latter plane, and exchange \(a\) for \(b\) and \(A\) for \(B\).
Expanding

\[ Q = Q_0 + a^2 \frac{\partial Q}{\partial a^2} + b^2 \frac{\partial Q}{\partial b^2} + (p - p_o) \frac{\partial Q}{\partial p} \]  

(14)

\[ \Omega = \Omega_0 + a^2 \frac{\partial \Omega}{\partial a^2} + b^2 \frac{\partial \Omega}{\partial b^2} + (p - p_0) \frac{\partial \Omega}{\partial p}, \]  

(15)

and introducing the frequency spreads \( \psi^+ \) (eq. (12)),

\[ \psi^+_a = (n + Q_0) \frac{\partial \Omega}{\partial a^2} + \Omega_0 \frac{\partial Q}{\partial a^2}, \]  

(16a)

\[ \psi^+_b = (n + Q_0) \frac{\partial \Omega}{\partial b^2} + \Omega_0 \frac{\partial Q}{\partial b^2}, \]  

(16b)

the ratios \( \eta^+ = \frac{\psi^+_b}{\psi^+_p} \), \( \mu^+ = \frac{\psi^+_a}{\psi^+_p} \)

(17a)

(17b)

and the normalised frequency shift

\[ \nu^+ = \left[ \omega - (n + Q_0) \Omega_0 \right] / (\psi^+_p \delta_p), \]  

(18)

equation (13) can be written:

\[ 1 = (U + V + iV) \left[ \frac{1}{\psi^+_p \delta_p} - \frac{1}{\psi^-_p \delta_p} \right], \]  

(19)

where \( I^+ = \int_0^\infty dP \int_0^\infty dB \int_0^\infty dA \frac{F(P) G(B) H(A)}{\nu^+ - \eta^+ E - \mu^+ A - P} \)  

(20)
To apply this formalism to the PSB, we have to assume particular normalized distributions $F(P)$, $G(B)$ and $H(A)$. It seems reasonable to choose simple parabolic distributions:

$$F(P) = \frac{3}{4\sqrt{2}} (1 - \frac{P^2}{2})$$

$$G(B) = 1 - \frac{B}{2}$$

$$H(A) = 1 - \frac{A}{2}$$

which has the advantage that the integral (20) can be computed analytically.

2.1.2. Frequency spread due to momentum spread

The frequency spread due to momentum spread can be written (dropping the index $V$ for convenience):

$$\psi_{\pm} = \int [-(n \pm Q_o)\eta + \xi Q_o] \Omega_0 \frac{d\varphi}{p},$$

where

$$\eta = \gamma_{tr}^{-2} - \gamma^{-2}, \text{ and}$$

$$\xi = \frac{P}{Q_o} \frac{\partial Q}{\partial p}$$

is the total machine chromaticity. It is the sum of the intrinsic machine chromaticity $\xi_o$ and the chromaticity generated by the sextupoles

$$\xi_s = \frac{1}{4\pi} \frac{e}{p} \frac{R}{Q} \int \alpha_p \beta \frac{d^2\beta}{d\kappa^2}$$

or, for thin lenses

$$\xi_s = \frac{R}{4\pi\mathcal{E}_p} \frac{1}{Q} \sum \alpha_{pi} \beta_i \int_{\text{lens}} \frac{d^2\beta}{d\kappa^2} ds.$$
By definition, $\delta_p$ is given by $\delta_p = \frac{dp}{\sqrt{\mathcal{C}}}$ for the particular $F(P)$ given in (21) where $dp$ is the half width of the momentum distribution at the base.

Hence

$$\psi_p \delta_p = \left[ -(n \pm Q_0) \eta \pm \left( Q_0 \frac{R}{4\pi B_p} \sum_{i} \alpha_{i} \beta_i B_i'' \ell_i \right) \right] \Omega_0 \frac{dp}{p\sqrt{2}}. \quad (25)$$

2.1.3. Frequency spread due to amplitude spread

For the terms occurring in eq. (16) for the amplitude spread, we have

$$\frac{\partial \Omega}{\partial b^2} = -\Omega_0 \left( \frac{Q_0}{2R} \right)^2 \quad (26)$$

and, from eq. (9) of Ref. 14 (setting the C.O. amplitude to zero)

$$\frac{\partial \Omega}{\partial b^2} = -\frac{1}{32\pi} \frac{1}{R} \frac{1}{B_p} \sum_i \beta_{Vi} B_i'' \ell_i, \quad (27)$$

$$\frac{\partial \Omega}{\partial b^2} = -\frac{1}{16\pi} \frac{1}{R} \frac{1}{B_p} \sum_i \beta_{Vi} B_i'' \ell_i, \quad (28a)$$

the summation $i$ being made over all octupoles. For our $G(B)$ and $H(A)$, $\delta_a^2 = a_i^2/2$ and $\delta_b^2 = b_i^2/2$, $a_i$ and $b_i$ being the beam radii at the position of the $i$ octupole. Hence we get from (16) for the amplitude spreads:

$$\psi_a \delta_a^2 = \left[ -(n \pm Q_0) \frac{\sqrt{\mathcal{C}}}{2R} a_i^2 \pm \frac{1}{16\pi B_0} \sum_i \beta_{Vi} B_i'' \ell_i a_i^2 \right] \Omega_0 \frac{a^2}{2} \quad (28a)$$

$$\psi_b \delta_b^2 = \left[ -(n \pm Q_0) \frac{\sqrt{\mathcal{C}}}{2R} b_i^2 \pm \frac{1}{32\pi B_0} \sum_i \beta_{Vi} B_i'' \ell_i b_i^2 \right] \Omega_0 \frac{b^2}{2} \quad (28b)$$

where $a^2$ and $b^2$ are averaged along the circumference.
Since the multipoles in the PSB are at the same position in each machine period, we can replace the summation over \( i \) by multiplication with the number of machine periods (16 for the PSB). The first term in the brackets can always be neglected (being \( \left( \frac{Q_0}{2R_b} \right)^2 a^2 \Omega_0 \approx 14 \text{ s}^{-1} \) at injection).

The large space charge forces in the PSB make it necessary to consider also the influence of space charge on the frequency spreads; in fact, the spread generated by space charge is dominating in the range of moderate octupole currents (Figs. 3b and 3d).

We assume a parabolic charge density distribution

\[
p(x, z) = \frac{eN}{abR_m^2} \left( 1 - \frac{x^2}{a^2} - \frac{z^2}{b^2} \right).
\]

This distribution is nearly consistent with the distribution functions of betatron amplitudes used (eq. (22)). "Nearly" in this context means that a distribution function of the factorized type \( g(b)h(a) \) actually corresponds to a beam of rectangular cross-section (with vanishing density near the corners). But at least the projected densities in both planes agree when derived either from amplitude or from spatial distribution, as long as one neglects the additional horizontal amplitude due to momentum dispersion.

*) The quadrupole term (giving linear focussing forces) of this potential was used to calculate the incoherent Q-shift, just as the octupole term was used in eq. (31) to calculate the Q-spreads. The appropriate image coefficients are included to obtain

\[ U = (\Delta Q_0 - \Delta Q_1) \Omega_0 \text{ at ten equidistant points within a machine period.} \]

The resulting \( U \) gives values about 25\% larger than the ones calculated in Section 1.
The resulting expression for the amplitude-dependent Q shift is*)

\[ \Delta Q_y = \frac{N_{r_p}}{2\pi \beta^2 \gamma^3 \varepsilon_y} \left\langle k^2 \frac{2ab}{(a+b)^2} + \lambda^2 \frac{(a+2b)b}{(a+b)^2} \right\rangle \]  

(30)

where \( 0 \leq k, \lambda \leq 1 \) denote relative amplitudes with respect to the maximum horizontal and vertical amplitudes respectively. The brackets indicate averaging along the circumference.

Keeping again only the second term of eq. (16), we obtain

\[ (\gamma_a^\pm \delta_a^2)_{s.c.} = \pm \frac{N_{r_p} \Omega_o}{4\pi \beta^2 \gamma^3 \varepsilon_y} < \frac{2ab}{(a+b)^2} > \]  

(31a)

\[ (\gamma_b^\pm \delta_b^2)_{s.c.} = \pm \frac{N_{r_p} \Omega_o}{4\pi \beta^2 \gamma^3 \varepsilon_y} \frac{(a+2b)b}{(a+b)^2} \]  

(31b)

which we have to add to the spreads due to external octupoles.

2.2. Sextupole and/or octupole strengths required

For guidance in setting these strengths we report on

(i) results from a computer program (see Appendix) plotting complete information on the thresholds in an octupole current vs sextupole current diagram, growth rates at arbitrarily chosen multipole currents and so on, and

(ii) a "rule of thumb" in form of inequalities containing the U value, which is specially useful when damping with octupoles only.

*) The emittances and beam radii to be inserted in equations (28) to (31) refer to emittances containing 100% of the beam \( \varepsilon_{100} \) for a distribution following eq. (29). For this particular distribution we have to multiply the commonly used 90% or 95% emittances by the factors \( \varepsilon_{100} = 1.46 \varepsilon_{90} = 1.29 \varepsilon_{95} \).
2.2.1. Computer results

Using the input data listed in Tables 2 and 3, the stability diagrams shown in Fig. 4 were computed with the program described in the Appendix.

Table 3 - PSB parameters used in computation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>50 MeV</td>
</tr>
<tr>
<td>Protons per ring</td>
<td>$2.5 \times 10^{12}$</td>
</tr>
<tr>
<td>Machine radius</td>
<td>25 m</td>
</tr>
<tr>
<td>Emittance, horizontal (95%)</td>
<td>$130\pi \times 10^{-6} \text{ rad m}$</td>
</tr>
<tr>
<td>Emittance, vertical (95%)</td>
<td>$40\pi \times 10^{-6} \text{ rad m}$</td>
</tr>
<tr>
<td>Momentum spread $\frac{d\varphi}{dp}$</td>
<td>$\pm 1.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>Working point:</td>
<td></td>
</tr>
<tr>
<td>$Q_H$</td>
<td>4.6</td>
</tr>
<tr>
<td>$Q_V$</td>
<td>4.8</td>
</tr>
<tr>
<td>$\frac{dQ}{dp}$, vertical</td>
<td>- 8</td>
</tr>
<tr>
<td>$\frac{dQ}{dp}$, horizontal</td>
<td>- 4</td>
</tr>
<tr>
<td>Lattice function at position of the multiplies:</td>
<td></td>
</tr>
<tr>
<td>$\beta_H$</td>
<td>5.2 m</td>
</tr>
<tr>
<td>$\beta_V$</td>
<td>11.5 m</td>
</tr>
<tr>
<td>$R_{\alpha\beta}$</td>
<td>1.39 m</td>
</tr>
<tr>
<td>Number of multiplies</td>
<td>16</td>
</tr>
<tr>
<td>Nominal (max. d.c.) currents:</td>
<td></td>
</tr>
<tr>
<td>sextupoles</td>
<td>270 A</td>
</tr>
<tr>
<td>octupoles</td>
<td>270 A</td>
</tr>
<tr>
<td>$\int B'' d\gamma$ for one sextupole at this current</td>
<td>$2.052 \text{ T/m}^{15}$</td>
</tr>
<tr>
<td>$\int B'''' d\gamma$ for one octupole at this current</td>
<td>$80.41 \text{ T/m}^{215}$</td>
</tr>
</tbody>
</table>

The data required to determine $U,V$ are given in section 1.2.
Hence one can expect the external octupoles to be rather ineffective below a certain excitation: damping is entirely governed by space-charge non-linearities. Above this "critical" current, Landau damping by space-charge forces can be neglected, regardless of how strong they may be. (However, the value of the critical current depends, of course, on the space-charge forces).

For the PSB the critical octupole currents can be estimated as $+160$ A, $-42$ A respectively under reasonable conditions. Within this range external octupoles should have only a minor effect on vertical stability.

2.2.4. Horizontal stability

Considering horizontal stability, we obtain similar results, but the different regimes are less pronounced. Equations (32), (33) remain valid if we exchange $\Delta A$ and $\Delta B$ and replace $Q_V$ by $\omega_H$. The inequalities corresponding to (32) for vanishing momentum spread are

\[
\frac{\Delta A}{\Delta B} < \frac{U_H}{\omega_H} < 1 + \frac{\Delta A}{2\Delta B} \text{ for sign } \Delta A = \text{sign } \Delta B \tag{34a}
\]

\[
\frac{\frac{\Delta A}{\Delta B}}{\omega_H} < \frac{U_H}{\omega_H} < 1 - \frac{\Delta A}{\omega_H} \text{ for sign } \Delta A \neq \text{sign } \Delta B \tag{34b}
\]

Under nominal PSB conditions, we have typical ratios:

$\Delta A_{s.c.} \approx 2.5 \Delta B_{s.c.} < 0$ and $\Delta A_{ext.} \approx -\Delta B_{ext.}$

As long as the external octupoles are not too strong, one has sign $\Delta B = \text{sign } \Delta A = -1$. We obtain, using the ratios $\Delta A/\Delta B$ given above:

\[-0.5 \Delta A_{ext.} + 0.7 \Delta A_{s.c.} < U_H < (1.2 \Delta A_{s.c.} - 0.5 \Delta A_{ext.}).\]
Orte recognizes that space charge and external octupoles have roughly the same effect.

In the range where external octupoles dominate, one has sign $\Delta A \neq \text{sign } \Delta B$ and the inequalities become

$$-0.5 \Delta A_{\text{ext.}} + 0.6 \Delta A_{\text{s.c.}} < U_H \left( -0.5 \Delta A_{\text{ext.}} - 0.6 \Delta A_{\text{s.c.}} \right)$$

for $\Delta A_{\text{ext.}} > 0$.

$$-2 \Delta A_{\text{ext.}} - 0.6 \Delta A_{\text{s.c.}} < U_H \left( \Delta A_{\text{ext.}} + 0.6 \Delta A_{\text{s.c.}} \right).$$

Note that in the latter case $\Delta A_{\text{s.c.}} < 0$ weakens the influence of external octupoles. The "critical" octupole currents separating the two regimes of space charge or external octupole dominance respectively are $-41 \, \text{A}$ and $104 \, \text{A}$.

3. **EXPERIMENTAL RESULTS AND CONCLUSIONS**

   PSB running-in has so far been essentially aimed at getting the machine to work and providing beam for CPS tests. Precision measurements and space charge studies are planned for 1973.

   Nevertheless, vertical and horizontal instabilities were observed parasitically, having growth times as computed here (in the $V$ plane to better than a factor two). We have shown in this report that their thresholds depend critically on a number of parameters such as beam emittances and momentum spread. We therefore postpone the further study of thresholds until the pertinent experimental data is available. In the meantime one can tentatively state that the introduction of the amplitude spreads in both planes seems to lead to a better agreement between theoretical and experimentally observed thresholds.
ACKNOWLEDGEMENTS

It is a pleasure to thank B. Zotter for letting us use his computer program, C. Arnaud for informing us on his Aquadac resistivity measurements, and E.C. Raka for giving us advance information on measurements of PSB instabilities.
DESCRIPTION OF THE COMPUTER PROGRAM

The program "LANDAU" calculates the range of stability of a coasting beam interacting with a non-resonant environment.

The results are plotted as a contour of the range of stability in the octupole current versus sextupole current plant, as shown in Figs. 3 to 5. In addition, it calculates the growth rates at an arbitrary point \((I_{\text{sext}}, I_{\text{oct}})\) in this plane and writes the results into the diagram (this was suppressed in the figures presented here). Plotting both the contours for vertical and horizontal stability must be run as an MJOB on the CDC 6600 computer (about 180 s. CP time). For fast information, one can suppress the contour and only determine stability and growth rate (in this case about 10 s. CP time are needed). There is also an option to plot a family of curves of constant growth rate; the necessary CP time is about 90 s. per per contour.

The values of \(V\) are calculated by an adapted version of a program for calculation of dispersion relation coefficients supplied by B. Zotter.

The \(U\) coefficient may be calculated from the same program or optionally by calculating the difference between coherent and incoherent Q-shift, including images.

In the case of the horizontal plane, we used the image coefficients given in Ref. 18 for the magnet chamber (R-CB).

Space charge is included in the calculation of Q-spreads as indicated in section 2. In order to calculate the octupole moment due to the space-charge non-linearity, the influence of the vacuum chamber is taken into account but is found to be of minor importance.
REFERENCES


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3. D. MühI, Intensity limitations in the present and future PS, MPS/DL Note 72-6: a) p. 17
   b) p. 25


5. K.H. Reich, Memorandum to W. Unterlerchner, Effect of PSB vacuum chambers on space charge limits, 1.10.1969.


15. K. Lohmann, Private communication.


FIG. 1: Total PSB dispersion relation coefficients (U above, V below, 
$\Delta \omega = U + V + jV$) as a function of mode number $n$, for $Q_v = 4.8$ and
$2.5 \times 10^{12}$ protons (per ring).
FIG. 2: Total PSB dispersion relation coefficients as a function of $Q_V$, for various mode numbers $5 < n < 500 \ (2.5 \times 10^{12} \text{ protons})$
FIG. 3: PSB stability diagram for the vertical plane (nominal currents are 270 A.)

a) amplitude distribution in V plane only, no space charge
b) idem, space charge included
c) amplitude distribution in both planes, no space charge
d) idem, space charge included
**FIG. 4a**

MODE NUMBER N = 5, 2.5E+12 P.P.P., DP/P = 1.6E-03

- EU = 40 PI
- QU = 4.00
- UN = -500097
- UN = 859.0

**FIG. 4b**

MODE NUMBER N = 5, 2.5E+12 P.P.P., DP/P = 1.6E-03

- EU = 40 PI
- QU = 4.00
- UN = -103712
- UN = 859.0

**FIG. 5**

Curves of constant growth rates (e-folding times given in msec) for nominal PSB parameters. Stability boundary same as in Fig. 4b).

**FIG. 4**

Stability diagrams using nominal PSB parameters and including parabolic amplitude distributions in both planes and space charge forces

a) calculated using linear term of space charge force (including image force).

b) $U_V$ set equal to value resulting from calculation in section 1.

c) $U$ calculated from $\left(\Delta Q_1 - \Delta Q_2\right)/\rho_0$ for elliptical cross-section with constant density.