One spectrum to rule them all?

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We show that in absence of entropy or effective anisotropic stress the freedom in the choice of the initial energy scale of inflation implies the existence of an infinite family of slow-roll parameters histories which can produce the same spectrum of comoving curvature perturbations. This implies that in general there is no one-to-one correspondence between the spectrum and higher order correlation functions. We give some numerical examples of expansion histories corresponding to different initial energy scales, with the same spectrum of curvature perturbations but different bispectra and different spectra of primordial gravitational waves. The combined analysis of data from future CMB and gravitational wave experiments could resolve this degeneracy.

\textbf{Introduction} Cosmic microwave background (CMB) observations are compatible with an approximately scale invariant primordial curvature perturbations power spectrum\textsuperscript{11,12}. Higher order correlation functions allow to distinguish between inflationary models producing the same spectrum, and it is important to investigate the relation between them. In this paper we consider what are the general conditions for different single field inflationary models to produce the same spectrum of comoving curvature perturbations, showing that the freedom in choosing the initial energy scale of inflation gives an infinite class of dual models. Our results are model independent and can be applied to any model for which the anisotropy and entropy effects are negligible, including multifields models in the regime in which isocurvature perturbations are not important. We give some examples of dual models with the same spectrum and different bispectra of curvature perturbations, and also different spectrum of primordial gravitational waves. In particular we show that there can exist dual models producing a scale invariant spectrum of curvature perturbation but completely different bispectra, and different primordial gravitational waves spectrum.

We also consider the case of models with local features in the expansion history, which are related to other types of features such as features of the inflaton potential\textsuperscript{13–41}, and can provide a better fit to observational data at the scales where the spectrum shows some deviations from power law\textsuperscript{16,18,22,24,32–34,42–46}. We discuss the implications of the existence of dual models on the violation of consistency relations between the spectrum and bispectrum of primordial curvature perturbations\textsuperscript{11,12,20,29}.

\textbf{Primordial curvature perturbations} The study of primordial scalar perturbations is attained by expanding perturbatively the action with respect to the background Friedmann-Lemaître-Robertson-Walker (FLRW) solution\textsuperscript{47,48}. In absence of entropy perturbations or effective anisotropic stress, the second-order action for scalar perturbations in the comoving gauge is

\begin{equation}
S^\mathcal{R}_c = M^3_{Pl} \int dt d^3x \left[ \frac{\epsilon a^3}{c^2} \mathcal{R}_c^2 - a\epsilon \partial \mathcal{R}_c \right],
\end{equation}

where $M_{Pl}$ is the reduced Planck mass, $a$ is the scale factor, $c_s$ is the scalar sound speed, $\mathcal{R}_c$ is the curvature perturbation on comoving slices, and we denote the derivatives with respect to time with dots. Throughout this paper we use units of $\hbar = c = 1$. For the slow-roll parameters we use the following definitions

\begin{equation}
\epsilon \equiv \frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\dot{\epsilon}}{\dot{H}}, \quad \phi \equiv \frac{\dot{\eta}}{H},
\end{equation}

where $H \equiv \dot{a}/a$ is the Hubble parameter. The Euler-Lagrange equations for this action give

\begin{equation}
\frac{\partial}{\partial t} \left( \frac{a^3 \epsilon}{c^2} \frac{\partial \mathcal{R}_c}{\partial t} \right) - a \epsilon \frac{\partial \mathcal{R}_c}{\partial x^i} \frac{\partial^2 \mathcal{R}_c}{\partial x^i \partial x^j} = 0.
\end{equation}

This equation is quite general and can also be derived from the perturbed Einstein’s equations. It can in fact be shown that Eq. (3) is satisfied by the curvature perturbations produced by an arbitrary physical system described by an effective energy momentum tensor with no entropy perturbations or effective anisotropic stress. This is for example the case of a quintessence. Given the wide class of physical systems to which this equation can be applied, also our results are quite general.

Taking the Fourier transform of the previous equation we obtain the equation of motion for the primordial curvature perturbation

\begin{equation}
\mathcal{R}_c''(k) + 2 \frac{\eta'}{z} \mathcal{R}_c'(k) + c_s^2 k^2 \mathcal{R}_c(k) = 0,
\end{equation}

where $z \equiv a \sqrt{2\epsilon}/c_s$\textsuperscript{49}, $k$ is the comoving wave number, and primes denote derivatives with respect to conformal time $dt \equiv dt/a$. As initial conditions for Eq. (4) we take the Bunch-Davies vacuum\textsuperscript{49,51}

\begin{align}
\mathcal{R}_c(\tau_i, k) &= \frac{v_i(k, c_s)}{z_i M_{Pl}}, \quad (5) \\
\mathcal{R}_c'(\tau_i, k) &= \frac{1}{M_{Pl}} \left( \frac{v_i'(k, c_s)}{z_i} - \frac{v_i(k, c_s) z_i'}{z_i} \right),
\end{align}

where $v_i(k, c_s)$ is the initial value of the adiabatic mode at scale factor $a_i$.
where from now on we denote quantities evaluated at initial time \( t_i \) or \( \tau_i \) by the subscript \( i \) and

\[
v(k, c_p) = \frac{e^{-ic_p k t}}{\sqrt{2 c_p k}} \left( 1 - \frac{i}{c_p k t} \right),
\]

where \( c_p \) is the sound speed of the perturbation.

**Primordial spectrum of curvature perturbations**

The two-point correlation function of primordial curvature perturbations is given by \[^9 \, ^10\]

\[
\langle \hat{R}_c(k_1, \tau_i) \hat{R}_c(k_2, \tau_i) \rangle = (2\pi)^3 \frac{2\pi^2}{k^3} P_{R_c}(k) \delta^{(3)}(k_1 + k_2),
\]

where \( \tau_i \) is the exit horizon time and the power spectrum of primordial curvature perturbations is defined as

\[
P_{R_c}(k) = \frac{2k^3}{(2\pi)^2} |R_c(k, \tau_i)|^2.
\]

**Primordial bispectrum of curvature perturbations**

The three-point correlation function is given by \[^47 \, ^52\]

\[
\langle \hat{R}_c(\tau_i, \vec{k}_1) \hat{R}_c(\tau_i, \vec{k}_2) \hat{R}_c(\tau_i, \vec{k}_3) \rangle = 2(2\pi)^3 B_{R_c}(k_1, k_2, k_3) \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3),
\]

where

\[
B_{R_c}(k_1, k_2, k_3) = 23 \left[ R_c(\tau_i, k_1) R_c(\tau_i, k_2) R_c(\tau_i, k_3) \right. \\
- k_1^2 R_c^*(\tau_i, k_1) R_c^*(\tau_i, k_2) R_c^*(\tau_i, k_3) \\
+ \left. \text{perms.} \right],
\]

is the bispectrum and “perms.” means the other two permutations of \( k_1, k_2 \) and \( k_3 \).

In this work we will study the degeneracy in the bispectrum using the usual \( f_{NL} \) quantity which in the case of our definition of the spectrum takes the form

\[
f_{NL}(k_1, k_2, k_3) = \frac{10}{3} \frac{(k_1 k_2 k_3)^3}{(2\pi)^4} \frac{B_{R_c}(k_1, k_2, k_3)}{P_{R_c}(k_1) P_{R_c}(k_2) k_3^4 + \text{perms.}}.
\]

**Primordial gravitational waves**

The primordial tensor perturbations satisfy the equation \[^9 \, ^10\]

\[
h''(\tau_i, k) + 2 \frac{c_s}{z} h'(\tau_i, k) + c_s^2 k^2 h(\tau_i, k) = 0,
\]

where \( c_s \) is the tensor sound speed \[^51 \, ^54\], \( z_i \equiv a/c_s \), and now the initial conditions for the tensor modes are

\[
h(\tau_i, k) = \frac{\sqrt{2} \delta_{i,3}}{z_i, M_{Pl}}
\]

\[
h'(\tau_i, k) = \frac{\sqrt{2}}{M_{Pl}} \left( \frac{v_i(k, c_s)}{z_i, \gamma} - \frac{v_i(k, c_s)}{z_i, \gamma} \right).
\]

**Primordial spectrum of gravitational waves**

The two-point correlation function of tensor perturbations is

\[
\langle h^\nu(\vec{k}_1, \tau_i) h^\nu(\vec{k}_2, \tau_i) \rangle \equiv (2\pi)^3 \frac{\pi^2}{2k^3} P_h(k) \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \delta_{ss'},
\]

where \( s = \pm \) is the helicity index and the power spectrum of primordial tensor perturbations is defined as

\[
P_h(k) \equiv \frac{2k^3}{\pi^2} |h_k(\tau_i)|^2.
\]

**How many expansion histories can give the same spectrum?**

As can be seen from Eq. \[^4\] the evolution of the primordial curvature perturbation \( R_c \) is entirely determined by \( c_s \), the background quantity \( z'/z \), and the initial conditions in Eqs. \[^5\] and \[^6\], which are in terms of \( z_i \) and \( z_i' \).

For a given spectrum how much freedom is left in specifying the slow-roll parameters history or equivalently the expansion history? Mathematically this question corresponds to ask under what conditions we can obtain the same solution for the curvature perturbation equation, and the answer is that this is possible as long as \( c_s \), the coefficient \( z'/z \) of the equation, and the initial conditions are the same.

After choosing a reference function \( z_{ref} \), we can find the family of expansion histories such that the corresponding equation of curvature perturbation has the same coefficient, i.e.

\[
\frac{z'}{z} = \frac{z'_{ref}}{z_{ref}}.
\]

A general solution for \( z \) is

\[
z(\tau) = C z_{ref},
\]

where \( C \) is an arbitrary integration constant. From now on we will assume that \( c_s \) is the same for the dual and the reference models. In order for the spectrum to be the same also the initial conditions have to be the same, which implies that

\[
z(\tau) = z_{ref}(\tau).
\]

This means that different models could have the same spectrum of primordial curvature perturbations as long as \( z \) is the same at all times.
The interesting fact is that models with the same $z$ can have different slow-roll parameters, or equivalently expansion histories. Note that the coefficients of the curvature perturbations equation depend only on the sound speed and the scale factor, and for this reason, in the spirit of the effective field theory of inflation \cite{45}, we can make a completely model independent analysis focusing on them without having to specify the Lagrangian of the model. This model independent analysis can be carried out by using the relation between $z$ and the scale factor in terms of the cosmic time $t$

$$z = \frac{a\sqrt{2\epsilon}}{c_s} = \frac{1}{c_s} \sqrt{2 \left( a^2 - \frac{a^3 \ddot{a}}{\dot{a}^2} \right)}, \quad (21)$$

For a given $z_{\text{ref}}$ the scale factor evolution is not uniquely determined. From the above relation we get in fact a second-order differential equation for the scale factor

$$a^2 - \frac{a^3 \ddot{a}}{\dot{a}^2} = \frac{1}{2} z_{\text{ref}}^2 c_s^2. \quad (22)$$

The initial value of the scale factor has no physical importance since it can always be arbitrarily fixed, but the initial condition for the first time derivative is physically important since it corresponds to consider background histories with different initial Hubble parameters $H_{i}$, and consequently different initial energy scales. We will parametrize this difference in the initial energy scale with the dimensionless quantity $\delta H = H_i / H_{\text{ref}, i}$.

This freedom in choosing $H_i$ while keeping the same evolution of the function $z(\tau)$ is the origin of the existence of an infinite set of expansion histories producing the same spectrum of curvature perturbations, which was found in some specific class of models in \cite{55}. Note that while here we will focus on the case of an expanding Universe, the freedom in the choice of the sign of $H_i$ can also lead to contracting Universes producing the same spectrum of curvature perturbations of expanding Universes, a well known phenomenon, previously interpreted as a duality \cite{56,57}, which is the basis for attempting to construct observationally viable bouncing cosmological models.

Note that we have derived the conditions to obtain any arbitrary spectrum for $\mathcal{R}_c$, while in the past \cite{56} the conditions to have an exactly scale invariant spectrum were investigated. For scale invariance the condition is weaker and it is enough to study the Sasaki-Mukhanov variable \cite{31} $u = a \delta \phi = -\text{sign}(\phi) \mathcal{R}_c z M_{\text{Pl}}$ which satisfy the equation

$$u_k'' + \left( c_s^2 k^2 - \frac{z''}{z} \right) u_k = 0. \quad (23)$$

In this case the condition is that $z''/z$ has to be the same, giving $z = z_{\text{ref}} \int d\tau z_{\text{ref}}^{-2}$, while to get the same spectrum the condition is $z = z_{\text{ref}}$. Our results are consistent with those obtained in \cite{56}, since the sign of the initial condition $\dot{a}_i$ corresponds to expanding and contracting background solutions. We have shown that in general there is an infinite class of expansion histories (contracting or inflationary), which in the context of the study of spectra with features has some important phenomenological implications in regard to the relation with the bispectra or other higher-order correlation functions.

In general in fact models with the same spectrum could have different bispectra.

**An example of a dual model** Let us consider a model with constant slow-roll parameter $\epsilon$, corresponding to the scale factor

$$a_{\text{rel}}(t) = \left( 1 + \epsilon H_{\text{rel}, i} t \right)^{1/\epsilon_c}, \quad (24)$$

where $H_{\text{rel}, i} = \dot{a}_{\text{rel}, i}/a_{\text{rel}, i}, \epsilon_c$ is a constant, and we have chosen the initial condition $a(t_i = 0) = 1$. For this model analytic expressions can be derived for the background quantities and the spectrum of primordial curvature and tensor perturbations, facilitating the comparison with the dual models obtained numerically, following these steps

- solve equation Eq. (22) for the dual $a$, where $z_{\text{rel}}$ is calculated using $a_{\text{rel}}$ from Eq. (24)
- different dual expansion histories giving the same $z_{\text{ref}}$ correspond to different initial conditions for the dual scale factor $\dot{a}_i$ or equivalently different initial value of the dual Hubble parameter $H_i$.
- from the dual $a$ we compute the corresponding slow-roll parameters histories, and from them the spectra of curvature and tensor perturbations and the bispectrum of curvature perturbations.
As shown in the next section the spectra of curvature perturbations of the dual models are exactly the same but the spectra of tensor perturbations and the bispectra of curvature perturbations are different.

In order to obtain an analytic formula for the scalar spectrum we first write Eq. (24) in terms of conformal time

\[ a_{\text{ref}}(\tau) = -H_{\text{ref},i}(1 - \epsilon_c \tau)^{1/(\epsilon_c - 1)}, \]  

(25)

such that

\[ \frac{z''_{\text{ref}}(\tau)}{z'_{\text{ref}}(\tau)} = \frac{2 - \epsilon_c}{(1 - \epsilon_c)^2 \tau^2}. \]  

(26)

From this expression, and using the Sasaki-Mukhanov variable in Eq. (23), we can obtain an exact solution when \( \epsilon_s = 1 \) for the curvature perturbation [55 60]

\[ R_c(\tau, k) = \frac{\sqrt{-k^2}}{2M_{Pl}} \left[ C_1 J_\nu(-k\tau) + C_2 Y_\nu(-k\tau) \right], \]  

(27)

where \( J_\nu(z) \) and \( Y_\nu(z) \) are the Bessel functions of the first and second kind, respectively, \( \nu = (\epsilon_c - 3)/(2(\epsilon_c - 1)) \), and \( C_1 \) and \( C_2 \) are given by the initial conditions in Eqs. [5] and [0]. A similar expression is found for the tensor modes \( h_k \) [58 60].

We show our results in Figs. [1-7], where \( N \equiv \ln[a/a_i] \) is the number of \( \epsilon \)-folds after the beginning of inflation. The values of the parameters used in the figures are \( H_{\text{ref},i} = 4.22 \times 10^{-6} M_{Pl}, \epsilon_c = 10^{-4}, c_s = c_\gamma = 1 \), and \( \delta H = 0.98 \). In the plots we use the reference scale \( k_0 \), corresponding to modes exiting the horizon at time \( t_0 \).

For the squeezed limit \( f_{NL} \) we use \( k_1 = k_2 = k_3 = 1000 k_0 \). It can be seen that, even though the curvature spectrum is the same for dual models, they can be distinguished at the bispectrum level.

Since the coefficients of Eq. (13) depend on the scale factor in a form different than in the equation for comoving curvature perturbation, we expect that also the spectra of primordial gravitational waves of dual models will be different, as confirmed in Fig. [5].

**An example of a dual models with features**  In order to understand the implications of the existence of dual models in presence of features [13-41] we consider a local [25 26] modification of the scale considered in the previous section, given by

\[ a_{\text{ref}}(t) = \left( 1 + \epsilon_c H_{\text{ref},i} t \right)^{1/\epsilon_c} \left[ 1 + \lambda e^{-\left(\frac{-t}{\tau_0}\right)^2} \right], \]  

(28)
where $\lambda$ and $\sigma$ are parameters that control the magnitude and width of the feature, and $t_0$ is the feature time. The results are shown in the bottom panel of Figs. 1 - 7. As can be seen, even in the presence of features, the spectrum of curvature perturbations of dual models is exactly the same, while the tensor spectrum and the curvature bispectrum are different.

**Violation of consistency conditions** Some relations between the spectrum and the bispectrum have been derived \[20, 29\] under some simplifying assumptions about dependence of $z$ on slow-roll parameters. It is easy to derive from Eq. (23) this relation in terms of slow-roll parameters,

$$\frac{z''}{z} = a^2 H^2 \left(2 - \epsilon + \frac{\eta}{2}\right) \left(1 + \frac{\eta}{2}\right) + \frac{aH}{2} \eta' \quad (29)$$

$$\equiv 2a^2 H^2 \left(1 + \frac{1}{2} \delta_H\right),$$

where we have assumed $c_s = 1$ and

$$\delta_H = -\epsilon - \frac{1}{2} \epsilon\eta + 3 \eta + \frac{1}{4} \eta^2 + \frac{1}{2aH} \eta'. \quad (30)$$

In \[29\] it was argued that for sudden variations of the background it can be further simplified to this form

$$\delta_H \approx -\frac{1}{2} \tau \eta'. \quad (31)$$

In Fig. 8 we plot the numerically computed $\delta_H$, showing that the above equation is not a good approximation for the case of dual models. In Ref. \[29\] this approximation
FIG. 7: The numerically computed $f_{NL}$ functions in the equilateral limit corresponding to the scale factor $a$ obtained from Eq. (22) (dashed red) are compared to $f_{NL}^\text{ref}$ (blue). On top (bottom) we plot the results obtained from Eq. (22) and Eq. (24) (Eq. (28) bottom).

is used to reconstruct the time evolution of the slow-roll parameter $\eta$ from the spectrum. The $\eta$ parameter is then used to compute the relation between the spectrum and bispectrum.

If these relations were always valid models with the same spectrum should also have the same bispectrum, but as shown in Figs. 6 and 7 this is not true. In general there is an infinite family of slow-roll parameter histories which do not satisfy the approximation in Eq. (31) and consequently violate the consistency relations which are derived assuming its validity.

A violation of these consistency conditions would not rule out all single field inflationary models, but only the ones which satisfy the approximation given in Eq. (31). As shown in the previous section in fact there exists an infinite set of models which can violate it.

**Conclusions** We have shown that there is an infinite family of slow-roll parameters histories which can produce the same spectrum of comoving curvature perturbations. This degeneracy is related to the freedom in the choice of the initial conditions for the second order differential equation relating the coefficients of the curvature perturbation equations to the scale factor. This freedom implies that in general there is no one-to-one correspondence between the spectrum and higher order correlation functions, unless some special conditions are satisfied by the slow-roll parameters. We have given some examples of dual models with the same spectrum but different bispectra, and different primordial gravitational wave spectra. A combined analysis of data from future CMB experiments such as the CMB-S4 and space gravitational detectors such as the Laser Interferometer Space Antenna (LISA) and Evolved Laser Interferometer Space Antenna (eLISA) could allow to distinguish between different models and resolve this degeneracy.
