Anode wire in cylindrical cathode tube

Destabilizing electrostatic force

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Abstract

A two-dimensional – cross-sectional – discussion suffices. The tube is offset, and the electrostatic potential is found analytically with perturbative methods. Then, the force is established with the Maxwell stress tensor. Alternatively, trying to find the force with energy methods, fails. Finally, finite element tests are performed in order to report on the degree of non-linearity for large offsets.

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Versions:
1. 12 May 2017: initial
2. 21 Feb. 2018: simple update
Conventions

Consider a (conductive) wire with diameter $2a$ perfectly centred inside a (conductive) tube with diameter $2b$. Whereas the force is zero because of symmetry, the configuration is unstable, i.e. associated with a **negative transverse stiffness**. An analytical solution to this problem is found by means of perturbative methods.

A global coordinate system (cartesian $x$-$y$-$z$ resp. cylindrical $r$-$\varphi$-$z$) is attached to the wire. We will offset the tube vertically by an amount $h$, solve for the (scalar) potential field $V$, derive the (vector) electric field $E$, then find the force, which indeed will exhibit the “wrong” sign. This force will turn out to be proportional to $h$; this should come as no surprise because we will have deployed first-order developments (in $h$). Force and stiffness (and energy and charge) will be per unit depth $z$. Indeed, we will only work out the two-dimensional cross-sectional problem, i.e. we assume that a sagging wire does not give an appreciable $z$-component for the electric field (or the potential).

Some of the notations, and the perturbation reasoning, have been taken from the textbook [Ref.1]. The way(s) of finding the force, is (are) distinctively different, however.

**Potential**

The equation governing the potential $V$ in the volume between wire and tube skins, filled with a non-solid linear \(^1\) medium with permittivity $\varepsilon$, is:

\[
\text{div} \left( \varepsilon(r) \ \text{grad} \ V \right) = 0
\]

Now, if the fluid is isotropic ($\varepsilon$-tensor reduces down to scalar) and homogeneous ($\varepsilon$ independent of location defined by vector $r$ above \(^2\)), the equation simplifies to:

\[
\nabla^2 V = 0
\]

The solution is subject to Dirichlet boundary conditions on the wire skin \(V = V_0\) and on the tube skin \(V = 0\).

In case of perfect centring \((h = 0)\), the problem is axisymmetric and hence dependent on the polar radius $r$ only. Solution:

\[
V_{\text{axi}} = V_0 \ln \frac{r}{b}
\]

In case of offset \((h \neq 0)\), we will try to find solutions based upon separation of variables:

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1. Permittivity independent of field strength.
2. Not to be confused with the scalar $r$ which is the cylindrical (polar) radial coordinate.
\[ V(r, \phi) = R(r) \Phi(\phi) \]

The Laplacian in polar coordinates is:
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}, \]
and the governing equation becomes (after multiplying by \( r^2/(R\Phi) \)):
\[ \frac{r^2}{R} \frac{d^2R}{dr^2} + \frac{r}{R} \frac{dR}{dr} + \frac{1}{R \Phi} \frac{d^2\Phi}{d\phi^2} = 0, \]
to be satisfied all over the domain. Hence:
\[ \frac{r^2}{R} \frac{d^2R}{dr^2} + \frac{r}{R} \frac{dR}{dr} = -\frac{1}{R \Phi} \frac{d^2\Phi}{d\phi^2} = \text{const.} = n^2, \]
That the constant must be positive, can be seen from the polar-angle equation:
\[ \frac{d^2\Phi}{d\phi^2} + n^2 \Phi = 0, \]
with solutions:
\[ \{ \cos n\phi, \sin n\phi \}. \]

Had instead a negative sign for the separation constant been withheld (const. = -n^2), then nonsensical exponential/hyperbolic functions would have resulted.

We must restrict even further. The necessary \( 2\pi \) periodicity in the solution is enforceable by restricting \( n \) to being integer only.

Now to the radial equation:
\[ \frac{r^2}{R} \frac{d^2R}{dr^2} + \frac{r}{R} \frac{dR}{dr} - n^2 R = 0. \]
This is a traditional Euler equation, with solution space:
- \( \ln r, \text{const.} \) in case \( n = 0 \);
- \( r, 1/r \) otherwise.

The grand total is formalized as “mode” superposition:
\[ V = \left( \ldots \ln \frac{r}{b} + \ldots \right) + \sum_{n=1}^{\infty} \left( \ldots r + r \frac{1}{r} \ldots \right) \left| \cos n\phi + \sin n\phi \right|, \]
where one notes a normalisation of the sine term\(^3\), for the sake of avoiding over-abundance of integration constants (denoted as \( \ldots \)). For the same reason, only positive values of \( n \) have been withheld (remembering that \( \cos -u = \cos u \) and \( \sin -u = -\sin u \)).

Now to the boundary conditions. The \( r \) coordinate of the tube wall equals:
\[ r_1 + r_2 = h \sin \phi + \left| b + O(h^2) \right|, \]
and thus, to first order (in \( h \)), the tube boundary condition is:
\[ V(r, \phi)|_{r=b+h \sin \phi} = 0 \quad (\text{for all } \phi). \]
We now try to bring the simplest possible form of the potential – keeping only the terms \( n=0 \) and \( n=1 \) – in agreement with the two boundary conditions.
\[ V = \left( \alpha \ln \frac{r}{b} + \beta \right) + \left( \gamma r + \frac{\delta}{r} \right) \left| n \cos \phi + \sin \phi \right| \quad \text{ansatz!} \]
\[ \text{3\footnote{Doing this on the cosine term instead, would not have been a good idea, as we see further.}} \]
Firstly, the wire b.c.:
\[ V(r, \varphi)|_{r=a} = V_0 \quad \text{(for all } \varphi) \]
and thus:
\[ \alpha \ln \frac{a}{b} + \beta = V_0 \quad ; \quad \gamma a + \frac{\beta}{a} = 0 \]

We anticipate that:
\[ \alpha = O(h^8) \]

Now to the tube b.c. Keeping in mind:
\[ \ln(1+u) = u + O(u^2) \quad , \quad \frac{1}{1+u} = 1 - u + O(u^2) \]

one obtains after a few manipulations (and substituting \(\delta\)):
\[ \frac{\alpha h}{b} \sin \varphi + \beta + \gamma \left[ b \left(1 - \frac{a^2}{b^2}\right) + h \left(1 + \frac{a^2}{b^2}\right) \sin \varphi \right] \eta \cos \varphi + \sin \varphi = 0 \quad \text{(for all } \varphi) \]

from which we can readily enforce:
\[ \beta = 0 \quad , \quad \eta = 0 \]

After multiplying out, a term in \(\sin^2 \varphi\) shows up, and appears to become a show stopper. However, this term can elegantly be discarded, because it is of second order in \(h\) ! Indeed, let us anticipate again:
\[ \gamma = O(h) \]

The final result, after straightforward operations, is neatly in agreement with the anticipations:
\[ \gamma = -\frac{\alpha h}{b b^*} \]
\[ b^* = b \left(1 - \frac{a^2}{b^2}\right) \]
\[ V(r, \varphi) = \alpha \left[ \ln \frac{r}{b} + \frac{h}{b b^*} \left(-r + \frac{a^2}{r}\right) \sin \varphi \right] \]
\[ \alpha = \frac{V_0}{\ln \frac{a}{b}} = -\frac{V_0}{\ln \frac{b}{a}} \]

Note that \(\alpha\) has the dimensions of voltage, and a sign opposite to that of \(V_0\). For many real-life configurations \((b >> a)\), it is only a mild function of geometry.

The potential field can also be written as:
\[ V = V_{\text{axi}} + V_{\text{pert}} \]
in which the first term is the aforementioned perfectly centred configuration, and the second term – the perturbation term – is neatly proportional to \(h\) – the perturbation parameter. This is obviously thanks to our consistently resorting to first-order developments.
**Electric field**

Now the differentiating to the electric field:

\[
E = (E_r, E_\phi, 0) = - \nabla V = \left(- \frac{\partial V}{\partial r}, - \frac{1}{r} \frac{\partial V}{\partial \phi}, 0 \right)
\]

Components:

\[
\frac{E_r}{\alpha} = -\frac{1}{r} + \frac{h}{bb} \left(1+\frac{a^2}{r^2}\right) \sin \phi
\]

\[
\frac{E_\phi}{\alpha} = \frac{h}{bb} \left(1-\frac{a^2}{r^2}\right) \cos \phi
\]

Magnitude (squared):

\[
\frac{E^2}{\alpha^2} = \frac{1}{r^2} - 2 \frac{h}{bb} \frac{a^2}{r^2} \sin \phi + O(h^2)
\]

Interestingly, the total charge on either electrode is not influenced by an offset \( h \) ..., again, to first order 4! This is easily demonstrated by applying Gauss’ law on a fishnet in the medium, encircling the wire, and taking the limit where the net shrinks to lay tight to the wire's surface. There, the \( E \)-field is strictly radial, and the second term – the one proportional to \( h \) – vanishes after \( \phi \)-integrating over \( 2\pi \).

**Force via Maxwell stress tensor**

The field emanating from an equipotential surface, is normal to that surface. As a result, the Maxwell stress on a conductor skin is purely tensile:

- normal to the equipotential surface;
- pulling 5 the conductor skin towards the dielectric medium.

\[\sigma_n = \frac{\varepsilon E^2}{2}\]

Applied to the wire skin, where:

\[E_r = |E|\]

\[\frac{E^2}{\alpha^2} = \frac{1}{a^2} - 4 \frac{h}{bb} \frac{a^2}{r^2} \sin \phi\]

Force on wire per unit depth (\( z \)) : the net force only has a vertical component:

\[f_{y,wire} = \int_0^{2\pi} \sigma_n \sin \phi \ a \ d\phi = - \frac{2\pi \varepsilon}{bb} \alpha^2 \ h = - \frac{2\pi \varepsilon}{bb} \frac{V_0}{(\ln \frac{b}{a})^2} \ h\]

Cross-checking this result by application of Maxwell tensions to the tube skin, would be costly amusement, not in the least because, there, normal is not (minus-)radial. We will refrain from it.

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4 We will soon learn about a terrible trap with this..

5 ... whichever the field’s “polarity”!
Force via virtual work

Alternatively, the virtual work concept could be deployed ...? Let \( \mathbf{E} \) be the electric field with offset \( h \), and \( \mathbf{E}_{\text{aug}} \) the field under an incrementally augmented offset \( h+\delta h \). Then,

\[
\delta u = f_{y,\text{win}} \delta h + V_0 b q ,
\]

where \( u \) is the stored electric energy and \( q \) is the wire charge, both per unit depth.

Firstly, consider \( u \):

\[
\delta u = u_{\text{aug}} - u , \quad u = \iint_{A} \frac{\varepsilon \mathbf{E}^2}{2} dA , \quad u_{\text{aug}} = \iint_{A_{\text{aug}}} \frac{\varepsilon \mathbf{E}_{\text{aug}}^2}{2} dA .
\]

One shall take into consideration that the integration domain is altered by the offset increment.

We are confronted with two integration tasks, each corresponding to one term of the above expression of the field squared-magnitude. The first is:

\[
\int_{\psi=0}^{2\pi} \int_{r=a}^{b} \frac{b+h \sin \psi}{a} r d\psi = \frac{2\pi}{\psi} \ln \left( \frac{1+\frac{b+h \sin \psi}{a}}{\ln \frac{b}{a}} \right) d\psi
\]

\[
\Rightarrow \int_{\psi=0}^{2\pi} \frac{b \sin \psi + \ln \frac{b}{a}}{a} d\psi = 2\pi \ln \frac{b}{a} .
\]

The second task is:

\[
\int_{\psi=0}^{2\pi} \int_{r=a}^{b} \left(1+\frac{a^2}{r^2}\right) \sin \psi d\psi = \int_{\psi=0}^{2\pi} I(\psi) \sin \psi d\psi = \pi \left(1+\frac{a^2}{b^2}\right) h + O(h^3)
\]

This is because:

\[
I(\psi) = \int_{a}^{b+\psi \sin a} \left( \frac{1+\frac{a^2}{r^2}}{r} \right) dr = b + h \sin \psi - a - a^2 \left( \frac{1}{b+h \sin \psi} - \frac{1}{a} \right) = b + h \sin \psi - \frac{a^2/b}{1+h/b \sin \psi}
\]

\[
\Rightarrow b + h \sin \psi - \frac{a^2/b}{\frac{1}{1+\frac{a^2}{b^2}}} = \text{const.} + \left(1+\frac{a^2}{b^2}\right) h \sin \psi .
\]

So we obtain:

\[
u = u_0 + \frac{k_{\text{wrong}} h^2}{2} \quad , \quad k_{\text{wrong}} = -\frac{2\pi \varepsilon \mathbf{E}^2}{bb^2} \left(1+\frac{a^2}{b^2}\right)
\]

a hopelessly wrong result! Not only is the magnitude wrong – by a factor \((1+a^2/b^2)\) – but so is the sign!

In reality, the stored energy increases with \(|h|\). This is shown convincingly by a simple finite element test. It would also have become obvious if we had resorted to the specialized virtual work equations from electrostatics:

\[
f_h = \left(\frac{\partial u}{\partial h}\right)_{\text{constant voltage}} = -\left(\frac{\partial u}{\partial h}\right)_{\text{constant charge}} ,
\]

which we know must be positive (the destabilizing force points towards the positive \(h\)-axis).

It appears that we have not been able to avoid all possible pitfalls. The first devil is in the energy differentiation. Let us write down the (stored) energy function \( u \) very generally... but fully developed in \( h \):

\[
u = C_0 + C_1 h + C_2 h^2 + C_3 h^3 + ...
\]

It is then immediately seen that:

\[
\frac{\partial u}{\partial h} = C_1 + 2 C_2 h + O(h^2)
\]
Now, the $C_1$ term shall vanish, because $h=0$ is the equilibrium point. Hence, the energy expression must have a full account of all second-order contributions (in the degree of freedom according to the increment), in order to make the virtual work method correctly applicable! Clearly, we have not satisfied this requirement.

The second danger is in the work done by the external supply. We would have been tempted to discard it, because the charge would not change… yes, but to first order! Consider this:

$$q = q_0 + \frac{Bh^2}{2} + \ldots \quad \Rightarrow \quad \delta q = \left[ B h + O(h^2) \right] \delta h .$$

Mechanical work delivered by the system, amounting to one energy unit, does not result in a decrease of the stored energy, but rather an increase, also by one unit, and this to the expense of the source, which had to deliver work amounting to two units. This is the typical signature for the constant-voltage situation in electrostatics.

With the ansatz:

$$u = u_0 + \frac{k h^2}{2} , \quad k \text{ positive} ,$$

we obviously have:

$$\delta u = k h \delta h .$$

Let us recover the general expression of virtual work:

$$\delta u = f_{y, \text{wire}} \delta h + V_0 \delta q ,$$

where the first term is the work done by an external force, on the wire, keeping the electrostatic force in balance. It will all fall neatly in place if (with the previous ansatz for $q$):

$$f_{y, \text{wire}} = -kh , \quad V_0 B = 2k .$$

The multiplier $k$ shall be nothing else but the magnitude of the negative stiffness, per unit depth ($z$), which we had earlier found to be:

$$k = \frac{2\pi\varepsilon}{b b^*} \frac{V_a^2}{(\ln\frac{b}{a})^2} .$$

We have forced our way towards the correct energy expression, and the stiffness was imported from the Maxwell stress exercise. Of course, we would rather have preferred to obtain a result independently, with the virtual work exercise; our attempts bitterly failed. This discussion of energy is still useful however, because stored energy can reliably be obtained in a finite element context; see further.

In summary: analytical energy methods may fail if they rely on post-processing of lower-order techniques.

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7 And constant-current in magnetostatics.
8 The sign is correct. The external force $f_y$ would be positive in the sense of increasing $h$, so: downward. The vectorial-opposite $f_{y,\text{wire}}$ is positive upward.
Numerical tests

A finite element model of a (right-)half cross-section has been made, and symmetry is exploited; obviously, an appropriate multiplier will be applied in the post-processing, in order to present results on a full section basis.

Quadratic elements have been used to model the dielectric; they are well-suited to capture high gradients near the wire skin, and to follow the curvature of the equipotential boundaries (wire and tube skin).

For a fixed $a/b$ ratio, the offset $h/b$ is gradually increased from 0 to 0.20 in steps of 0.02. For each $h$-step $i$, the force is obtained in 3 different ways:

- energy method: use central-difference numerical differentiation (thus needing an extra $u$ data point at 0.22):
  \[ f_{h,i} = \frac{du}{dh} \bigg|_i \text{ approximated as } \frac{u_{i+1} - u_{i-1}}{2\Delta h}; \]

- straightforward summing of Maxwell stress-force portions; this is done on the wire skin and on the tube skin.

Rather than plotting the forces (as function of offset), we will give the secant stiffness:

\[ k_{sec,i} = \frac{f_{h,i}}{h_i}. \]

Both the energy and the Maxwell stress methods are delicate, not in the least because they both involve the square of a derived quantity, making them very sensitive to errors resulting from the finite element approximation itself. Great care has been paid to arrive at a high-quality mesh, following, in a smooth way, the offset $d$, and sufficiently fine-grained, especially in cases of a very thin wire. Other numerical pitfalls:

- energy method: the subtraction of two energy values possibly very close in magnitude;
- Maxwell stress integration: again a delicate subtraction with the goal of reliably arriving at an end result possibly very small (think of small offset cases).

Mesh details:

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$n_r$</th>
<th>$n_\varphi$</th>
<th>$r_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>12</td>
<td>9</td>
<td>3.</td>
</tr>
<tr>
<td>0.1</td>
<td>14</td>
<td>10</td>
<td>6.</td>
</tr>
<tr>
<td>0.03</td>
<td>16</td>
<td>11</td>
<td>36.</td>
</tr>
<tr>
<td>0.01</td>
<td>26</td>
<td>12</td>
<td>72.</td>
</tr>
<tr>
<td>0.003</td>
<td>40</td>
<td>12</td>
<td>300.</td>
</tr>
<tr>
<td>0.001</td>
<td>60</td>
<td>12</td>
<td>600.</td>
</tr>
</tbody>
</table>

$n_r$: number of elements along $r$

$2n_\varphi$: number of elements along $\varphi$ from 6:00 to 12:00

$r_r$: ratio of element size along $r$ between biggest (near tube) and smallest (near wire) element

The language of slope (stiffness, modulus,...) in a non-linear world. In a working point $p$, clear distinction exists between “tangent” and “secant”. The notion of “initial” is self-explaining. The definitions are unambiguous only if there is no path dependence (hysteresis). The force and deflection axes are on a linear scale. In our case, the “deflection” is (normalized) $h$. The curve to the left depicts a “softening” response. In contrast, we will find a (very mildly) “hardening” behaviour for the problem at hand.

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9 It is the scalar potential that is solved for.

10 It is the addition of $y$-components ... north and south companions about equal in magnitude and of opposite sign.
Top left: the mesh for $a/b=0.3$, deformed by $h/b=0.22$.

Bottom: gradual zoom-in into the extreme mesh of an extreme case: $a/b=10^{-3}$ ($h=0$).

The obtained secant stiffness is plotted below, normalized to the stiffness $k$ obtained analytically. Legend to the curves:

- full line: energy method;
- (long) dashed-dotted: Maxwell stress method on wire surface;
- (short) dashed: Maxwell stress method on tube surface.
Below are the same data repeated once more, but zoomed-in (ordinate).

One would be tempted to conclude that energy and Maxwell-tube are reliable. Maxwell-wire fails, except for extremely thick wires. The (main) culprit is the finite element approximation error, see Appendix 1.

The non-linearity is very mild.

**Appendix 1 : Tests on mesh refinement**

Let us fix: \( a/b = 0.01 \quad h/b = 0.02 \quad r_r = 72 \),

but play with: \( n_r = M \cdot 26 \quad n_\phi = M \cdot 12 \),

where the multiplier \( M \) now becomes the variable. Below are the Maxwell-wire results. The quality of the result improves significantly with increasing \( M \).

It shall be kept in mind that the number of elements is proportional to \( M^2 \). For \( M=5 \), we had 15600 elements...
Appendix 2: ANSYS® command input stream

/nopr
/plop,info,1
/prep7
immed,0

bb = 1. ! radius of tube
aa = .01 ! radius of wire
hh = .02 ! y-offset of tube w.r.t. wire
multi = 5
nr = multi*26 ! number of elements in r
nphi = multi*12 ! number of elements in a 90-degree sector
ratr = 72. ! ratio element size along r
V0 = 1. ! high voltage on wire
epsr = 1/(8.85e-12) ! relative permittivity of gas
pi = 3.1415926536
local,11,cart, 0.,hh ! 11 : offset cartesian system
local,12,cylin,0.,hh ! 12 : offset cylindrical system
csys,0
et,1,plane121
mp,perx,1,epsr
delphi = 45./nphi
ninc = 2*nr+1
csys,1
n,1,aa,-90. $ngen,4*nphi+1,ninc,1,,,0.,delphi
csys,12
n,2*nr+1,bb,-90. $ngen,4*nphi+1,ninc,2*nr+1,,,0.,delphi
csys,0
fill,1,2*nr+1,2*nr-1,2,1,4*nphi+1,ninc,ratr
type,1 $mat,1
i = ninc $ti = 2*ninc
en,1,1,3,t,i+3,t,i+1,2,i+3,t,i+2,i+1
i= $ti=
ngen,2,nr,2,1
ngen,2,ninc,2*nphi,2*ninc,1,2*nr-1
nsle $nsel,inv$ $ndel,all $nsel,all
csys,1 $nsel,s,loc,x,aa $d,all,volt,V0 $nsel,all
csys,12 $nsel,s,loc,x,bb $d,all,volt,0. $nsel,all
csys,0
fini
fini

/post1

! stored energy
etab,sene,sene
ssum
! Maxwell force on wire
fysum = 0.

*do,i,istart,istop,2 ! only corner nodes
  *i,ieq,istart,or,i,ieq,istop,then
    dc = .5*delcirc
  *else
    dc = delcirc
  *endif
  *get,efsum,node,nn,ef,sum
  ! get magnitude E-field
  del = .5*efsum**2*dc ! only IF total permitt = 1
  yi = ny(nn)
  sinphi = yi/aa
  delf = del*sinphi
  fysum = fysum+delf
*endo
/gopr $Maxwell_wire = fysum $/nopr
! Maxwell force on tube
csys,11
defsum = 0. 
defsum = pi*bb/(2*nphi)
*do,i,istart,istop,2
  *i,ieq,istart,or,i,ieq,istop,then
    dc = .5*delcirc
  *else
    dc = delcirc
  *endif
  *get,efsum,node,nn,ef,sum
  ! get magnitude E-field
  del = .5*efsum**2*dc
  xi = nx(nn) $yi = ny(nn)
  sinpsi = yi/sqrt(xi**2+yi**2)
  delf = -del*sinpsi
  fysum = fysum+delf
*endo
csys,0
/gopr $Maxwell_tube = fysum $/nopr
fini
/solu
solv
fini

Reference


11 Finite element code ; a trademark of ANSYS, Inc., Canonsburg, PA, USA.