Coherent States and Squeezed States, 
Supercoherent States and Supersqueezed States

Michael Martin Nieto

Theoretical Division, Los Alamos National Laboratory
University of California
Los Alamos, New Mexico 87545, U.S.A.¹

ABSTRACT

This article reports on a program to obtain and understand coherent states for general systems. Most recently this has included supersymmetric systems. A byproduct of this work has been studies of squeezed and supersqueezed states. To obtain a physical understanding of these systems has always been a primary goal. In particular, in the work on supersymmetry an attempt to understand the role of Grassmann numbers in quantum mechanics has been initiated.

1 Introduction

In 1926, Schrödinger published a paper [1] which described what we today call the “coherent states.” This paper was separate from his fundamental series on “Quantization as an Eigenvalue Problem.” Schrödinger sent a copy of this paper to Lorentz in response to Lorentz’s objection to using wave packets to represent particles (since the packets must spread out with time) [2].

From what we would call a knowledge of the generating function for the Hermite polynomials, Schrödinger was able to show that in a harmonic oscillator potential, a general Gaussian wave with the width of the ground state could have arbitrary energy and momentum, follow the classical motion of a classical particle in the potential, and not change its shape with time. This insight eventually became formulated in one of the standard ways to define the coherent states (of the harmonic oscillator), as minimum uncertainty coherent states (MUCS).

In the early 1960’s, stimulated by the classic work of Klauder [3,5], Sudarshan [4,5], and Glauber [6], there was a renewed interest in these states in the context of quantum optics. By using the boson operator techniques of these authors, the coherent states can be defined as displacement operator coherent states (DOCS) or as annihilation operator coherent states (AOCS). The displacement operator method uses symmetry, or group theory, techniques.

For the harmonic oscillator all three methods are equivalent, but for other systems they are not, in general. After reviewing these methods for the harmonic oscillator,
I will discuss extensions of them to general potentials (MUCS) and arbitrary symmetries (DOCS). In these contexts, the place of the “squeezed states” of the harmonic oscillator then follows.

Although my own work for bosonic systems has strongly emphasized the physically intuitive minimum-uncertainty approach, when I and my colleagues came to consider supercoherent states, we reached the conclusion that the best approach there was to use the more abstract DOCS method. This method provided an explicitly defined mathematical approach which we hoped would yield physical insight into what such a bosonic-fermionic system means, something that, a priori, was opaque.

In the final sections I will review this work for coherent states, announce some new results for squeezed states, and discuss some physical conclusions which can be inferred from this work. This includes an indication of what the role of Grassmann numbers in quantum mechanics might be.

2 Coherent states

The harmonic oscillator Hamiltonian,

\[ H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2, \tag{1} \]

is quadratic in the operators, \( x \) and \( p \), which classically vary as \( \sin(\omega t) \) and \( \cos(\omega t) \).

The commutation relation of the associated quantum operators (\( \hbar = 1 \))

\[ [x, p] = i, \tag{2} \]

defines an uncertainty relation

\[ (\Delta x)^2(\Delta p)^2 \geq 1/4. \tag{3} \]

I: (MUCS). The minimum uncertainty coherent states for the harmonic oscillator potential can be defined as those states which minimize the uncertainty relation (3), subject to the added constraint that the ground state is a member of the set.

Those states which minimize the uncertainty relation (3) are

\[ \psi(x) = [2\pi\sigma^2]^{-1/4} \exp \left\{ -\left(\frac{x - x_0}{2\sigma}\right)^2 + i\sigma_0 x \right\}, \tag{4} \]

\[ \sigma = s\sigma_0 = s/[2m\omega]^{1/2}. \tag{5} \]

When \( s = 1 \), these Guassians have the width of the ground state, so they are the coherent states. The states are labeled by two parameters, \( x_0 = \langle x \rangle \) and \( p_0 = \langle p \rangle \).
Now consider the displacement operator approach, which was first championed by Klauder [7]. Consider the oscillator algebra defined by $a, a^+, a^a, a^a$, and $I$. The displacement operator is the unitary exponentiation of the elements of the factor algebra, spanned by $a$ and $a^+$:

$$D(\alpha) = \exp[\alpha a^+ - \alpha^* a] = \exp\left[-\frac{1}{2}|\alpha|^2\right]\exp[\alpha a^+]\exp[-\alpha^* a],$$

where the last equality comes from using a Baker-Campbell-Hausdorff relation.

II: (DOCS). The displacement operator coherent states are obtained by applying the displacement operator $D(\alpha)$ on an extremal state, i.e., the ground state.

Specifically, this yields

$$D(\alpha)|0\rangle = \exp[\alpha a^+ - \alpha^* a]|0\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv |\alpha\rangle,$$

where $|n\rangle$ are the number states. With the identifications

$$Re(\alpha) = [m\omega/2]^{1/2}x_0, \quad Im(\alpha) = p_0/[2m\omega]^{1/2},$$

these are the same as the MUCS.

The third definition I mention here is

III: (AOCS). The annihilation operator coherent states are the eigenstates of the destruction operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$ (For the harmonic oscillator, III can be shown to follow from II.)

3 Coherent states for general potentials and symmetries

At the end of his 1926 paper [1], Schrödinger predicted that similar (coherent) states could be constructed for the hydrogen atom, but that it would be difficult. (He never returned to the problem.) Some 50 years later, Pete Carruthers asked me if I thought it could be done for arbitrary potentials. In striving to answer Pete’s question Mike Simmons, Vincent Gutschick, and myself developed our generalization of the minimum-uncertainty method.

Consider an arbitrary classical Hamiltonian and find those classical variables, $X_c(x_c, p_c)$ and $P_c(x_c, p_c)$, that vary as $\sin(\omega t)$ and $\cos(\omega t)$. Our first ansatz was
that the classical Hamiltonian would be quadratic in these variables. (For the many systems we studied, it was.) Now turn these classical variables into quantum operators, $X$ and $P$, by turning $x_c$ and $p_c$ into the quantum operators $x$ and $p$ and then appropriately symmetrizing them in the functionals $X$ and $P$. These operators define a commutation relation and hence an uncertainty relation

$$[X, P] = iG, \quad (\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle G \rangle^2.$$ (10)

The state which minimizes the uncertainty relation (10) is given by the solution to the eigenvalue equation

$$\left(X + \frac{i\langle G \rangle}{2(\Delta P)^2}P\right)\psi_{\text{mus}} = \left(\langle X \rangle + \frac{i\langle G \rangle}{2(\Delta P)^2}\langle P \rangle\right)\psi_{\text{mus}}.$$ (11)

Note that of the four parameters $\langle X \rangle$, $\langle P \rangle$, $\langle P^2 \rangle$, and $\langle G \rangle$, only three are independent because they satisfy the equality in Eq. (10).

Our second ansatz is our definition for

I: General MUCS for Arbitrary Potentials. The coherent states are the states $\psi_{\text{mus}}$, subject to the restriction that the ground state solution of the Schrödinger equation be a member of the set. (It always turned out to work.) This means that the three independent parameters are now reduced to two.

As indicated, this definition worked for every solvable potential we tried. (In WKB approximation it holds in general.) The results are described in a series of papers [8], and a summary article is reprinted in the book by Klauder and Skagerstam [9]. We even produced a 16 mm color-sound film on the time evolution of these coherent states. It was later made into a video [10]. (Unfortunately, I have not found it on the shelves at any Blockbuster Video Store.) As hoped for, these states can be shown to follow the classical motion. They disperse with time, as they have to, since the eigenenergies are in general not commensurate. The variation of decoherence time from system to system can also be understood. These states maintain their coherence as well as or better than those from other methods. In the end this is not too surprising since they were physically designed to do so. It is this physically intuitive basis for these states which is one of their advantages.

Numerous times John Klauder expressed an interest in this program, and his “Doubting Thomas,” penetrating questions led to specific results on at least two occasions: i) the numerical comparison of the time evolution of our states with continuous representation states, and ii) an explicit demonstration of the resolution of the identity [11].

Continuing, the generalization of the DOCS method to other symmetries is clear. Its application to arbitrary Lie groups has been discussed by many people [7,12].
II: General DOCS for Arbitrary (Lie) Symmetries. The general DOCS are those states obtained by applying the displacement operator, which is the unitary exponentiation of the elements of the factor algebra, on the extremal state. This is the extremal weight vector for noncompact groups. (I’d call it the ground state but John would wince.)

One important advantage of this method is that it presents a well-defined mathematical procedure to obtain the states.

4 Squeezed states

The squeezed states of the harmonic oscillator are very easy to obtain from the MUCS point of view [13]. Look back to Eqs. (4-5). Simply let $s \neq 1$, and you have the “squeezed states.” That is, they are minimum uncertainty Gaussians whose widths are not necessarily that of the ground state. This is a continuous three-parameter set of states.

Similarly, the generalization of squeezed states for arbitrary potentials works the same. These states are one step back from the general MUCS. They are the three parameter set of states defined in Eq. (11), $\psi_{\text{mus}}$.

The displacement operator squeezed states for the harmonic oscillator are more complicated. Consider the “unitary squeeze operator”

$$S(z) = \exp \left[ \frac{1}{2} z a^+ a - \frac{1}{2} z^* a a \right]$$

$$= \exp \left[ \frac{1}{2} e^{i\phi} (\tanh r) a^+ a^+ \right] \left( \frac{1}{\cosh r} \right)^{(1+a^+ a)} \exp \left[ -\frac{1}{2} e^{-i\phi} (\tanh r) a a \right],$$

where Eq. (13) is obtained from a BCH relation [14]. The squeezed states equivalent to the $\psi$ of Eqs. (4-5) are obtained by operating on the ground state by

$$T(\alpha, z) |0\rangle = D(\alpha) S(z) |0\rangle \equiv |\alpha, z\rangle,$$

$$z \equiv r e^{i\phi}, \quad r = \ln s.$$  

[$\phi$ is a phase which defines the starting time, $t_0 = (\phi/2\omega)$, and $s$ is the wave-function squeeze of Eq. (5).] But here one sees a difference. $S(z)$ by itself can be considered to be the displacement operator for the group SU(1,1) defined by

$$K_+ = \frac{1}{2} a^+ a^+, \quad K_- = \frac{1}{2} a a, \quad K_0 = \frac{1}{2} (a^+ a + \frac{1}{2}),$$

so that the $S(z) |0\rangle$ by themselves are the “SU(1,1) coherent states.”
It is not clear how one generalizes DOCS squeezed states to other systems, e.g., to the \( \cosh^{-2} \) potential. (Obviously, squeezed states represent a more complicated symmetry than coherent states.) But the MUCS generalization is clear.

Bob Fisher, Vern Sandberg, and myself looked at the possibility of naively generalizing the harmonic oscillator squeezed states to higher order in \( a \) and \( a^+ \) [15]. It turned out this could not be done. (Once again John’s hand crept in, helping us along the way.) This background [13-15] was later an aid to Alan Kostelecký, Rod Truax, and myself, when we came to the problem of “supersqueezed states.”

5 Supercoherent states

The introduction of graded Lie algebras was an important milestone in the study of combined internal and space-time symmetries. This led to the development of supersymmetric theories which predict the existence of boson and fermion partner states: e.g., for the photon there is a partner photino, etc.

Up to now none of the supersymmetric partners have been found, so that supersymmetry, if extant, is broken. All evidence for supersymmetry has been found in the low energy regime: e.g., in nuclei [16] in atomic systems [17] and in WKB calculations [18]. But these results, although exciting and indicative, do not unequivocally prove the need for a fundamental supersymmetry. Rather, they are tantalizing hints. The proof would be in the discovery of supersymmetric particles, perhaps at the SSC.

Our own interest in physical manifestations of supersymmetry in atomic systems [17], combined with our interest in coherent states, led Kostelecký, Truax, and myself to consider how one should define “supercoherent states.” We came to the conclusion that we should not seek to generalize the minimum-uncertainty method, because just how that should be applied to the fermion sector was unclear.

In the MUCS program, physical intuition had led to the mathematics. But here we felt we should let the mathematics lead us to the physical intuition. That is to say, by using the displacement operator method, but with supergroups, we had a method that was well-defined and which we could use because we had already developed the theory and application of super-BCH relations [19]. This gave us a tool that other workers in the field did not have [20].

Joined by Beata Fatyga, Alan’s graduate student, we derived our general supercoherent states [21]. (As before, John Klauder could be found giving advice.) The definition is the same as the DOCS method for general Lie groups given above, only one uses supergroups and their associated (anti)commutation relations. We presented three examples, with physical models for each: i) The super oscillator algebra defines the supersymmetric harmonic oscillator. ii) A supersymmetric quantum-mechanical system was given which has a Heisenberg-Weyl algebra plus another bosonic degree of freedom. This represents an electron in a constant magnetic field. iii) An OSP(1/2) supersymmetry represents the electron-monopole system.

Of the three examples, the first is the simplest and I use it to demonstrate the technique. The super oscillator algebra is defined by

\[
[a, a^+] = I, \quad \{b, b^+\} = I. \tag{17}
\]
Using super-BCH relations, one obtains that the super displacement operator is
\[
D(g) = \exp[Aa^+ - \overline{A}a + \theta b^+ + \overline{\theta}b] 
\]
\[
= \left( \exp[-\frac{1}{2}|A|^2] \exp[Aa^+] \exp[-\overline{A}a] \right) \left( \exp[-\frac{1}{2}\overline{\theta}\theta] \exp[\theta b^+] \exp[\overline{\theta}b] \right) 
\]
\[
\equiv D_B(A) D_F(\theta). 
\]

The \( B \) and \( F \) subscripts denote the fact that our supersymmetric displacement operator can be written as a product of "boson" and "fermion" (more properly, even and odd) displacement operators.

\( \theta \) and \( \overline{\theta} \) are odd Grassmann numbers. They are nilpotent (they only contain a "soul"). They satisfy anticommutation relations among themselves and with the fermion operators \( b \) and \( b^+ \). \( A \) and \( \overline{A} \) are complex, even, Grassmann numbers. Because \( a^+ \) and \( a \) are pure even elements, we associate the "body" of \( A \) and \( \overline{A} \) with the \( \alpha \) and \( \alpha^* \) of the ordinary coherent states. The "soul" part of \( A \) must be even in the Grassmann numbers. For example, \( A \) could be of the form
\[
A = \alpha + \tau \overline{\theta} \theta, 
\]
although technically \( A \) is not restricted to the two odd basis elements we have.

Explicit calculation yields
\[
D_S(g)|0, 0\rangle = [1 - (1/2)\overline{\theta}\theta]|A, 0\rangle + \theta|A, 1\rangle \equiv |Z\rangle. 
\]

The two labels of \( |0, 0\rangle\) in Eq. (22) represent the even (bosonic) and odd (fermionic) sectors. The bosonic sector has the form of an ordinary coherent state \( |A\rangle \) and the fermionic sector has zero or one fermions. (I refer the reader to Ref. [21] for further details.) If the bosonic displacement is turned off, then the fermionic displacement produces states with zero and one fermion, but no bosons. (In Sec. 7, I will return to an idea on how to physically interpret these states.)

### 6 Supersqueezed states

From the previous sections, it is clear that the supersymmetric generalization of the SU(1,1) squeeze operator of Eqs. (12-13) is what is desired. The group involved is the supergroup OSP(2/2). In addition to the SU(1,1) elements of Eq. (16), it has five more:

\[
M_0 = \frac{1}{2}(b^+b - \frac{1}{2}), \quad Q_1 = \frac{1}{2}a^+b^+, \quad Q_2 = \frac{1}{2}ab, \quad Q_3 = \frac{1}{2}a^+b, \quad Q_4 = \frac{1}{2}ab^+. 
\]
The commutation relations follow, and so the supersqueeze operator can in principle be written as ($\hat{g}$ is the factor algebra)

$$S(g) = \exp \left[ \sum_{i=1}^{6} \alpha_i \hat{g}_i \right] \quad (24)$$

$$= \prod_{i=1}^{8} \exp[\beta_i \hat{g}_i]. \quad (25)$$

The above can be solved by using super-BCH relations. They yield twenty coupled differential equations. (The even operators have separate equations for order zero, two and four Grassmann numbers, and the odd operators have separate equations for orders one and three.)

We have just finished solving these equations and are in the process of performing some final calculations [22]. For now it is useful to note that the squeeze operator can again be separated into a product of bosonic and fermionic pieces:

$$S(g) = S_B(g) S_F(g). \quad (26)$$

The fermionic squeeze operating by itself on $|0,0\rangle$ produces the states $|0,0\rangle$, $|1,1\rangle$, and $|2,0\rangle$.

Therefore, one finally obtains that the supersqueezed states are, in general, of the form

$$T(g)|0,0\rangle = DS|0,0\rangle$$

$$= D_B(A)D_F(\theta)S_B(g)S_F(g)|0,0\rangle = [D_B(A)S_B(g)][D_F(\theta)S_F(g)]|0,0\rangle$$

$$\equiv T_B(g)T_F(g)|0,0\rangle. \quad (27)$$

The general operator produces states with arbitrary numbers of bosons and zero or one fermion.

7 Discussion

If supersymmetric partners were actually found, then fundamentally one would need to give a physical interpretation to Grassmann numbers. This situation would be similar to the problem of the physical interpretation of imaginary numbers when quantum mechanics was discovered. When Schrödinger described his “coherent states” [1], he thought that the physics was contained in the real part of his wave solutions. The realization that the complex phase had physical information came later.

Recently I addressed this problem [23], although I was by no means the first to do so [24]. Looking at the supercoherent states of Eq. (22), I suggested that the odd part describes the existence of a coherent, massless fermion; i.e., a "photino" with energy $E$, it being coherent with the various n-photon states in the bosonic sector.
As to the Grassmann numbers, I made the suggestion that the fermion sector “phase” relative to the boson sector is defined by the $c$ in $\theta \equiv c\zeta$, $\zeta$ being a Grassmann basis vector and labeling the fermionic part of the state. The probability of finding a supercoherent state that has a bosonic sector coherent with one photino is $c^*c$, with the $\overline{\zeta}\zeta$ in $\overline{\theta}\theta$ labeling the probability as being for a fermion. Thus, the probability of finding a bosonic sector without a coherent photino is $(1 - c^*c)$, from $(1 - \overline{\theta}\theta)$. Note that one must have $|c| \leq 1$. Then the probabilities for the coherent state having or not having a fermion are both $\leq 1$.

The restriction on the value of $c$ is analogous to physical restrictions placed on ordinary quantum mechanics. For example, one demands that all state probabilities $P_n = |a_n|^2 \leq 1$ and one disallows unnormalizable solutions of the Schrödinger equation. (See Ref. [23] for more details.) Even so, I must emphasize that at this point this restriction on $c$ is based on physical intuition rather than on mathematical rigor.

Now that we have our supersqueezed states [22], I hope to pursue this line of thought, taking into account the insight available from studying the Grassmann structure of the supersqueezed states. It appears that a similar restriction can be made for the quantities analogous to $c$ in the supersqueezed states. Just as as was the case for Ref. [23], I expect useful comments to come from John Klauder.

Acknowledgements

Of course, in a review of a program such as this, I must once again acknowledge my colleagues and friends with whom I have enjoyed working over the years.

But most all, this is John Klauder’s celebration. As the reader will have noted, the influence of John Klauder can be seen throughout this work. It is an honor, but most of all it is a great personal pleasure, to congratulate John on the occasion of his 60th birthday. I eagerly await our next discussion, perhaps on Grassmann numbers and supersqueezed states, when I can once again hear John say, “But Michael, . . .”

References


