THE LUMINOSITY FOR BEAM DISTRIBUTIONS WITH ERROR AND WAKEFIELD EFFECTS IN LINEAR COLLIDERS

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We derive the expression of the luminosity for three types of beam distributions: Gaussian distributions in the Eulerian coordinates \((x, y)\), Gaussian distributions in the TRANSPORT variables \((x, x', y, y', c\tau, \delta)\), and \(\tau\)-dependent Gaussian distributions in \((x, x', y, y', \delta)\) as occurring in linear colliders when wakefield effects are taken into account. In all cases we assume unequal distributions, with the most general offset vectors and envelope matrices with coupling terms, for the two colliding beams and we do not take the beam-beam forces into account. We illustrate these results with specific examples and by studying the timing jitter and vibration tolerances of the CLIC final focus system.

KEY WORDS: Luminosity, Wakefields, Linear Colliders, CLIC

1 INTRODUCTION

The expression of the luminosity integrated over the head-on collision of two identical bunched beams with Gaussian widths \(\sigma_x\) and \(\sigma_y\), is well known to be

\[
\bar{\mathcal{L}} = \frac{N_1 N_2}{4\pi \sigma_x \sigma_y}
\]

(1)

where \(N_1\) and \(N_2\) are the bunch populations. Generalizations of this expression have been derived in many papers\(^1,2,3,4,5,6,7\) for various cases including unequal beam sizes, transverse offsets, crossing angle, \(xy\) coupling, and ‘hour-glass’ effect for flat beams when the bunch length \(\sigma_z\) can no more be considered small as compared to the depth of focus set by \(\beta^*\). However, to our knowledge, the case of a general Gaussian distribution in 6-dimensional phase space, which includes and further generalizes the preceding examples, has not been studied.

A good reason for deriving the expression of the luminosity for this case can be found, in the context of \(e^+e^-\) linear colliders, in the study of errors. Indeed in a final focus section, the effect of misalignments and field errors is, at the lowest order, to offset the beam centroid and to introduce coupling terms in the 6-dimensional beam matrix.
The analytic calculation of the resulting luminosity reduction would for instance speed up the derivation of tolerance specifications and of correction algorithms, for which it is necessary to repeatedly evaluate the luminosity for a large number of different beam distributions at the IP. The alternative method for calculating the luminosity is based on multiparticle simulation and involves tracking through the imperfect lattice and simulation of the beam-beam luminosity. It is therefore very time consuming. On the other hand, it permits to take into account the luminosity enhancement due to the pinch effect, while the analytic calculation can only be done for vanishing beam-beam forces. However, it has been recently realized that a strong disruption has mostly detrimental consequences like the degradation of the c.m. energy resolution of the collision and the creation of backgrounds of beamstrahlung photons and e⁺-e⁻-pairs. In the designs where the energy resolution is an important issue for physics, multi-bunching and/or flat beams must be used to keep the pinch enhancement factor of the luminosity below 2. Furthermore, the required precision on the luminosity reduction factor is not so high as to prevent one from assuming that the luminosity enhancement factor for perfect beam lines may be used. The analytic calculation of the luminosity, for vanishing beam-beam forces, provides then a useful and rapid estimate of the expected luminosity.

The main purpose of this paper is to derive analytic expressions of the luminosity including the effect of errors, for bunches which are assumed rigid (ie. without disruption) during the interaction taking place in a drift space. The last assumptions are necessary not to spoil the Gaussian nature of the distributions to be analytically integrated in the expression of the luminosity. It is in fact the luminosity integrated over one bunch crossing \( L \) which is of interest. It relates the total interaction rate \( N \) to the cross section \( \sigma \) through

\[
N = \sigma \cdot L
\]  

As shown in Section 1 the integrated luminosity can be written in an explicitly Lorentz-invariant way, as a functional of the two opposing-beam distributions. We have considered here two types of distributions, either Gaussian in the position and velocity, so-called Eulerian coordinates \((x, \bar{v})\), or Gaussian in the TRANSPORT\(^8\) variables \((x, x', y, y', -c\tau, \delta)\) where \(x\) and \(y\) are the transverse coordinates, \(\tau\) is the time delay and \(\delta\) the relative energy difference with respect to the reference trajectory. We have derived the expression of the integrated luminosity for the most general 6-dimensional offset vectors and coupled beam matrices distinct for the two beams. In the case of the Eulerian coordinates, considered in Section 3, the calculation can be performed exactly till the end with the only assumption that one bunch is relativistic. The case of the TRANSPORT coordinates, considered in Section 4, further requires the assumption that both bunches are paraxial with respect to the same axis. It is nevertheless of more practical use. Taking the example of the CLIC\(^9\) final focus system, we analyze first the effect of time jitter and then the effect of horizontal and vertical vibrations on the CLIC luminosity. In the second case a comparison is made with the results obtained by direct tracking simulations.

One of the effects of misalignment in the FFS is the creation of a non-zero dispersion at the IP. The tolerance on this dispersion depends of course on the energy distribution in the bunch. Realistic distributions have to take into account the explicit longitudinal dependence introduced by the longitudinal wakefields in the linac. In fact, transverse
misalignments and chromatic effects in the linac introduce such a dependence also for the beam transverse distributions. In order to investigate this case, we have calculated in Section 5 the expression of the integrated luminosity in terms of the $\tau$-dependent Gaussian distribution in $(x, x', y, y', \delta)$ of two beams exiting from the linacs, and in terms of the maps of the imperfect FFS lattices. The luminosity is given by a double integral and non-Gaussian effects originating from the 2nd and higher order terms in the FFS maps can be treated perturbatively.

Finally, Section 6 contains our conclusions and outlook.

2 GENERALITIES

2.1 Definitions

The first problem one meets when trying to calculate the luminosity for arbitrary distributions, is the one of the definition of the luminosity generalizing the well-known expressions for the special frames of reference where either the center of mass or one of the bunches is at rest. The expression of the interaction rate in an arbitrary reference frame was derived many years ago\textsuperscript{10}. For homogeneous velocity distributions as depicted in Fig. 1, it is given by

$$\frac{d^2N}{dVdt} = \sigma \rho_1 \rho_2 \left[ (\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2} \right]^{1/2}$$

where $\vec{v}_1$ and $\vec{v}_2$ are the velocity vectors of bunch 1 and 2, $\rho_1$ and $\rho_2$ their density, $\sigma$ the cross section and $dV$ the volume element of the interaction region. In this definition it is assumed after Møller\textsuperscript{10}, that the total cross section is Lorentz invariant. Then from the definition of the luminosity $\mathcal{L}$

$$dN = \sigma \mathcal{L} dt$$

the Lorentz invariance of the total interaction rate $N$ translates to the integrated luminosity $\overline{\mathcal{L}}$ which is given by

$$\overline{\mathcal{L}} = \int \mathcal{L} dt = \frac{N}{\sigma} = \int dV \ dt \ \rho_1 \rho_2 \left[ (\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2} \right]^{1/2}$$
Introducing the current-density Lorentz vector

\[ \mathbf{J}(x) = \left( \rho c, \rho \vec{v} \right) \quad \text{with} \quad x = (ct, \vec{x}) \] (6)

the integrated luminosity can be written in a manifestly Lorentz invariant way as

\[ \bar{\mathcal{L}} = \frac{1}{c^2} \int d\mathbf{x} \left[ (\mathbf{J}_1 \cdot \mathbf{J}_2)^2 - \mathbf{J}_1^2 \mathbf{J}_2^2 \right]^{1/2} \] (7)

with \( d\mathbf{x} = cdt \, d\vec{x} \) the 4-dimensional volume element.

This expression can be simplified without spoiling the Lorentz invariance when at least one bunch is relativistic. Since

\[ \mathbf{J}^2 = \rho^2 (c^2 - \vec{v}^2) = \rho^2 c^2 / \gamma^2 \] (8)

the term \( \mathbf{J}_1^2 \cdot \mathbf{J}_2^2 \) in Eq. (7) is of order \( 1/(\gamma_1 \gamma_2)^2 \) and can be neglected to recover the well-known expression

\[ \bar{\mathcal{L}} = \frac{1}{c^2} \int d\mathbf{x} \mathbf{J}_1 \cdot \mathbf{J}_2 = \frac{1}{c^2} \int d\mathbf{x} \rho_1(\vec{x}, t) \rho_2(\vec{x}, t) (c^2 - \vec{v}_1 \cdot \vec{v}_2) \] (9)

For inhomogeneous velocity distributions it is convenient from the point of view of machine physics to assume that the total cross section is constant over the range of variation of the velocities \( \vec{v}_1 \) and \( \vec{v}_2 \), so that it factors out from the expression of the total interaction rate like in Eq. (4). The integrated luminosity, in the relativistic case, is then given by

\[ \bar{\mathcal{L}} = \frac{1}{c^2} \int d\mathbf{x} d\vec{v}_1 d\vec{v}_2 \rho_1(\vec{x}, \vec{v}_1, t) \rho_2(\vec{x}, \vec{v}_2, t) (c^2 - \vec{v}_1 \cdot \vec{v}_2) \] (10)

with the bunch densities \( \rho_1 \) and \( \rho_2 \) re-defined such that

\[ \int d\vec{x} d\vec{v} \rho_n(\vec{x}, \vec{v}, t) = N_n \quad n = 1, 2 \] (11)

The bunch densities \( \rho_n \) in Eq. (10) can be contemplated from two points of view: either as \textit{time-dependent} distributions in the Eulerian coordinates \( (\vec{x}, \vec{v}) \) or, in order to make the connection with the classical beam optics formalism, as \textit{z-dependent} distributions in the TRANSPORT variables, with \( z \) the coordinate along a given longitudinal axis.

### 2.2 The Eulerian coordinate point of view

The time dependence of the distribution is emphasized by writing

\[ \rho(\vec{x}, \vec{v}, t) = \rho(X, t) \quad \text{with} \quad X = \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix} \] (12)

In a drift space, and neglecting the beam-beam forces, one can relate the beam distribution at time \( t \) to the distribution at a given origin in time \( t = 0 \) by

\[ \rho(X, t) = \rho(\mathbb{T}_t^{-1} \cdot X, t = 0) := \rho^*(\mathbb{T}_t^{-1} \cdot X) \] (13)
where the 6x6 matrix $T_t$ describes the particle motion in free space over a time $t$

$$T_t \cdot X = \begin{pmatrix} x + t \vec{v} \\ \vec{v} \end{pmatrix}$$  (14)

The consequence is that if $\rho^*(X)$ is a Gaussian distribution in $X$, so is $\rho(X,t)$ and the luminosity can be integrated analytically as shown in Section 3.

### 2.3 The TRANSPORT coordinate point of view

In this case, we consider directly the case of two relativistic beams. The dependence on the longitudinal coordinate $z$ is singled out by writing

$$x' = \frac{x}{v_x/v_z}$$

$$y' = \frac{y}{v_y/v_z}$$

$$-c \tau = z - ct$$

$$\delta = E/E_0 - 1$$

where $\tau$ is the time delay with respect of the reference trajectory in the forward $z$-direction, and $\delta$ is the relative energy difference with respect to the nominal energy $E_0$. Assuming again a free motion in a drift space, the beam distribution at the coordinate $z$ is related to the one at a given origin $z = 0$ by

$$\tilde{\rho}(Y,z) = \tilde{\rho}(T_z^{-1}(Y), z = 0) := \tilde{\rho}^*(T_z^{-1}(Y))$$  (16)

where $T_z$ maps the coordinates $Y$ from $z_0$ to $z_0 + z$, namely

$$T_z(Y) = \begin{pmatrix} x + x'z \\ x' \\ y + y'z \\ y' \\ -c \tau - cz(v_z^{-1} - c^{-1}) \\ \delta \end{pmatrix}$$  (17)

The linearity of the map $T_z(Y)$ breaks down in the transformation of the fifth coordinate since $1/v_z$ is a non-linear function of $x', y'$ and $\delta$. It is only when both beams are paraxial, i.e. for $x', y' \ll 1$, with respect to a common axis that one can replace $T_z(Y)$ by the linear transformation $T_z$ defined by

$$T_z \cdot Y = \begin{pmatrix} x + x'z \\ x' \\ y + y'z \\ y' \\ -c \tau \\ \delta \end{pmatrix}$$  (18)

Under this assumption, Gaussian distributions in $Y$ stay Gaussian over the longitudinal evolution and the analytic calculation of the luminosity for Gaussian beams is possible, as shown in Section 4. However, one loses the possibility of treating cases with large crossing angles.
2.4 The difference between the two points of view

It is important, before going into the calculations, to understand that the two points of view presented above are actually different, i.e. that Gaussian distributions in Eulerian coordinates are not equivalent to Gaussian distributions in TRANSPORT coordinates. It is enough to use simple examples where we consider only one transverse dimension. A typical Gaussian distribution in the Eulerian coordinates for a bunch moving along the z-axis with velocity \( c \) is, up to a normalization factor, given by

\[
\rho_E(x, v_x, z, t) = \exp \left( -\frac{(x - v_x t)^2}{2\sigma_x^2} - \frac{(z - ct)^2}{2\sigma_z^2} - \frac{v_x^2}{2\sigma_{v_x}^2} \right)
\]  

(19)

At any given time \( t \) it describes a Gaussian bunch in \((x, v_x, z)\).

On the other hand, a typical Gaussian distribution in the TRANSPORT coordinates for a bunch moving in the same way, is given by

\[
\rho_T(x', x, z, t) = \exp \left( -\frac{(x - x' z)^2}{2\sigma_x^2} - \frac{(z - ct)^2}{2\sigma_z^2} - \frac{x'^2}{2\sigma_{x'}^2} \right)
\]  

(20)

The essential difference is that at a given time \( t \), it is not a Gaussian distribution in \((x, x' = v_x/c, z)\) because of the product \( x' z \). It is therefore not equivalent to \( \rho_E \). In particular the bunch described by \( \rho_T \) shows the typical hour-glass dependence of its transverse width upon \( z \). On the contrary \( \rho_T \) is a \( z \)-dependent Gaussian distribution in \((x, x', \tau)\) which is more familiar to accelerator optics. The issue of knowing which is the most valid description of an actual bunch is not the subject of this paper, since it depends essentially on the type of accelerator considered.

There is however a case when the two types of distributions are equivalent, namely for isotropic beams. The homogeneous velocity distribution is then described by a \( \delta \)-function which is the limit of a Gaussian function for zero standard deviation. The most general offset vector and beam matrix are restricted to the 3-dimensional \((x, y, z - ct)\) coordinate space, which shows clearly that both points of view can be adopted in this case. The expression of the integrated luminosity for isotropic beams is derived in Section(3.2.2) while examples involving isotropic beams are given in Sections(3.3.1-3).

2.5 Notations and convention

In this paper we use arrows for 3-dimensional vectors like \( \vec{x} \), bold-face character for 4-dimensional Lorentz vectors like \( x \), and capital letters for 6-dimensional vectors (or 5-dimensional ones in section 5) like \( X \). For matrices, we use capital letters like \( \Sigma \) or \( P \) for operators in or into 3-dimensional space, and like \( T \) for operators in 6-dimensional space (or 5-dimensional ones in section 5). \( M^T \) stands for the transpose of \( M \).

Moreover in the following sections, Gaussian distributions for beam 2 will be characterized by the offset vectors and beam matrices of the distributions symmetrized with respect to transverse plane containing the interaction point. For instance, the case of symmetric beams will be described by identical parameters for beam 1 and beam 2, including the average velocity. This convention is adopted to facilitate the calculation of the luminosity reduction factor when both beam parameter sets are calculated by an optics program for two independent beams moving in the same direction.
3 THE LUMINOSITY FOR COUPLED AND OFFSET GAUSSIAN DISTRIBUTIONS IN EULERIAN COORDINATES

3.1 Calculation of the luminosity

With the notation introduced in Eq. (12), namely \( X = (\vec{x}, \vec{v}) \), the integrated luminosity given by Eq. (10) becomes

\[
\overline{\mathcal{L}} = \frac{1}{c^2} \int c dt \, d\vec{x}_1 \, d\vec{x}_2 \, d\vec{v}_1 \, d\vec{v}_2 \, \delta(\vec{x}_1 - \vec{x}_2) \rho_1(X_1, t) \rho_2(X_2, t) \, (c^2 - \vec{v}_1 \cdot \vec{v}_2)
\]  

(21)

Using the Fourier representation of the \( \delta \)-function, one gets

\[
\overline{\mathcal{L}} = \frac{1}{c^2(2\pi)^3} \int c dt \, d\vec{p} \, J_1(\vec{p}, t) \cdot J_2(\vec{p}, t)
\]  

(22)

where \( J_n(\vec{p}, t) \) is the Fourier transform of the current density \( J_n(X, t) \) for vanishing conjugate variables of \( \vec{v} \), namely

\[
J_n(\vec{p}, t) = \int dX \, \left( \frac{c}{\nu} \right) \rho_n(X, t) \exp(i\vec{p} \cdot \vec{x}) \quad n = 1, 2
\]  

(23)

The beam distributions at the IP, i.e. at a given origin in time \( t = 0 \), are assumed Gaussian with the most general offset vector \( X^o = (\vec{x}^o, \vec{v}^o) \) and beam matrix \( S \)

\[
\rho_n^*(X^*) = \frac{N_n}{(2\pi)^3 \det^{1/2}(S_n)} \exp \left[ -\frac{1}{2} (X^* - X^o_n)^\top S_n^{-1} \cdot (X^* - X^o_n) \right] \quad n = 1, 2
\]  

(24)

In view of the convention adopted in Section(2.5) for beam 2, the time-dependent distributions \( \rho_n(X, t) \) can be deduced from \( \rho_n^*(X^*) \) by

\[
\begin{align*}
\rho_1(X, t) &= \rho_1^*(X^*) \\
\rho_2(X, t) &= \rho_2^*(P_z \cdot X^*)
\end{align*}
\]  

with \( X = \mathbb{T}_t \cdot X^* \)  

(25)

where \( \mathbb{P}_z \) is the symmetry operator which flips the sign of the coordinates \( z \) and \( v_z \) in \( X \), and \( \mathbb{T}_t \) is given by Eq. (14).

The calculation of \( J_1(\vec{p}, t) \) and \( J_2(\vec{p}, t) \) requires only the evaluation of 6-dimensional Gaussian integrals. It leads to

\[
J_n(\vec{p}, t) = N_n \left( \frac{c}{\vec{j}_n(\vec{p}, t)} \right) \exp \left( i\vec{p} \cdot \vec{A}_n(t) - \frac{1}{2} \vec{p} \cdot \Sigma_n(t) \cdot \vec{p} \right) \quad n = 1, 2
\]  

(26)

with the following definitions of the 3-dimensional vectors \( \vec{j}_n(\vec{p}, t) \)

\[
\begin{align*}
\vec{j}_1(\vec{p}, t) &= P_v \cdot (X_1^o + iS_1 \cdot \mathbb{T}_t^\top \cdot P_x^\top \cdot \vec{p}) \\
\vec{j}_2(\vec{p}, t) &= P_v \cdot \mathbb{P}_z \cdot (X_2^o + iS_2 \cdot \mathbb{T}_t^\top \cdot \mathbb{P}_z \cdot P_x^\top \cdot \vec{p})
\end{align*}
\]  

(27)
and of the 3-dimensional vectors $\tilde{A}_n(t)$ and square matrices $\Sigma_n(t)$

$$
\tilde{A}_1(t) = P_x \cdot T_t \cdot X_1 = \tilde{x}_1^o + \tilde{v}_1^o t
$$

$$
\tilde{A}_2(t) = P_x \cdot P_z \cdot T_t \cdot X_2 = P_z \cdot (\tilde{x}_2^o + \tilde{v}_2^o t)
$$

$$
\Sigma_1(t) = P_x \cdot T_t \cdot S_1 \cdot T_t^T \cdot P_x^T
$$

$$
\Sigma_2(t) = P_x \cdot P_z \cdot T_t \cdot S_2 \cdot T_t^T \cdot P_z \cdot P_x^T
$$

In these expressions $P_x$ and $P_v$ are the projection operators defined by

$$
P_x \cdot X = \tilde{x}, \quad P_v \cdot X = \tilde{v}
$$

and $P_z$ is the 3-dimensional symmetry operator which flips the sign of the coordinate $z$ in $\tilde{x}$ or $\tilde{v}_z$ in $\tilde{v}$. It is easy to show that the following identities $P_z \cdot T_t = T_t \cdot P_z$ and $P_v \cdot T_t = P_v$ hold.

Substituting Eq. (26) into Eq. (22) leads to

$$
\overline{\mathcal{E}} = \frac{N_1 N_2}{c^2 (2\pi)^3} \int \frac{d\vec{p}}{d\Omega} \exp \left( i \vec{p} \cdot \tilde{A}(t) - \frac{1}{2} \vec{p} \cdot \Sigma(t) \cdot \vec{p} \right) \times \left( c^2 - \tilde{v}_1^o \cdot P_z \cdot \tilde{v}_2^o - i \vec{p} \cdot \tilde{B}(t) - \vec{p} \cdot \Pi(t) \cdot \vec{p} \right)
$$

with

$$
\tilde{A}(t) = \tilde{A}_1(t) - \tilde{A}_2(t)
$$

$$
\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)
$$

and

$$
\tilde{B}(t) = P_x \cdot T_t \cdot (S_1 \cdot P_v \cdot P_z \cdot X_2^o - P_z \cdot S_2 \cdot P_z \cdot P_v \cdot X_1^o)
$$

$$
\Pi(t) = P_x \cdot T_t \cdot S_1 \cdot P_v \cdot P_z \cdot S_2 \cdot T_t^T \cdot P_x^T
$$

Integrating over $\vec{p}$, one gets the general expression of the integrated luminosity when at least one beam is relativistic

$$
\overline{\mathcal{E}} = \frac{N_1 N_2}{c (2\pi)^{3/2}} \int dt \left( c^2 + v_1^o v_2^o \cos \alpha + \tilde{B}(t) \cdot \Sigma(t)^{-1} \cdot \tilde{A}(t) - \text{tr} \left[ \Pi(t) \cdot \Sigma(t)^{-1} \right] \right)

+ \tilde{A}(t) \cdot \Sigma(t)^{-1} \cdot \Pi(t) \cdot \Sigma(t)^{-1} \cdot \tilde{A}(t) \exp \left( -\frac{1}{2} \tilde{A}(t) \cdot \Sigma(t)^{-1} \cdot \tilde{A}(t) \right) / \sqrt{\det \Sigma(t)}
$$

with $\alpha$ the average crossing angle determined by $\tilde{v}_1^o$ and $\tilde{v}_2^o$. The integral over $t$ can easily be done numerically since the integrand is proportional to the overlap of the two distributions and therefore decays exponentially.

### 3.2 Simplifications

We now consider simpler situations where both beams have a small angular divergence.
3.2.1 Beams with small angular opening

We first assume that each beam is relativistic and paraxial with respect to its average propagation axis, namely

\[(v_{x,y} - v_{x,y}^0) \ll c\]  \hspace{1cm} (34)

The second assumption is easily fulfilled when the transverse emittance is small with respect to the transverse spot size of the beams. In this case one can approximate

\[\vec{v}_1 \cdot \vec{v}_2 \simeq -c^2 \cos \alpha\]  \hspace{1cm} (35)

in Eq. (10), leading to

\[\vec{L} = (1 + \cos \alpha) \int dx \, d\vec{v}_1 \, d\vec{v}_2 \, \rho_1(\vec{x}, \vec{v}_1, t) \rho_2(\vec{x}, \vec{v}_2, t)\]  \hspace{1cm} (36)

It is easy to see that Eq. (33) then simplifies as follows

\[\vec{L} = \frac{N_1N_2}{(2\pi)^{3/2}} (1 + \cos \alpha) \int cdt \, \exp \left( -\frac{1}{2} \vec{A}(t) \cdot \Sigma(t)^{-1} \cdot \vec{A}(t) \right) / \sqrt{\det \Sigma(t)}\]  \hspace{1cm} (37)

3.2.2 Isotropic beams

This is the extreme case where all particles have the same velocity. This implies that transverse emittances are zero because all trajectories are parallel and also entails that no hour-glass effect appears. Contrary to the preceding section we do not assume both beams to be relativistic anymore. The distribution for isotropic beams is given by

\[\rho_n(X^\ast) = \frac{N_n}{(2\pi)^{3/2} \det^{1/2} S_n} \frac{\delta(\vec{v}_1 - \vec{v}_n^0)}{\delta(\vec{x} - \vec{x}_n^0)} \exp \left[ -\frac{1}{2} (\vec{x} - \vec{x}_n^0) \cdot S_n^{-1} \cdot (\vec{x} - \vec{x}_n^0) \right] \quad n = 1, 2\]  \hspace{1cm} (38)

in such a way that the 6-dimensional beam matrix \(S_n\) can be written in term of the 3 x 3 matrix \(S_n\) as

\[S_n = \begin{pmatrix} S_n & 0 \\ 0 & 0 \end{pmatrix}\]  \hspace{1cm} (39)

From Eqs.(28,31) the matrix \(\Sigma(t)\) is then independent of time and is given by

\[\Sigma = S_1 + P_z \cdot S_2 \cdot P_z\]  \hspace{1cm} (40)

while the vector \(\vec{A}(t)\) can be decomposed as

\[\vec{A}(t) = \vec{A}_x + t\vec{A}_v\]  \hspace{1cm} (41)

with

\[\vec{A}_x = \vec{x}_1^0 - P_z \cdot \vec{x}_2^0, \quad \vec{A}_v = \vec{v}_1^0 - P_z \cdot \vec{v}_2^0\]  \hspace{1cm} (42)

It is also easy to show that

\[\Pi(t) = 0 \quad \text{and} \quad \vec{B}(t) = 0\]  \hspace{1cm} (43)
FIGURE 2: Collision with crab crossing

The integrated luminosity is given in Eq. (33) by a Gaussian integral leading to

\[ \bar{\mathcal{L}} = \frac{N_1 N_2}{(2\pi)^{3/2}c^2} \left( \frac{c^2 + \rho_1 \rho_2 \cos \alpha}{\Sigma^{-1} \cdot \bar{\mathcal{A}}_v} \right) \left( \frac{\det \Sigma}{\bar{\mathcal{A}}_v \cdot \Sigma^{-1} \cdot \bar{\mathcal{A}}_v} \right)^{1/2} \times \]
\[ \exp \left( -\frac{1}{2} \left[ (\bar{\mathcal{A}}_x \cdot \Sigma^{-1} \cdot \bar{\mathcal{A}}_x) - (\bar{\mathcal{A}}_r \cdot \Sigma^{-1} \cdot \bar{\mathcal{A}}_r)^2/(\bar{\mathcal{A}}_v \cdot \Sigma^{-1} \cdot \bar{\mathcal{A}}_v) \right] \right) \]  

(44)

3.3 Applications

3.3.1 Crab-crossing luminosity  We consider the head-on collision of identical relativistic beams with crab-crossing angle \( \alpha/2 \) as depicted in Fig. 2. The beam distributions are supposed to be isotropic and diagonal in space, in such a way that

\[ \bar{\mathcal{A}}_x = 0 \quad \bar{\mathcal{A}}_v = \begin{pmatrix} 0 \\ 0 \\ 2c \cos(\alpha/2) \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix} \]  

(45)

Eq. (44) then leads to

\[ \bar{\mathcal{L}} = \frac{N_1 N_2}{4\pi \sigma_x \sigma_y} \cos(\alpha/2) = \frac{N_1 N_2}{4\pi (\sigma_x)_{cm} \sigma_y} \]  

(46)

where \( (\sigma_x)_{cm} = \sigma_x / \cos(\alpha/2) \) is the beam width as seen in the c.m. frame of reference. This is in agreement with the Lorentz invariance of the integrated luminosity.

3.3.2 Crossing angle and transverse offset  We now consider the collision with an ordinary crossing angle as shown in Fig. 3, of two isotropic beams with identical beam
We also introduce a 3-dimensional offset for both beams given by $x_1^\circ$ and $x_2^\circ$. Eq. (42) then gives

$$A_x = \left( \begin{array}{c} x_1^\circ - x_2^\circ \\ y_1^\circ - y_2^\circ \\ z_1^\circ + z_2^\circ \end{array} \right) = \left( \begin{array}{c} \delta x^\circ \\ \delta y^\circ \\ \delta z^\circ \end{array} \right)$$

while $A_v$ is still given by Eq. (45). Introducing the matrix $R_{\alpha/2}$ which describes the rotation of angle $\alpha/2$ in the $xz$ plane, one can write

$$S_1 = S_2 = R_{\alpha/2} \cdot S \cdot R_{\alpha/2}^{-1}$$

in such a way that

$$\Sigma = R_{\alpha/2} \cdot S \cdot R_{\alpha/2}^{-1} + R_{\alpha/2}^{-1} \cdot S \cdot R_{\alpha/2}$$

$$= 2 \left( \begin{array}{ccc} \sigma_x^2 \cos^2(\alpha/2) + \sigma_z^2 \sin^2(\alpha/2) & 0 & 0 \\ 0 & \sigma_z^2 & 0 \\ 0 & 0 & \sigma_z^2 \cos^2(\alpha/2) + \sigma_x^2 \sin^2(\alpha/2) \end{array} \right)$$

where we have used the fact that $P_z \cdot R_{\alpha/2} = R_{\alpha/2}^{-1} \cdot P_z$. After some algebra, one gets

$$\bar{L} = \frac{N_1 N_2}{4\pi \sigma_x \sigma_y} \frac{1}{\sqrt{1 + (\sigma_z \tan(\alpha/2))/\sigma_x^2}} \times$$
\[
\exp \left[ -\frac{\left( \frac{\Delta x^\circ}{2} \right)^2}{\sigma_x^2 \cos^2(\alpha/2) + \sigma_y^2 \sin^2(\alpha/2)} + \frac{1}{\sigma_y^2} \right] \quad (51)
\]

Notice that the luminosity as expected does not depend on the longitudinal offsets \( z_1^\circ \) and \( z_2^\circ \).

### 3.3.3 Coupling in xy-plane

We consider the collision at zero crossing angle of two isotropic beams with transverse offset and \( xy \) coupling. From Eq. (42) one can write

\[
\vec{A}_x = \begin{pmatrix} x_1^\circ - x_2^\circ \\ y_1^\circ - y_2^\circ \end{pmatrix} = \begin{pmatrix} \delta x_1^\circ \\ \delta y_1^\circ \end{pmatrix} \quad \text{and} \quad \vec{A}_v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (52)
\]

We assume that the beam matrices \( S_n \) are given by

\[
S_n = \begin{pmatrix} S_{\perp,n} & 0 \\ 0 & \sigma_z^2 \end{pmatrix}
\]

where \( S_{\perp,n} \) are coupled matrices in the transverse space. From Eq. (40) one gets

\[
\Sigma = \begin{pmatrix} S_{\perp,1} + S_{\perp,2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Sigma_\perp \\ 0 \\ 0 \end{pmatrix} \quad (54)
\]

The calculation of the integrated luminosity from Eq. (44) then leads to

\[
\bar{L} = \frac{N_1 N_2}{2\pi} \exp \left( -\frac{1}{2} \frac{1}{\Delta x_1^\circ \cdot \Sigma_\perp^{-1} \cdot \Delta x_1^\circ} \right) / \sqrt{\det \Sigma_\perp} \quad (55)
\]

This expression was already obtained in 6.

### 3.3.4 The 3-dimensional hour-glass effect

As a final example we consider the case of the head-on collision of two relativistic beams with different diagonal 6-dimensional beam matrices. By taking into account the finite transverse and longitudinal emittances of the beams, this example generalizes the 1-dimensional hour-glass effect discussed in the introduction to 3 dimensions. For head-on collisions one has

\[
\vec{A}_x = 0 \quad \text{and} \quad \vec{A}_v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{A}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (56)
\]

The 6-dimensional beam matrices can be written, skipping the off-diagonal elements, as

\[
S_n = (\sigma_x^2 \quad \sigma_y^2 \quad \sigma_z^2 \quad \sigma_{x} \quad \sigma_{y} \quad \sigma_{z})
\]

For diagonal matrices \( S_n \) it is easy to get from the definitions in Eqs.(28,31)

\[
\Sigma(t) = 2 \begin{pmatrix} \tilde{\sigma}_x^2 + t^2 \tilde{\sigma}_{x}^2 \\ 0 \\ \tilde{\sigma}_y^2 + t^2 \tilde{\sigma}_{y}^2 \\ 0 \\ 0 \\ \tilde{\sigma}_z^2 + t^2 \tilde{\sigma}_{z}^2 \end{pmatrix} \quad (58)
\]
with the notation
\[ \sigma^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \] (59)

After some algebra, Eq. (37) leads to

\[ \bar{\mathcal{L}} = \frac{N_1 N_2}{4\pi \sigma_x \sigma_y} H_A(\bar{\Delta}_x, \bar{\Delta}_y, \bar{\Delta}_z) \] (60)

where the reduction factor \( H_A \) is given by the integral

\[ H_A(\bar{\Delta}_x, \bar{\Delta}_y, \bar{\Delta}_z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \frac{\exp \left(-u^2/(1 + u^2\bar{\Delta}_z^2)\right)}{\sqrt{(1 + u^2\bar{\Delta}_x^2)(1 + u^2\bar{\Delta}_y^2)(1 + u^2\bar{\Delta}_z^2)}} \] (61)

with
\[ \bar{\Delta}_x = \frac{\sigma_z \sigma_{v_x}}{c \sigma_x} , \quad \bar{\Delta}_y = \frac{\sigma_z \sigma_{v_y}}{c \sigma_y} , \quad \bar{\Delta}_z = \frac{\sigma_{v_z}}{c} \] (62)

4 THE LUMINOSITY FOR COUPLED AND OFFSET GAUSSIAN DISTRIBUTIONS IN TRANSPORT COORDINATES

4.1 Calculation of the luminosity

The calculation of the luminosity is similar although much simpler than for the Eulerian coordinates. Indeed with the assumption of paraxial relativistic beams made in Section (2.3), the integrated luminosity is simply given from Eq. (10) by

\[ \bar{\mathcal{L}} = 2 \int cdt \: d\vec{x} \: d\vec{v}_1 \: d\vec{v}_2 \: \rho_1(\vec{x}, \vec{v}_1, t) \: \rho_2(\vec{x}, \vec{v}_2, t) \] (63)

In a first step towards the TRANSPORT variables, we introduce the vector

\[ \vec{u} = \left( \begin{array}{c} x \\ y \\ -ct \end{array} \right) \] (64)

so that one can rewrite the integrated luminosity as

\[ \bar{\mathcal{L}} = 2 \int dz \: d\vec{u}_1 \: d\vec{u}_2 \: d\vec{v}_1 \: d\vec{v}_2 \: \rho_1(\vec{u}_1, \vec{v}_1, z) \: \rho_2(\vec{u}_2, \vec{v}_2, z) \: \delta(\vec{u}_1 - \vec{u}_2) \] (65)

Using the Fourier representation of \( \delta \)-function, one gets

\[ \bar{\mathcal{L}} = \frac{2}{(2\pi)^3} \int dz \: d\vec{p} \: I_1(\vec{p}, z) \: I_2^*(\vec{p}, z) \] (66)

with \( I_n(\vec{p}, z) \) a partial Fourier transform of the bunch density \( \rho_n(\vec{u}, \vec{v}, z) \)

\[ I_n(\vec{p}, z) = \int d\vec{u} \: d\vec{v} \: \rho_n(\vec{u}, \vec{v}, z) \: \exp(i\vec{p} \cdot \vec{u}) \quad n = 1, 2 \] (67)
The beam distributions \( \rho_n(\vec{u}, \vec{v}, z) \) can be calculated from the ones at the IP, i.e. at a given origin \( z = 0 \). The latter are assumed Gaussian in TRANSPORT coordinates and are given with the notations of Section 2.3, by

\[
\tilde{\rho}_n^*(Y^*) = \frac{N_n}{(2\pi)^{3/2} |S_n|} \exp \left[ -\frac{1}{2} (Y^* - Y_n^o) \cdot S_n^{-1} \cdot (Y^* - Y_n^o) \right] \quad n = 1, 2
\]  

(68)

where \( Y_n^o \) and \( S_n \) are the 6-dimensional offset vectors and beam matrices at the IP. The change of variables from \((\vec{u}, \vec{v})\) to \( Y^* \) is such that

\[
\exp(i\vec{\rho} \cdot \vec{u}) = \exp(\mp ip_z z) \exp(i\vec{\rho} \cdot P_u \cdot T_{\pm z} \cdot Y^*)
\]

(69)

where \( T_z \) is given in Eq. (18) and \( P_u \) is the projection operator defined by

\[
P_u \cdot Y = \begin{pmatrix} x \\ y \\ -cT \end{pmatrix}
\]

(70)

The sign in Eq. (69) and the following one, depends on the direction of propagation of the beams: it is the upper sign for beam 1 moving forward in \( z \), and the lower one for beam 2 moving backward. The integration over \( Y^* \) is then Gaussian and leads to

\[
I_n(\vec{p}, z) = N_n \exp \left( \mp ip_z z + i\vec{\rho} \cdot P_u \cdot T_{\pm z} \cdot Y_n^o \right) \times
\]

\[
\exp \left( -\frac{1}{2} \vec{\rho} \cdot P_u \cdot T_{\pm z} \cdot S_n \cdot P_u \cdot \vec{\rho} \right) \quad n = 1, 2
\]

(71)

Once again the integration over \( \vec{\rho} \) in Eq. (66) is Gaussian and leads to the expression of the luminosity for general Gaussian distributions in TRANSPORT coordinates with offset and coupling, namely

\[
\overline{\mathcal{L}} = \frac{2N_1 N_2}{(2\pi)^{3/2}} \int dz \exp \left( -\frac{1}{2} \tilde{A}(z) \cdot \Sigma(z)^{-1} \cdot \tilde{A}(z) \right) / \sqrt{\det \Sigma(z)}
\]

(72)

where the 3-dimensional vector \( \tilde{A}(z) \) and matrix \( \Sigma(z) \) are given by

\[
\tilde{A}(z) = \begin{pmatrix} x_1^o + x_1'z - x_2^o - x_2'z \\ y_1^o + y_1'z - y_2^o - y_2'z \\ cT_2^o - cT_1^o - 2z \end{pmatrix}
\]

(73)

and

\[
\Sigma(z) = P_u \cdot (T_z \cdot S_1 \cdot T_z^T + T_{-z} \cdot S_2 \cdot T_{-z}^T) \cdot P_u^T
\]

(74)

It is important to notice that the operators \( P_u \cdot T_{\pm z} \) project the beam matrices \( S_1 \) and \( S_2 \) onto the 5x5 sub-matrices acting in the space of the coordinates \((x, x', y, y', -cT)\), so that neither \( \Sigma(z) \) nor \( \overline{\mathcal{L}} \) actually depend on the matrix elements related to the 6th dimension, i.e. energy. In the case of isotropic beams the relevant part of the beams matrices further reduces to their projection \( P_u \cdot S_n \cdot P_u^T \) onto the 3-dimensional \( \vec{u} \) vector space. As discussed in Section 2.4 in this case Eulerian and TRANSPORT Gaussian distributions are equivalent.
4.2 Applications

4.2.1 The 2-dimensional hour-glass effect with time jitter  We consider the head-on collision of two beams with a 6-dimensional diagonal beam matrices given by

$$S_n = (\sigma_x^2 \quad \sigma_y^2 \quad \sigma_z^2 \quad \sigma_x^2 \quad \sigma_y^2 \quad \sigma_z^2)_n$$  \(75\)

where we skipped the zero off-diagonal elements. We also assume that the offset vectors \(Y^o_n\) are zero but for their 5th component \(-c\tau^o_n\) describing a time delay at the IP. With these assumptions Eqs.(73,74) lead straightforwardly to

$$\vec{A}(z) = \begin{pmatrix} 0 \\ 0 \\ c\delta\tau^o - 2z \end{pmatrix}$$  \(76\)

with \(\delta\tau^o = \tau^o_2 - \tau^o_1\), and

$$\Sigma(t) = 2 \begin{pmatrix} \sigma_x^2 + z^2 \sigma_x' \quad 0 \\ 0 \\ \sigma_y^2 + z^2 \sigma_y' \quad 0 \\ 0 \\ 0 \end{pmatrix}$$  \(77\)

with the notation defined in Eq. (59). Substituting Eqs. (76, 77) into the expression of the integrated luminosity given by Eq. (72) leads to

$$\overline{\mathcal{L}} = \frac{N_1 N_2}{4\pi\sigma_x\sigma_y} H_A(\tilde{A}_x, \tilde{A}_y, B_t)$$  \(78\)

where the reduction factor \(H_A(\tilde{A}_x, \tilde{A}_y, B_t)\) due to the hour-glass effect in the transverse dimensions, is given by the integral

$$H_A(\tilde{A}_x, \tilde{A}_y, B_t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \frac{\exp\left(-\frac{(u - B_t)^2}{2}\right)}{\sqrt{(1 + u^2\tilde{A}_x^2)(1 + u^2\tilde{A}_y^2)}}$$  \(79\)

with

$$\tilde{A}_x = \frac{\tilde{\sigma}_x \tilde{\sigma}_x'}{\sigma_x} = \quad \tilde{A}_y = \frac{\tilde{\sigma}_y \tilde{\sigma}_y'}{\sigma_y} = \quad \tilde{B}_t = \frac{c\delta\tau^o}{2\sigma_z}$$  \(80\)

For \(B_t = 0\), this reduction factor coincides with the one obtained\(^7\).

Assuming that \(\tilde{A}_x \simeq 0\) as is the case for flat beams when the horizontal beta-function \(\beta^*_x\) at the IP is much bigger than the bunch length \(\sigma_z\), we recover for vanishing time delay \(\delta\tau^o = 0\), the closed form already found in \(^3\) namely

$$H_A(0, \tilde{A}_y, 0) = \frac{1}{\sqrt{\pi\tilde{A}_y}} \exp(1/2\tilde{A}_y^2) K_0(1/2\tilde{A}_y^2)$$  \(81\)

where \(K_0\) is the modified Bessel function of order zero.

If we consider now a time jitter described by a Gaussian distribution of time delays \(\tau^o_1\) and \(\tau^o_2\) with a standard deviation \(\sigma_t\), the luminosity is given by

$$\mathcal{L} = \int d\tau^o_1 \quad d\tau^o_2 \exp\left(-\frac{(\tau^o_1)^2}{2\sigma_t^2}\right) \exp\left(-\frac{(\tau^o_2)^2}{2\sigma_t^2}\right) \overline{\mathcal{L}}(\delta\tau^o)$$  \(82\)
where \( f \) is the repetition rate of the collisions and \( \mathcal{L}(\delta \tau^0) \) is the luminosity integrated of one bunch crossing for a relative time delay \( \delta \tau^0 \). With the integrated luminosity given by Eq. (78), the average reduction factor \( \mathcal{H}_A = \mathcal{L}/\mathcal{L}_0 \) is given by

\[
\mathcal{H}_A(A_x, A_y, u = B) = \frac{1}{2\sqrt{\pi} \sigma_t} \int d(\delta \tau^0) \exp \left( \frac{(\delta \tau^0)^2}{4\sigma_t^2} \right) \mathcal{H}_A(A_x, A_y, B)
\]

with \( C_t = c\sigma_x/\sigma_z \).

To illustrate this relation we have considered the 500 GeV c.m. energy CLIC parameters\(^1\) at the IP, namely

\[
\begin{align*}
\sigma_x &= 90 \text{ nm} \\
\sigma_y &= 8 \text{ nm} \\
\sigma_z &= 170 \mu\text{m}
\end{align*}
\]

and

\[
\begin{align*}
\beta_x^* &= 2.20 \text{ mm} \\
\beta_y^* &= 157 \mu\text{m}
\end{align*}
\]

The luminosity reduction factor \( H_A \) and the average reduction factor \( \overline{H}_A \) are plotted in Fig. 4 as functions of \( B \) and \( C_t \). As expected the reduction factor averaged over Gaussian distributed time delays, decreases less rapidly than the one for a single bunch crossing. The two curves intersect on the vertical axis around 0.85, the value of the 2-dimensional hour-glass reduction factor for vanishing time delay, i.e. for perfectly synchronized beams.
4.2.2 Vibration tolerances for the CLIC final focus system To illustrate the analytic calculation of the integrated luminosity, we investigate the dependence of the luminosity upon the amplitude of ground vibrations in the CLIC final focus section\textsuperscript{12} for the same set of parameters as in Eq. (84). These vibrations introduce errors in the optics and in the transfer map of the FFS. Assuming perfect bunches at the entrance of the FFS and truncating the map up to first order, i.e. keeping the constant and linear parts of the map, these errors propagate into offset and coupling terms for the beam distribution at the IP, which are then used to calculate the luminosity according to Eq. (72). We have considered independent sets of errors for the two final focus sections and hence for the two colliding beams. We have used the code MAD\textsuperscript{13} to generate the errors and to calculate the transfer matrix by tracking a few trajectories.

Fig. 5 and Fig. 6 show the dependence of the luminosity with respect to the amplitude of the horizontal and vertical vibrations assumed equal for 28 quadrupoles of the line, except for the last doublet which needs a special treatment and which is kept fixed. In both figures the analytic results averaged over 10 different seeds, are compared with the ones obtained with the same seeds from the direct tracking method. The latter are derived with 4,000 macro-particles in each bunch with a moderate Gaussian energy spread of 0.1% and not taking synchrotron radiation nor beam-beam effect into account. As discussed above, the luminosity reduction observed in the analytic results comes from errors at the zeroth order, namely transverse mis-steering, crossing-angles and timing.
jitter, and at the first order, namely jitter in the beam matrix including waist motion and transverse coupling. The agreement with the numerical results which take into account higher order terms, shows that the non-linear aberrations are small in this case. However the important point for tolerances is not that the two methods coincide, but rather that they lead to the same reduction factors with respect to the perfect case.

For the study of jitter, different seeds can be interpreted as representing successive bunches. The interest of the analytic method is to permit a rapid average of the luminosity reduction factor over a large number of seeds. In the figures we compare the results averaged over 10 seeds showing large fluctuations, to the ones averaged over 50 seeds showing a smoother behaviour.

5 THE LUMINOSITY WITH WAKEFIELD EFFECTS

5.1 The beam distributions with wakefields

As discussed in the introduction, wakefields along the linac modify the beam distributions by introducing an explicit longitudinal dependence of the transverse and energy distributions. For Gaussian distributions, this means that the beam centroid is moved and the beam matrix rotated according to the position along the bunch. In order to take this effect into account in conjunction with errors in the FFS, we proceed as follows.
We chose a point of reference along the beam line, for instance the transition point from the linac to the final focus section, and we calculate the integrated luminosity in terms of two basic ingredients:

1. the distribution $\rho^{(r)}$ of the beam at the reference point assuming that it is shaped by the upstream generated wakefields

2. the downstream map $\mathcal{M}$ from the reference point to the IP, eventually containing the effect of errors

As another example one can choose the reference point to be the IP, in which case the map is the identity, and study the effect of the wakefields generated in the collimation or final focus sections.

Using the TRANSPORT coordinates, the beam distributions showing explicit longitudinal dependence upon the variable $c\tau$, but Gaussian in the other coordinates $Z = (x, x', y, y', \delta)$, can be written as follows

$$
\rho_n^{(r)}(Y) = \frac{N_n}{(2\pi)^{5/2} \det^{1/2} S_n(c\tau)} \rho_{\parallel, n}(c\tau) \times 
\exp \left[ -\frac{1}{2} (Z - Z_n^o(c\tau)) \cdot S_n(c\tau)^{-1} \cdot (Z - Z_n^o(c\tau)) \right] \quad n = 1, 2 \quad (85)
$$

where the longitudinal distribution $\rho_{\parallel, n}(c\tau)$ is normalized to one

$$\int c\tau \rho_{\parallel, n}(c\tau) = 1 \quad n = 1, 2 \quad (86)$$

$Z_n^o(c\tau)$ and $S_n(c\tau)$ are the 5-dimensional offset vectors and beam matrices depending upon the longitudinal position in the beam.

The maps $\mathcal{M}_n$ relate the coordinates at the reference point to the ones at the IP, denoted by $Y^*$, through

$$Y^* = \mathcal{M}_n(Y) \quad (87)$$

We assume that they leave the longitudinal coordinate $c\tau$ invariant, and do not couple it to the other coordinates, as expected in the paraxial approximation for static systems. When restricted to the 5-dimensional vector $Z$, these maps can be expanded in the usual way as

$$\mathcal{M}_n(Z) = \delta Z_n + R_n \cdot Z + T_n \cdot (Z \otimes Z) + \ldots \quad (88)$$

where $\delta Z_n$ is the constant term, $R_n$ is the transfer matrix and $T_n$ the TRANSPORT second-order matrix, and so on.

### 5.2 Calculation of the luminosity

The calculation of the integrated luminosity proceeds in the same way as for the Gaussian distributions in TRANSPORT coordinates in Section 4.1. After a first change of integration variables of Eq. (67) from $(\bar{u}, \bar{v})$ to $Y^*$ like in Eq. (69), the volume-preserving
maps $\mathcal{M}_n$ enable one to perform a second change from $Y^*$ to the coordinates $Y$ of the reference point, leading to

$$I_n(\vec{p}, z) = e^{\mp ip_z} \int dY \exp \left( i \vec{p} \cdot \mathbf{P}_u \cdot T_{\pm z} \cdot \mathcal{M}_n(Y) \right) \rho_n^{(r)}(Y) \quad n = 1, 2 \tag{89}$$

with $dY = cd\tau \, dZ$. To proceed further we assume that the maps $\mathcal{M}_n$ are truncated up to first order. Higher order terms of the maps can be treated perturbatively. At first order the integration over $Z$ in Eq. (89) is Gaussian and once it is performed $I_n(\vec{p}, z)$ can be written as a one-dimensional integral over $c\tau$. Reporting these integrals into Eq. (66) and performing the Gaussian integration over $\tau_1$, leads to the following 2-dimensional integral expression for the integrated luminosity

$$\bar{\mathcal{L}} = \frac{N_1 N_2}{\pi} \int dz \, cd\tau_1 \, cd\tau_2 \, \delta(2z + c\tau_1 - c\tau_2) \rho_{||,1}(c\tau_1) \rho_{||,2}(c\tau_2) \times$$

$$\exp \left( -\frac{1}{2} \mathbf{A}(c\tau_1, c\tau_2, z) \cdot \Sigma(c\tau_1, c\tau_2, z)^{-1} \cdot \mathbf{A}(c\tau_1, c\tau_2, z) \right) / \sqrt{\det \Sigma(c\tau_1, c\tau_2, z)} \tag{90}$$

The 2-dimensional vector $\mathbf{A}(c\tau_1, c\tau_2, z)$ is given by

$$\mathbf{A}(c\tau_1, c\tau_2, z) = \mathbf{P}_\perp \left[ T_z \cdot (\delta Z_1 + \mathbf{R}_1 \cdot Z_i^2(c\tau_1)) - T_{-z} \cdot (\delta Z_2 + \mathbf{R}_2 \cdot Z_i^2(c\tau_2)) \right] \tag{91}$$

and the 2-dimensional matrix $\Sigma(c\tau_1, c\tau_2, z)$ by

$$\Sigma(c\tau_1, c\tau_2, z) = \mathbf{P}_\perp \cdot \left[ T_z \cdot \mathbf{R}_1 \cdot S_1(c\tau_1) \cdot \mathbf{R}_1^T + T_{-z} \cdot \mathbf{R}_2 \cdot S_2(c\tau_2) \cdot \mathbf{R}_2^T \cdot T_{-z}^T \right] \cdot \mathbf{P}_\perp^T \tag{92}$$

where $\mathbf{P}_\perp$ is the projection operator defined by

$$\mathbf{P}_\perp \cdot Z = \begin{pmatrix} x \\ y \end{pmatrix} \tag{93}$$

and $T_z$ is identified to its 5-dimensional restriction acting on $Z$.

We have calculated in perturbation the first correction to Eq. (90) coming from the 2nd order terms in the map expansion, i.e. the term linear in the TRANSPORT matrices $T_n$. The expression obtained is of course quite complicated and a bit disappointing because it does not incorporate the effect of sextupole-strength error which mainly motivated its calculation. Indeed a simple parity argument shows that for a perfectly aligned system with zero offset, the luminosity contains no term linear in $T$ since it would originate from cubic terms in $Z$ and vanish under integration. The effect of the sextupole strength must be found in the term quadratic in $T$ and quartic in $Z$. However to be consistent, at this order one has to include also the contribution from the third order TRANSPORT matrix $\mathcal{U}$.

### 5.3 Emittance growth and luminosity reduction factor in linacs

A particular application of this expression is the study of emittance growth due to wakefields in linacs. In such studies it is customary to divide the bunch longitudinally in
transverse slices and to compute the emittance resulting from tracking through the linac. It is also meaningful to calculate the luminosity reduction factor from the beam distributions transported to the IP. Of course the dependence of the offsets and beam matrices upon $\epsilon \tau$ is now discrete and the luminosity is given by a double summation over the slices indexed by $i$ for beam 1 and by $j$ for beam 2, as follows

$$
\bar{\mathcal{L}} = \sum_{ij} \frac{N_i N_j}{2\pi} \left[ \exp \left( -\frac{1}{2} \tilde{A}_{ij}(z) \cdot \Sigma_{ij}(z)^{-1} \cdot \tilde{A}_{ij}(z) \right) / \sqrt{\det \Sigma_{ij}(z)} \right]_{z = c \delta \tau_{ij}/2}
$$

(94)

where $N_i$ and $N_j$ are the population of the slices, and $\delta \tau_{ij}$ is the time separating the arrivals at the IP, i.e. $z = 0$, of the slice $j$ in bunch 2 from the slice $i$ in bunch 1. The vectors $\tilde{A}_{ij}(z)$ can be written in terms of the coordinates of the slice center at the IP, as

$$
\tilde{A}_{ij}(z) = \begin{pmatrix} x_i^* + z x_{ij}^* - (x_j^* - z x_{ij}^*) \\ y_i^* + z y_{ij}^* - (y_j^* - z y_{ij}^*) \end{pmatrix}
$$

(95)

On the other hand if the beam matrices of the slices show neither coupling nor space-energy correlations at the IP, the matrices $\Sigma_{ij}(z)$ are diagonal with diagonal elements given by

$$
\left( \Sigma_{ij}(z) \right)_{11} = (S_{i,xx}^* + 2z S_{i,xx'}^* + z^2 S_{i,x'x'}^*) + (S_{j,xx}^* - 2z S_{j,xx'}^* + z^2 S_{j,x'x'}^*)
$$

$$
\left( \Sigma_{ij}(z) \right)_{22} = (S_{i,yy}^* + 2z S_{i,yy'}^* + z^2 S_{i,y'y'}^*) + (S_{j,yy}^* - 2z S_{j,yy'}^* + z^2 S_{j,y'y'}^*)
$$

(96)

and Eq. (94) can be easily calculated.

Finally if one neglects the ‘hour-glass’ effect, that is for short bunches such that

$$
\sigma_z \ll \beta_{x,y}^*
$$

(97)

the integrated luminosity further simplifies to

$$
\bar{\mathcal{L}} = \sum_{ij} \frac{N_i N_j}{4\pi \sigma_{x,ij} \sigma_{y,ij}} \exp \left( -\frac{\delta x_{ij}^2}{4\sigma_{x,ij}^2} - \frac{\delta y_{ij}^2}{4\sigma_{y,ij}^2} \right)
$$

(98)

with

$$
\delta x_{ij} = (x_i - x_j) \ , \quad \delta x_{ij}^2 = \frac{1}{2} (S_{i,xx} + S_{j,xx})
$$

$$
\delta y_{ij} = (y_i - y_j) \ , \quad \delta y_{ij}^2 = \frac{1}{2} (S_{i,yy} + S_{j,yy})
$$

(99)

The interpretation of this expression is quite clear: $\bar{\mathcal{L}}$ is the sum of the instantaneous luminosities produced by each pair $(i,j)$ of opposing slices and calculated from the transverse-rms quadratic average with an exponential dependence on the relative offsets of the slices, like in Eq. (51) for instance. The advantage of this expression is that the reduction factor does not depend on the transverse de-magnifications achieved after the linac. It can therefore be evaluated at the reference point itself, assuming that its phase-advance separation from the IP is a multiple of $\pi$. This is why we have skipped the subscript * in the above definitions. Naturally if the de-magnifications are too strong, the hypothesis in Eq. (97) does not hold because the beta-functions at the IP are too small and Eq. (94) should be used for calculating the luminosity reduction factor.
6 CONCLUSIONS

We have calculated the integrated luminosity produced by a single collision of relativistic bunched beams with general 6-dimensional Gaussian distributions, either in the space-velocity or in the TRANSPORT coordinates. The first ones are the most adapted to the complete analytical calculation of the luminosity, in particular when large crossing angles or angular opening of the bunch are considered. We have illustrated this by considering a few simple cases, mainly for isotropic beams, for which the result was sometimes already known. The second ones are of course better adapted to analyze and post-process data coming from standard accelerator optics programs. The expressions derived are, however, restricted to paraxial beams with respect to a common axis. Taking advantage of the fact that the effect of misalignment shows up mainly in the lowest order moments of the beam distributions, i.e. offset and beam matrix, we showed that the analytic calculation of the luminosity is a reliable way to evaluate tolerances to misalignment. We have also derived the expression for the luminosity when the beam distributions have an explicit non-Gaussian dependence on the longitudinal position in the bunch. These expressions are appropriate for studying the tolerance to dispersion at the IP when the energy distribution is given by the superposition of the RF-wave and longitudinal wake potential, and also to analyze the luminosity reduction associated to emittance growth in linacs.

Non-Gaussian effects like higher order terms in the maps, synchrotron radiation or beam-beam forces are difficult to incorporate in this analytical approach. We have made an attempt to include the effect of the sextupole strength by calculating in perturbation the first correction to the Gaussian luminosity due to the second order TRANSPORT matrix, but we found that it appears only at the next order together with the third order TRANSPORT matrix. The use of the approach proposed in\textsuperscript{15} to treat non-Gaussian effects via the Stratonovich expansion was not yet investigated.

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