Application of the Methods of Optimum Control Theory to the RF System of a Circular Accelerator

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Abstract

The state-variable representation of a system and the associated linear quadratic optimum control are presented in the general case. Their application to the beam-cavity system of a circular accelerator (in the case of heavy beam loading) is given with a view to minimizing the beam disturbance at injection.

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1. INTRODUCTION

More and more, control science uses the state-variable approach which provides a general conceptual frame and naturally leads to the idea of optimum control. In the state-variable representation, a system is described by a set of parameters (state vector) which entirely characterizes its present state. The evolution of the system under command actions (described by the command vector) and its internal physical laws (described by the system equations and the state matrix) can be represented in an elegant way as the trajectory of a point (the state vector) in a $n^{th}$ dimensional space.

Optimum control is associated with the idea of a prescribed trajectory the system has to follow with a minimum error and at minimum cost, (for instance related to the power needed to bring the system on the specified trajectory). Obvious applications of the optimum control theory are in the aerospace field where the trajectory is usually in the three dimension space and where cost means the amount of fuel spent to follow the prescribed path. However, the theory is far more general and can be applied in many other areas of science.

Amongst the various possibilities to define what is meant by minimum error and minimum cost, we shall restrict ourselves to one class of problems where optimum control can be specified as the minimization of a mathematically defined criterion. This criterion is a time integral (generally from the origin of time to infinity) of a weighted sum of the squares of state and command variables. The part on state variables corresponds to the goal we have (error with the desired value) and the part on commands to the energy spent to reach the target. This leads to a feedback that is a linear combination of the state variables. This method, linear quadratic optimum control, which is related with R. E. Kalman's work [1], has become one of the major techniques used for the design of automatic-control processes. All types of industry - chemical, steel, food processing, aircraft, machine tool, textile - are operating with greater efficiency thanks to this optimum control. State-variable approach has already been applied to the accelerator field for the phase synchronisation of an RF system at low intensity [2] [3].

In the following, the system composed of the accelerating cavity and the circulating beam in a circular accelerator will be described using the state-variable approach. Linear quadratic optimum control can be applied to this system to minimize the undesired perturbations of the beam, even in the presence of strong beam loading.
2. STATE-VARIABLE REPRESENTATION OF A SYSTEM

2.1 State Notion

Every system can be described externally by the relation between its outputs $Y$ and its inputs $U$: $Y(t) = h(U(t))$ with $-\infty \leq \tau \leq t$.

If the system is invariant and linear, this relation is a convolution with the impulse response $h(t)$ of the system: $Y(t) = h(t) * U(t)$. The state of a system is a vector $X(t)$, minimal set of time functions $x_i(t), 1 \leq i \leq n$, defined at any time $t$, which represents all energies and informations in the system at time $t$. The state summarizes its past entirely and contains just enough information to determine its future evolution. The minimal size $n$ of $X$ is called order of the system.

To study the system is equivalent to study the evolution of its state. As the state summarizes its past totally, the knowledge of it at time $t_0$ and of the commands (inputs) applied to it between $t_0$ and $t$ determines its outputs entirely:

$$Y(t) = h(t_0, t, X(t_0), U(\tau)) \text{ with } t_0 \leq \tau \leq t.$$

The evolution of the state will be written in the same way:

$$X(t) = \varphi(t_0, t, X(t_0), U(\tau)) \text{ with } t_0 \leq \tau \leq t.$$

2.2 State and Measurement Equations

One can determine the relationship between $X(t)$ and $U(t)$, without taking into account the commands applied before $t$.

Knowing $X(t)$ and $U(\tau)$ for $t \leq \tau \leq t + \Delta t$, one gets:

$$X(t + \Delta t) = \varphi(t + \Delta t, t, X(t), U(\tau))$$

or at first order

$$\frac{X(t + \Delta t) - X(t)}{\Delta t} = \frac{\varphi(t + \Delta t, t, X(t), U(\tau)) - \varphi(t, t, X(t), U(t))}{\Delta t}.$$

The limit when $\Delta t$ goes to zero defines the functional $f$:

$$\frac{dX}{dt} = f(t, X(t), U(t)).$$

This equation is called the state equation. It determines, through the functional $f(t, X(t), U(t))$, the evolution of the state with respect to time.

In case of a time invariant system, $f$ does not depend explicitly on time: $f = f(X(t), U(t))$. 
If now we take $t_0=t$, we get the observation equation:

$$Y(t) = h(t, t, X(t), U(t)).$$

2.3 Case of Linear and Determinist Systems

In this case, the right sides of the state and observation equations are linear in $X$ and $U$. They can be written:

$$\begin{align*}
\dot{X}(t) &= A(t)X(t) + B(t)U(t) \\
Y(t) &= C(t)X(t) + D(t)U(t)
\end{align*}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$ are continuous matrices.

$A$, $B$, $C$, $D$ are constant matrices if the system is time invariant.

3. THE BEAM-CAVITY SYSTEM

The system we want to describe in terms of state variables is the combination of an RF accelerating cavity and the circulating beam.

We simulate the cavity by an R.L.C. parallel circuit driven by the beam and the generator currents. Both of them are represented by ideal current sources, respectively $I_b$ and $I_s$: $\bar{I} = \bar{I}_s + \bar{I}_b$ (Fig. 1a). If $Z$ is the equivalent impedance of the R.L.C. circuit, the total cavity voltage $\bar{V}$ is $\bar{V} = Z(\bar{I}_s + \bar{I}_b)$.

Following the notations used by F. Pedersen [4], the steady state can be represented in a vector diagram, $\bar{V}$ being the phase reference rotating at the reference RF frequency, as shown in Fig. 1b:

- $I_0$ is the current the generator should deliver if there were no beam and if the cavity were tuned at resonance: $I_0 = \frac{V}{R}$

- $\Phi_s$ is the stable phase

- $\Phi_z$ is the detuning angle of the cavity

- $\Phi_i$ is the generator phase angle.

The quantity $Y = \frac{I_b}{I_0}$ characterizes the beam loading effect.
4. DESCRIPTION OF THE SYSTEM UNDER STUDY

4.1. State Variables

The state of the system will be described by the deviations in amplitude and phase of the vectors from their equilibrium positions. However, we assume that the amplitude of \( I_s \) remains constant (\( I_s = 2I_{de} \)) which is the case for short bunches; therefore, \( I_s \) is only modulated in phase.

Four variables are sufficient to describe our system. They are, in terms of variations:

- the phase \( p_s \) of the beam and its derivative \( \dot{p}_s \) to characterize the beam.
- the phase \( p_v \) and the amplitude \( a_v \) of the accelerating voltage to characterize the cavity.

The state vector is therefore

\[
X = \begin{bmatrix}
  x_1 = p_s \\
  x_2 = \dot{p}_s \\
  x_3 = p_v \\
  x_4 = a_v
\end{bmatrix}.
\]

The commands are the phase \( p_s \) and the amplitude \( a_s \) of the generator current: the command vector is

\[
\begin{bmatrix}
  p_s \\
  a_s
\end{bmatrix}.
\]

We suppose that we can observe all the state variables: the observation vector is therefore simply

\[
Y = X = \begin{bmatrix}
  p_b \\
  \dot{p}_b \\
  p_v \\
  a_v
\end{bmatrix}.
\]

To find a state-space representation of the system, we have to determine four matrices \( A, B, C, D \) so that

\[
\begin{cases}
  \dot{X}(t) = A(t)X(t) + B(t)U(t) \\
  Y(t) = C(t)X(t) + D(t)U(t)
\end{cases}
\]

We make the assumption that our system is time invariant, that is to say that \( A, B, C, D \) do not depend on time.
4.2 State Equations of the Beam

They relate the evolution of the beam \((p_v, \dot{p}_v, \dot{p}_b, p_v, a_v)\) to the cavity voltage \((p_v, a_v)\). By definition, we have \(\dot{x}_1 = x_2\). The quantity \(x_2 = \dot{p}_b\) (frequency deviation) is proportional to the energy deviation of the beam and so \(\dot{x}_2 \propto \frac{dE}{dt}\).

If \(V\) is the steady state peak value of the RF voltage and \(\Phi_b\) the stable phase, the reference energy gain per turn is \(\Delta E_0 = V \sin \Phi_b\). If the cavity voltage is \(V(1+a_v)\) and its phase \(p_v\), and if the phase of the beam is \(p_b\), the beam sees an energy gain per turn:

\[
\Delta E = V(1+a_v) \sin(\Phi_b + p_b - p_v).
\]

As a result, the energy deviation of the beam changes per turn by

\[
\Delta E - \Delta E_0 = V(1+a_v) \sin(\Phi_b + p_b - p_v) - V \sin \Phi_b.
\]

If we develop at first order in \(a_v\), \(p_v\), and \(p_b\), it comes out:

\[
\Delta E - \Delta E_0 = V a_v \sin \Phi_b + V \cos \Phi_b (p_b - p_v).
\]

With the assumption \(\frac{dE}{dt} = \frac{\Delta E - \Delta E_0}{T}\), where \(T\) is the revolution period and if \(k\) is the proportionality constant between frequency deviation and energy deviation, one obtains:

\[
\dot{x}_2 = k (V a_v \sin \Phi_b + V \cos \Phi_b (p_b - p_v)).
\]

From the classical synchrotron oscillation theory, \(k\) is related to the synchrotron frequency \(\frac{\omega_s}{2\pi}\) by the relation: \(k V \cos \Phi_b = \omega_s^2\) and therefore \(\dot{x}_2 = -\omega_s^2 (x_1 - x_3) + \tan \Phi_b \omega_s^2 x_4\). Finally:

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\omega_s^2 x_1 + \omega_s^2 x_3 + \tan \Phi_b \omega_s^2 x_4.
\end{cases}
\]

This representation is equivalent to the beam transfer function used for instance in [4], which gives:

\[
p_b = \frac{\omega_s^2}{s^2 + \omega_s^2} (p_v + \tan \Phi_b a_v)
\]

or equivalently:

\[
\dot{p}_b = -\omega_s^2 p_b + \omega_s^2 p_v + \tan \Phi_b \omega_s^2 a_v.
\]
We have assumed here that the response of the beam to an excitation whose frequency is close to the beam revolution harmonics is negligible (single bunch case).

4.3 State Equations of the Cavity

They determine the evolution of the cavity voltage \((x_c, x_a)\) under the effect of the beam \((x_i)\).

At a given time \(t\), our system is described by the equilibrium vectors \(V\) and \(I_b\) and the instantaneous vectors \(V(t)\) and \(I_b(t)\) (Fig. 2a):

- When going through the cavity, the additional beam current \(I_{b,b}\) induces, in the time interval \(\delta t\), an extra voltage in quadrature with \(I_b\)

\[
\delta V = R \sigma \delta t I_{b,b}, \quad R \text{ being the cavity shunt impedance and } \sigma \text{ the half cavity bandwidth } \left( \sigma = \frac{\omega_c}{2Q} \right) \text{ (Fig. 2b).}
\]

The influence of this induced voltage on the amplitude is \(\delta V \cos \Phi_b\) and on the phase \(-\delta V \sin \Phi_b\).

- The cavity detuning \(\Delta \omega\) makes the voltage vector \(\tilde{V}(t)\) rotate by an angle \(\Delta \omega \delta t\). Its amplitude decreases by a factor \((1-\sigma \delta t)\) due to the finite quality factor of the cavity. As a result, the new components \(p'\) and \(a'\) are given by:

\[
\begin{cases}
V_{p'} = (1 - \sigma \delta t)(V_p + V_a \Delta \omega \delta t) \\
V_{a'} = (1 - \sigma \delta t)(V_a - V_p \Delta \omega \delta t)
\end{cases}
\]

or at first order:

\[
\begin{cases}
V_{p'} = V_p - V_p \sigma \delta t + V_a \Delta \omega \delta t \\
V_{a'} = V_a - V_a \sigma \delta t - V_p \Delta \omega \delta t
\end{cases}
\]

The two effects give finally:

\[
\begin{cases}
V_{p'} = V_p - V_p \sigma \delta t + V_a \Delta \omega \delta t - \delta V \sin \Phi_b \\
V_{a'} = V_a - V_a \sigma \delta t - V_p \Delta \omega \delta t + \delta V \cos \Phi_b
\end{cases}
\]
For a vanishing small $\delta t$:

$$
\begin{align*}
\dot{p}_v &= -\frac{R\sigma I_b}{V}\sin\Phi_b p_b - \sigma p_v + \Delta \omega \alpha_v, \\
\dot{a}_v &= \frac{R\sigma I_b}{V}\cos\Phi_b p_b - \Delta \omega \alpha_v - \sigma a_v,
\end{align*}
$$

or

$$
\begin{align*}
\dot{x}_3 &= -\frac{R\sigma I_b}{V}\sin\Phi_b x_1 - \sigma x_3 + \Delta \omega x_4, \\
\dot{x}_4 &= \frac{R\sigma I_b}{V}\cos\Phi_b x_1 - \Delta \omega x_3 - \sigma x_4.
\end{align*}
$$

With $\frac{RI_b}{V} = Y$ and $\Delta \omega = \sigma \tan \Phi_z$, the equations become:

$$
\begin{align*}
\dot{x}_3 &= -\sigma Y \sin \Phi_b x_1 - \sigma x_3 - \sigma \tan \Phi_z x_4, \\
\dot{x}_4 &= \sigma Y \cos \Phi_b x_1 - \sigma \tan \Phi_z x_3 - \sigma x_4.
\end{align*}
$$

### 4.4 State Matrix

Putting together the previous state equations for the beam and the cavity, we get the state matrix $A$:

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\omega_z^2 & 0 & \omega_z^2 & \omega_z^2 \tan \Phi_b \\
-\sigma Y \sin \Phi_b & 0 & -\sigma & \sigma \tan \Phi_z \\
\sigma Y \cos \Phi_b & 0 & -\sigma \tan \Phi_z & -\sigma
\end{bmatrix}
$$

In this matrix, we can distinguish four sub-matrices:

- $A_{11} = \begin{bmatrix} 0 & 1 \\ -\omega_z^2 & 0 \end{bmatrix}$ which characterizes the beam,

- $A_{22} = \begin{bmatrix} -\sigma & \sigma \tan \Phi_z \\ -\sigma \tan \Phi_z & \sigma \end{bmatrix}$ which characterizes the cavity,

- $A_{21} = \begin{bmatrix} -\sigma Y \sin \Phi_b & 0 \\ \sigma Y \cos \Phi_b & 0 \end{bmatrix}$ which characterizes the influence of the beam on the cavity,

- $A_{21} = \begin{bmatrix} 0 & 0 \\ \omega_z^2 & \omega_z^2 \tan \Phi_b \end{bmatrix}$ which characterizes the influence of the cavity on the beam.
The characteristic equation of the system is given by \( \det[A - sI] = 0 \). Looking for stable roots of the characteristic equation, one finds the classical Robinson stability criteria [5].

### 4.5 Command Matrix

The commands are the phase variations \( p_s \) and the amplitude variations \( a_s \) of the generator current. \( p_s \) acts in the same way as \( p_b \) (Fig. 3a).

The additional current \( I_p a_s \) gives a variation \( \delta V \) of the cavity voltage, the value of which is \( R \sigma I_p a_s \delta t \), or in terms of amplitude \( \delta V \sin \Phi_i \) and of phase \( \delta V \sin \Phi_i \).

Moreover, with \( I_s = \frac{I_x (1 + Y \sin \Phi_b)}{\cos \Phi_i} \) and \( \tan \Phi_i = \frac{\tan \Phi_z - Y \cos \Phi_b}{1 + Y \sin \Phi_b} \) (Fig. 1), it comes out:

\[
\begin{align*}
\frac{\delta V}{V} \cos \Phi_i &= \sigma (1 + Y \sin \Phi_b) \delta t p_s \\
\frac{\delta V}{V} \sin \Phi_i &= \sigma (\tan \Phi_z - Y \cos \Phi_b) \delta t
\end{align*}
\]

The influence of \( a_s \) is studied by the same method (Fig. 3b):

The additional current \( I_s a_s \) gives a variation of the cavity voltage \( \delta V \) the value of which is \( R \sigma I_s a_s \delta t \) or in terms of amplitude \( \delta V \cos \Phi_i \) and of phase \( \delta V \sin \Phi_i \).

Finally, we get the command matrix:

\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\sigma (1 + Y \sin \Phi_b) & -\sigma (\tan \Phi_z - Y \cos \Phi_b) \\
\sigma (\tan \Phi_z - Y \cos \Phi_b) & \sigma (1 + Y \sin \Phi_b)
\end{bmatrix}
\]

### 4.6 Final State-Space Representation

As we observe all the state variables, the observation matrices are simply \( C = [I]_4 \) and \( D = [0]_{4 \times 2} \).

The system under study is finally described by the state equations:

\[
\begin{align*}
\dot{X}(t) &= AX(t) + BU(t) \\
Y(t) &= CX(t) + DU(t)
\end{align*}
\]
with:
\[
\dot{X} = \begin{bmatrix} p_b & \dot{p}_b & p_v & a_v \end{bmatrix},
\]
\[Y = X,\]
\[\bar{U} = \begin{bmatrix} \bar{p}_b & a_v \end{bmatrix},\]
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\omega_s^2 & 0 & \omega_s^2 & \omega_s^2 \tan \Phi_b \\
-\sigma Y \sin \Phi_b & 0 & \sigma & \sigma \tan \Phi_z \\
\sigma Y \cos \Phi_b & \sigma \tan \Phi_z & -\sigma & -\sigma
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\sigma (1 + Y \sin \Phi_b) & -\sigma (\tan \Phi_z - Y \cos \Phi_b) \\
\sigma (\tan \Phi_z - Y \cos \Phi_b) & \sigma (1 + Y \sin \Phi_b)
\end{bmatrix},
\]
\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

5. **LINEAR QUADRATIC OPTIMAL CONTROL**

5.1 **Performance Index**

The system is described by its state equation:
\[
\dot{X}(t) = A(t)X(t) + B(t)U(t), \ X \text{ being a vector of } n \text{ components and } U \text{ of } m \text{ components: } X(t) = [x_i(t)]_{i=1}^{n} \text{ and } U(t) = [u_j(t)]_{j=1}^{m}.
\]

The problem is to determine the command \( U_{opt} \) which minimizes the performance index, or criterion:
\[
J = \int_{t_0}^{t_f} \left[ \sum_{i=1}^{n} a_i(t)x_i^2(t) + \sum_{j=1}^{m} b_j(t)u_j^2(t) \right] dt.
\]

The coefficients \( a_i(t) \) and \( b_j(t) \) are positive.
More generally, this criterion can be written:

\[ J = \int_{t_0}^{t_1} [X(t)Q(t)X(t) + U(t)R(t)U(t)] dt . \]

In this formula:

- \( Q(t) \) is a symmetric positive semidefinite matrix,
- \( R(t) \) is a symmetric positive definite matrix,
  (positive definite means that all eigenvalues are strictly positive)
- \( t_0 \), initial time and \( t_1 \), final time, are finite and fixed.
- \( X^T \) and \( U^T \) are the transposed matrices \( X \) and \( U \).

The aim of the regulator is to bring back in the best possible way, as specified by the performance index, the state of the system from its initial value \( X_0 \) to the reference value (\( x_i = 0 \)). The solution we are looking for here is a compromise between the errors with respect to the target (\( x_i = 0 \)) (first term of the criterion) and the cost (second term).

### 5.2 Optimum Control

The control theory demonstrates that the optimal command related to the problem formulated above is given by:

\[
U_{opt}(t) = -L(t)X(t) = -R^{-1}(t)B^T(t)P(t)X(t)
\]

where \( P(t) \) is the unique symmetric positive semidefinite solution of the Ricatti differential equation:

\[
-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t).
\]

The optimal value of the performance index is then:

\[
J_{opt}(t) = X^T(t)P(T)X(t).
\]

The command is linear in terms of the state \( X(t) \) and does not depend on the initial value \( X_0 \).

The system can be described by the diagram given on Fig. 4.
5.3 Case of an Invariant System

In this case, A, B, Q, R are constant matrices.

The performance index we want to minimize is

\[ J = \int_{t_0}^{t_f} [X^T(t)QX(t) + U^T(t)RU(t)] dt. \]

We have in this case the two following results:

If the system is stabilizable, then:

- the Ricatti solution \( P(t) \) is a constant and satisfies the algebraic Ricatti equation:
  \[ PA + A^TP - PBR^{-1}B^TP + Q = 0, \]
- the optimal command is: \( U_{opt}(t) = -R^{-1}B^TPX(t) = -LX(t), \) \( L \) being a constant.
- the optimal cost is: \( J_{opt} = X_0^TPX_0, \) \( X_0 \) being the initial value of the state.

Knowing a state space representation of the system, to find an optimal command, one has to:

- choose two matrices \( Q \) and \( R \) which define the criterion,
- solve the Ricatti solution to find the feedback coefficients of the matrix \( L; \) This can be done thanks to a program like Matlab™,
- The command is then a linear combination of the state variables. The new poles of the system are given by the eigenvalues of the matrix \( A - B* L. \)

6. OPTIMUM INJECTION OF A HIGH INTENSITY BEAM IN A CIRCULAR MACHINE

6.1 Initial Conditions

Trying to apply the results of linear quadratic optimum control to the beam-cavity system, we first specify the target, which will be the equilibrium situation (or \( x_i = 0 \)): the system should behave as a regulator. The disturbance, or the initial condition \( X_o \), will correspond to the situation just after the beam has
been injected into the machine. Here we shall consider two possible cases, the results of which could be combined.

1. Assume first an ideal high intensity injection for which the system does not see any disturbance. It corresponds to the case where \( \bar{I}_s = \bar{I}_s \) before injection, and where the drive current \( \bar{I}_s \) is jumped from its value before injection to its equilibrium value after injection. This happens at the same time as \( \bar{I}_s \) jumps from 0 (before injection) to its equilibrium value. In this ideal scenario, the total current \( \bar{I}_s = \bar{I}_s + \bar{I}_b \) is the same before and after injection and the system, assumed to be at equilibrium before injection, sees no disturbance. It is assumed here that the cavity tune is unchanged before and after injection.

This ideal scenario, however, assumes that \( \bar{I}_s \) is fully known in advance, so as to program the jump of \( \bar{I}_s \), in amplitude and phase. In reality, the injected beam current may have a phase error at injection (\( p_b \neq 0 \)), and the initial state \( X_0 \) would be described by a non-zero vector \( (x_1 \neq 0, x_2, x_3, x_4 = 0) \). Similarly, programming of \( \bar{I}_s \) may not be perfect with the result \( (x_1, x_4 \neq 0) \) for the initial state.

2. The scenario described above, which requires programming of \( \bar{I}_s \), may not be very realistic. Indeed, it would be far better if the system itself would generate the appropriate change of \( \bar{I}_s \) when \( I_s \) comes in. We assume therefore that \( \bar{I}_s = \bar{I}_s \) before injection and that, just after injection, when the vector \( \bar{I}_s \) has jumped from zero to its equilibrium value (no injection error) \( \bar{I}_s \) has not changed (no programming).

This initial situation can be described by an initial state vector \( X_0 \) with \( x_3 = p_b = \Phi_1 - \Phi_2 \) and with \( x_4 = a_s \), which corresponds to the relative difference in amplitude between \( \bar{I}_s \) and \( I_s \) (Fig. 5).

The initial condition for our system can therefore be described by either a non-zero state initial vector \( X_0 \neq 0 \) or by a non-zero input vector \( U_0 \neq 0 \), or possibly a combination of both. We assume here that all the other parameters remain constant, in particular \( \Phi_2 \). This is the case if the cavity tuning response is very slow.

6.2 The Performance Index

In the analysis of chapter 2, it is assumed that the beam behavior is entirely described by the evolution of the phase of the center of gravity of the bunches. Of course, this is an oversimplification which could be valid if RF focusing were perfectly linear. More precisely, in the case of a linear RF wave form (ideal sawtooth voltage), amplitude and phase modulations of \( \bar{V} \) would transform an initial elliptic contour in phase space into an other elliptic contour with the same area: the longitudinal beam emittance would be rigorously
preserved. In reality, the RF wave form is sinusoidal. Therefore, phase and amplitude modulations will result in contour distortions in phase space, which can not in practice be recovered and will ultimately lead to emittance blow-up. As a consequence, it appears that emittance blow up is linked to the amount of RF non linearity that the beam experiences during the injection transient, which itself is proportional to the quantity \((p_b - p_v)^2\).

Similarly, we may want to avoid changes in the amplitude of the RF voltage in order not to excite quadrupole oscillations on the injected beam. To that end, the performance index will include a non zero coefficient on the state variable \(x_4 = a_v\).

There is another parameter which we want to minimize during the injection transient, that is \(\dot{p}_b\), which is proportional to the radial displacement of the beam. There is of course a sharp absolute limit on \(\dot{p}_b\) which corresponds to the machine physical aperture, but we may also want the radial beam position not to deviate too much from the ideal orbit, for instance not to restrict the dynamic aperture of the machine.

With these two parameters, we can write the first part of the performance index:

\[
J_1 = \int \left[ q_1 (p_b - p_v)^2 + q_2 \dot{p}_b^2 + q_3 a_v^2 \right] dt
\]

which corresponds to the matrix \(Q = \begin{bmatrix} q_1 & 0 & -q_1 & 0 \\ 0 & q_2 & 0 & 0 \\ -q_1 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_3 \end{bmatrix}\).

\(q_1, q_2\) and \(q_3\) are weighting factors for the longitudinal beam blow-up and radial displacement.

The second part of the performance index involves the control parameters \(p_s\) and \(a_s\): \(J_2 = \int \left[ r_1 p_s^2 + r_2 a_s^2 \right] dt\).

In the following numerical examples, \(r_1\) and \(r_2\) are considered as free parameters, in the absence of a specific design of the RF system hardware. Obviously, \(r_1\) and \(r_2\) depend on the characteristics of the RF power amplifier. Even if the amplifier itself had no limit, the model would fail if \(r_1\) and \(r_2\) would be set to zero, because \(p_s\) and \(a_s\) (which we assumed to be small deviations to equilibrium values) would become very large.
6.3 Numerical Results

The following parameters have been used in the numerical calculations:

\[ \omega_s = 2\pi \times 1 \text{ kHz} \quad \sigma = 2\pi \times 50 \text{ kHz} \quad \Phi_s = 0 \]

They correspond to a typical proton machine where the synchrotron frequency is much smaller than the cavity bandwidth.

The beam loading parameter \( Y \), which we assume to be equal to \( \tan \Phi_z \) \( (\Phi_z = 0) \), can be varied from zero (very small beam current) to several units (very heavy beam loading).

The selected weighting factors for the longitudinal beam blow up \( q_s = 1 \) and \( q_s = 1 \) are such that a 0.1 radian phase error for \( p_s - p_n \) is as detrimental to the beam as a 10\% change of the RF voltage amplitude. In fact, numerical calculations have shown that the influence of \( q_s \) on the system behavior is fairly small. The weighting factor for \( \dot{p}_b \): \[ q_s = \frac{1}{(2\omega_s)^2} \], corresponds to the effect of an energy deviation equal to the bucket height \( (2\omega_s \text{ in terms of } \dot{p}_b) \), which has the same effect as a phase error \( (p_s - p_n) \) of one radian.

For a given beam loading parameter \( (Y = 1) \) and a given weight of the second part of the performance index \( (r_1 = r_2 = 1) \), the optimum response does not depend on which parameter \( (p_s, a_s, p_n) \) is initially perturbed. This is illustrated on Fig. 6a (perturbation on \( p_s \): \( p_s = 0.785 \)), on Fig. 6b (perturbation on \( p_n \) alone: \( p_s = 0.785 \)) and on Fig. 6c (perturbation on both \( p_s \) and \( a_s \): \( p_s = 0.785, a_s = -0.3 \)). The response of the system shows an initial fast jump on \( p_s - p_n \) (duration about one \( \mu s \), amplitude 0.3), followed by a slower damped response with a time constant comparable to the synchrotron period.

Keeping the same beam loading parameter but changing \( R \) \( (r_1 = r_2 = 10^{-3}) \), one obtains an initial jump of larger amplitude, which brings \( p_s - p_n \) immediately very close to zero, followed by an even slower response on \( \dot{p}_b \) (Fig 6d). Fig. 6e zooms on the initial fast jump. Obviously, the initial part of the performance index (related to the beam blow-up) is better in this case: \( p_s - p_n \) is brought to zero very rapidly and the amplitude of the \( \dot{p}_b \) excursion is smaller. This result is very well known to accelerator physicists: the best system against injection errors is a strong phase loop, with large gain, which locks immediately the RF onto the beam, combined with a sluggish radial loop to slowly bring the RF frequency to its equilibrium value.
The return matrices in the two cases \((r_1=r_2=1\text{ and } r_1=r_2=10^{-3})\)

\[
L = \begin{bmatrix}
-0.42 & 10^{-3} & 0.42 & 5 \times 10^{-3} \\
-8 \times 10^{-3} & 0.42 & 5 \times 10^{-3} & 0.417
\end{bmatrix} (r_1=r_2=1)
\]

\[
L = \begin{bmatrix}
-30.6 & 10^{-3} & 30.6 & 2 \times 10^4 \\
-2 \times 10^{-4} & 0 & 2 \times 10^{-4} & 30.6
\end{bmatrix} (r_1=r_2=10^{-3})
\]

differ essentially by the much larger coefficients of phase \((\phi_0, \phi_c)\) and amplitude \((a_r)\) for the \(R=10^{-3}\) case, while the radial coefficient \((\hat{p}_r)\) is left practically unchanged.

It is interesting to look at the **robustness** of the obtained control system.

For instance, we can calculate the response of the system optimized for \(Y=1\), when this parameter is changed to \(Y=10\) (Fig. 6f). This response shows that the system is robust against changes of this parameter.

Unfortunately, this points out some of the limitations of the model. It is known that an RF system with \(Y=10\) is very likely to be unstable [6], contrary to our findings with the state variable model. This discrepancy is presumably explained by an oversimplification of our model, which assumes RF power amplifiers of infinite bandwidth (the return paths \(P_s-P_c\) to \(P_s\) and \(A_r\) to \(A_s\) are simply linear). To include a band limited RF power amplifier (as has been done to some extent in the frequency analysis [4]), one would have to complete the model with two more variables, like \(\hat{p}_r\) and \(\hat{a}_s\), which would made the analysis far more complicated.

7. **CONCLUSION**

Optimum control theory has been applied to the beam - RF cavity system of a circular accelerator, with a view to minimizing the disturbances to the beam. The analysis confirms some old design practices in RF engineering for circular machines but seems cumbersome to apply to the more realistic case of limited bandwidth RF amplifiers. It would be very useful to extend the state-space variable model to the multibunch case to permit optimum damping of coupled bunch instabilities and injection transients in large machines. Unfortunately, the attempts of extending the model to the case of several bunches have failed until now.
REFERENCES


Fig. 1a  R.L.C. representation of the beam cavity system

Fig. 1b  Vector diagram (case below transition energy)
Fig. 2a  State at time $t$

Fig. 2b  Effect of $p_b$

Fig. 3a  Effect of $p_d$

Fig. 3b  Effect of $a_x$
Fig. 4 System representation and its optimum feedback $L(t)$

Fig. 5 Initial condition without programming of $I_s$
Fig. 6a  Perturbation on \( p_b \)

Fig. 6b  Perturbation on \( p_s \)
Fig. 6c Perturbation on $p_e$ and $a_e$

Fig. 6d Perturbation on $p_e$ with $R=10^{-3}$
Fig. 6e Perturbation on $p$, with $R=10^{-3}$, Zoom on the initial jump

Fig. 6f Robustness against changes of $Y$
APPENDIX

PROCEDURE FOR NUMERICAL CALCULATIONS

Simulations have been performed by using a program written with Matlab™ commands (Matlab™ is a software distributed by The MathWorks, Inc). Matlab™ is an interactive program designed for scientific and engineering numerical calculations. Associated with its control system toolbox, systems can be simulated and their optimum feedbacks determined.

The structure of the program is the following:

- Constant parameters of the system (synchrotron frequency, half bandwidth of the cavity, Φθ and Φz angles, beam loading parameter Y) are input first.

- The system is described by using its state-variable representation (matrices A, B, C, D).

- To determine the optimum feedback, the matrices Q and R of the performance index are input.

- The feedback matrix K is calculated using the "lqr" command which solves the Ricatti equation.

- The state matrix of the closed loop system A-B*K is calculated to study the behavior of the system.

- At this stage, to study the two foreseen scenarios, there are two different procedures. The step on the phase of the beam pθ is simulated with the command "initial" which calculates the outputs and the state variables for an initial state vector X0 (in this case [atan(Y) 0 0 0]T). If we want to make a step on both inputs pθ and aθ, these commands have to be stored in a matrix U and the behavior of the system is determined with the command "lsim".

- Finally, the four state variables and the command pθ are plotted in time domain.
clear;

%****************************************************************************Constants****************************************************************************

ws=1e3*2*pi;
 tb=0;
 phib=0;
tz=1;
 Y=tz;
 sig=50e3*2*pi;

%**************************************************************************State variable representation**************************************************************************

A=[0 1 0 0; %variables pb, pb., pv, av
 -(ws^2) 0 ws^2 ws^2*tb;
-sig*Y*sin(phib) 0 -sig sig*tz;
sig*Y*cos(phib) 0 -sig*tz -sig];

B=[0 0;
 0 0;
sig*(1+Y*sin(phib)) -sig*(tz-Y*cos(phib));
sig*(tz-Y*cos(phib)) sig*(1+Y*sin(phib))];

C=eye(4);

D=zeros(4,2);

%**************************************************************************Performance index matrices**************************************************************************

Q=[1 0 -1 0;
 0 1/(4*ws^4) 0 0 ;
-1 0 1 0;
 0 0 0 1];

R=1*eye(2);

%**********Determination of the optimum feedback**********

[K,S]=lqr(A,B,Q,R);

%**************************************************************************New state matrix**************************************************************************

Ar=A-B*K;
% Case of the step on pb

x0=[atan(Y) 0 0 0];
[y,x,t]=initial(Ar,B,C,D,x0);

% Case of a step on pg and ag

N=1e3;
for l=1:N
  t(l)=8e-3*(l-1)/(N-1);
  U(l,1)=atan(Y);
  U(l,2)=cos(atan(Y))-1;
end

[y,x]=lsim(Ar,B,C,D,U,t);

% Command vector of the closed loop system

up=-K(1,1)*x(:,1)-K(1,2)*x(:,2)-K(1,3)*x(:,3)-K(1,4)*x(:,4);
ua=-K(2,1)*x(:,1)-K(2,2)*x(:,2)-K(2,3)*x(:,3)-K(2,4)*x(:,4);

% Drawing procedure

clg
subplot(221), plot(t,x(:,1)-x(:,3)), title('pb-pv'), grid
subplot(222), plot(t,-x(:,2)), title('pbdot'), grid
subplot(224), plot(t,x(N,3)-x(:,3)), title('pv'), grid
subplot(223), plot(t,up), title('pg'), grid