On LHC Orbit Correction

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Abstract

Orbit correction for the LHC is discussed and proportionality constants relating r.m.s. transverse quadrupole displacements and r.m.s. or peak orbits with and without correction are calculated. This analysis can be used to determine required maximum orbit corrector strengths. Special attention is paid to interaction regions where quadrupoles and correctors affect both clockwise and counter clockwise circulating beams.

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1 Introduction

In storage rings the misalignment of quadrupoles can lead to large orbit excursions of the beam in the vacuum chamber, which can lead to beam loss, dispersion mismatch, and other undesirable effects. Usually beam position monitors (BPM) are used to detect these excursions and dipole orbit correctors are utilized to flatten the orbit. In this report we investigate what correction strength are required for the Large Hadron Collider (LHC) [1] in order to correct typical quadrupole misalignment distributions and what typical residual rms and peak orbits after correction are achievable. It turns out that both rms correction strength and residual rms orbit are proportional to the rms quadrupole misalignment, and we will explicitly calculate the proportionality coefficients for the arcs and the interaction regions of LHC. Furthermore we will study the distribution of achievable residual rms and peak orbits by correcting typically 1000 different quadrupole misalignment configurations and determine probabilities to which the rms or peak orbit are correctable. The entire analysis utilizes the collision optics of the linear uncoupled lattice of LHC, Version 2. The linearity is a prerequisite for the present analysis, which is based mainly on linear algebra concepts. The restriction to an uncoupled beam line is not, because this study is easily extendable to investigate coupled beam lines, that include solenoids, skew quadrupoles, or skew quadrupole components of dipole fields.

A special complication arises in the LHC interaction regions where two equally charged, but oppositely propagating beams share a common piece of beam line with common quadrupoles and orbit correctors. We will pay special attention to the resulting inter-dependence of correcting the beam lines for both beams simultaneously. We will show that a simple extension of existing correction algorithms can handle the more complicated two-beam case.

An equally important topic is the performance of the correction algorithm with imperfections such as defect BPMs or differences in the beam optics in the tunnel to the one used in the correction algorithm on the computer. We will devote some effort in specifying tolerances on the number of defect BPMs or BPM scale errors, which we will use to simulate the aforementioned model mismatch between tunnel and computer.

The present study is based on the concept of the response matrix which is a generalization of the transfer matrix to circular machines and is explained in detail in appendix A. Conceptually, response coefficients, or response matrix elements, are derivatives of the closed orbit position (or angle) with respect to a corrector kick or quadrupole displacement. It is shown in appendix A that in a linear beam line these coefficients can be calculated in terms of normal transfer matrices which are easily obtained from a computer. The corrector response coefficients can also be determined experimentally in a real machine by a difference orbit measurement at two different corrector excitations. In this way they are proportionality factors of the difference orbit to the corrector excitation.

The corrector response coefficients are convenient quantities to estimate tolerances for corrector strengths. We do so by calculating the response coefficients between a corrector and each BPM and calculate the rms over the BPMs. This yields the rms orbit that a given corrector produces. Performing yet another rms over all correctors yields the rms orbit amplification factor for the correctors, which turns out to be about 105 m for the LHC-Version 2 lattice, both horizontally and vertically. If we require that the total rms orbit variation due to power supply noise for the 298 correctors is in the order of the BPM electronic noise
or 50 µm we conclude for the corrector tolerance $50 \mu m/\sqrt{298} \approx 0.03 \mu rad$. Comparing this to the maximum corrector excitation of 15 kGm [1] which translates to a deflection angle of $56.2 \mu rad$ at 8 TeV, we arrive at a relative tolerance $\Delta B/B \lesssim 5 \times 10^{-4}$.

The quadrupole response coefficients are not experimentally accessible, unless one physically moves a quadrupole and measures the resulting closed orbit change. They have, however, the same interpretation as proportionality factors between difference orbit and quadrupole displacement and can easily be calculated on a computer. The top two graphs in fig. 1 show the rms of the proportionality factor over all 347 BPMs in LHC for each quadrupole. Performing a second rms over all rms proportionality factors gives the rms orbit amplification factor for the LHC lattice, which is stated on the right margin of fig. 1. So, an rms quadrupole misalignment of 1 mm typically results in an rms orbit of 3.97 mm horizontally and 4.74 mm vertically if many misalignment configurations are averaged. There are, however, large rms orbits deviations, due to the large beta functions, in the three experimental interaction regions IR 1, IR 2, and IR 7 where the spikes in fig. 1 are visible. The lower two graphs in fig. 1 show the peak orbit amplification factors which are the maximum of all the response coefficients between a given quadrupole and all BPMs. It clearly follows the same pattern as the rms, but the scale is stretched by about a factor of 6, which becomes even more apparent in fig. 2 where the ratio of the peak to rms response coefficients is plotted for each quadrupole. The ratio is remarkably constant near 6 for all quadrupoles;
Figure 2: The ratio of peak and rms orbit response coefficients for the uncorrected LHC lattice at 8 TeV.

only the vertical ratio is more ragged. The peak to rms ratio is a convenient measure for back-of-the-envelope calculations, because rms values are usually easily calculable, but peak values are sometimes more important in worst-case estimates.

In the remainder of this report we deal with the details of the correction algorithm, first for a simple beam line and then for the interaction regions where two beams share a common piece of beam line. We then look at the treatment of imperfections such as defect BPMs, and BPMs with scale factors. In the subsequent sections we apply the presented method to arcs and interaction regions of LHC. The treatment of more technical issues is deferred to the appendices, where we discuss response coefficients, the construction of the beam line for the second beam, and the solution of under-determined linear systems.

2 Correction Algorithm

In this section we want to elaborate on the details of the correction algorithm. The orbit position at BPM $i$ is given by

$$x^i = x_0^i + \sum_j C^{ij} \varepsilon^j + \sum_k \dot{C}^{ik} \Delta x^k$$  \hspace{1cm} (1)
where the first term $x^j_0$ represents the usually unknown (on the order of about 200 $\mu$m or so [8]) BPM offsets stemming from the positioning of the BPM in the beam pipe or imbalances of different channels in the BPM electronics. The second term represents the effect of each orbit corrector $j$ with excitation $\varepsilon^j$ on BPM $i$. The third term represents the effect of each quadrupole displacement $\Delta x^k$ of quadrupole $k$ on BPM $i$. As we are dealing only with the linear lattice in this note the relations in eq. 1 are necessarily linear. In eq. 1 BPM errors can be assigned to each BPM and each of the $i = 1, N_{\text{BPM}}$ equations is divided by that error. In this way a weight is assigned to each measurement. In particular BPMs can be switched off by making the error very large. In the subsequent sections, however, we will suppress these BPM errors to make the text more readable.

The task of a correction algorithm is the search for clever corrector strengths $\varepsilon^j$ that minimize some aspect of the left hand side of eq. 1, e.g. the rms or the peak orbit. In order to keep the analysis linear we choose to minimize the rms of the $x^i$. We will also use all available correctors, because we wish to investigate whether it is in principle possible to minimize a given quadrupole misalignment pattern. It were possible to choose another minimization scheme, e.g. minimizing the biggest appearing value (=peak) of the $x^i$ or choosing only the most effective correctors, like the MICADO algorithm [2] does. However, both modifications make the correction algorithm non-linear and less susceptible to the systematic analysis presented below. We do, however, perform a MICADO analysis for the beam lines discussed below in Appendix D in order to estimate the effect of using a reduced number of correctors on the required correction strengths.

Under the prerequisites stated above and under the assumption there being more BPM than correctors, eq. 1 can be solved for the $\varepsilon^j$ by calculating the $\chi^2$-pseudo inverse of the matrix $C$, namely $(C^T C)^{-1} C^T$ of which the definition is given in Appendix C. Note that the requirement for more BPMs than correctors is not stringent, because in Appendix C we show that it is possible to calculate a pseudo inverse of $C$ even if it is rank deficient using a singular value decomposition method. These considerations will prove to be useful for the analysis of the effect of defect BPMs on the orbit correction of the arcs where there are equal numbers of BPM and correctors.

However, under the above assumptions we obtain for the corrector strengths required to fix the orbit

$$\varepsilon^j = - \left( (C^T C)^{-1} C^T \right)^{ji} (x^j_0 + (\hat{C} \Delta x)^i) = \sum_{i=1}^{N_{\text{BPM}}} F^{ji} x^i_0 + \sum_{k=1}^{N_{\text{QUAD}}} K^{jk} \Delta x^k$$

(2)

where we introduce the matrices $F$ and $K$. The latter being the correction matrix relating the quadrupole displacements to the corrector strengths needed to fix the orbit. $F$ describes how bad BPM offsets "fool" the correctors into assuming erroneous values.

Assuming that the $x_0$ and the $\Delta x$ are distributed according to a gaussian distribution with zero mean and rms width $\sigma_B$ and $\sigma_Q$, respectively, the distribution of the $\varepsilon$ is also gaussian with width

$$\sigma^2 = \frac{\sigma_B^2}{\kappa_B^2} + \frac{\sigma_Q^2}{\kappa_Q^2}$$

$$\frac{1}{\kappa_B^2} = \frac{1}{N_{\text{COR}}} \sum_{j=1}^{N_{\text{COR}}} \sum_{i=1}^{N_{\text{BPM}}} (F^{ji})^2$$

(3)
\[ \frac{1}{\kappa_B^2} = \frac{1}{N_{\text{COR}}} \sum_{j=1}^{N_{\text{COR}}} \sum_{k=1}^{N_{\text{QUAD}}} (K^{jk})^2 \]

where we introduce the correctabilities \( \kappa_Q \) for the quadrupoles and \( \kappa_B \) for the BPMs. Large correctabilities mean that little corrector strength is needed to fix a given rms misalignment. Note that we can also calculate correctabilities for each corrector individually by summing only over the BPM and quadrupole indices. The correctability of corrector \( j \) is then given by

\[ \frac{1}{(\kappa_Q^j)^2} = \sum_{k=1}^{N_{\text{QUAD}}} (K^{jk})^2. \quad (4) \]

The fact that the \( \varepsilon \) are also distributed according to a gaussian distribution allows us to determine the probability that a corrector exceeds a given maximum value \( \varepsilon_{\text{max}} \). It is given by integrating the tails of the gaussian with width given by eq. 3, from \( \varepsilon_{\text{max}} \) to infinity with the result

\[ P(\varepsilon > \varepsilon_{\text{max}}) = \text{erfc} \left( \frac{\varepsilon_{\text{max}}}{\sqrt{2}\sigma_B^2/\kappa_B^2 + \sigma_Q^2/\kappa_Q^2} \right), \quad (5) \]

where \( \text{erfc} \) is the complementary error function. We will calculate the \( \kappa \) and the probabilities for different parts of LHC in later sections.

Now we can determine the corrected orbit by reinserting eq. 2 into eq. 1 and obtain

\[ x_{\text{cor}}^i = (1 - C(C^T C)^{-1} C^T)^{ij} (x_0 + \hat{C} \Delta x)^j. \quad (6) \]

We see that the residual orbit after correction is given in terms of the BPM and quadrupole misalignments through simple linear relations by the matrices

\[ A^{ij} = \left( 1 - C(C^T C)^{-1} C^T \right)^{ij} \]
\[ B^{ik} = \left( (1 - C(C^T C)^{-1} C^T) \hat{C} \right)^{ik} \quad (7) \]

which relate the corrected orbit at BPM \( i \) to the BPM offsets of BPM \( j \) and to the displacement of quadrupole \( k \).

Again assuming that the quadrupoles and BPMs are misaligned according to a gaussian distribution we can calculate the resulting rms orbit at each BPM after correction by averaging eq. 6 over samples of \( x_0 \) and \( \Delta x \) with width given above. We arrive at

\[ x_{\text{rms}}^i = \sqrt{\langle \alpha_B^i \rangle^2 \sigma_B^2 + \langle \alpha_Q^i \rangle^2 \sigma_Q^2} \]

\[ \langle \alpha_B^i \rangle^2 = \sum_{j=1}^{N_{\text{BPM}}} (A^{ij})^2 \]
\[ \langle \alpha_Q^i \rangle^2 = \sum_{k=1}^{N_{\text{QUAD}}} (B^{ik})^2 \quad (8) \]

where we define the orbit amplification factors after correction \( \alpha \) for BPM \( i \) for rms quadrupole and BPM misalignments. We can now proceed to calculate the rms of \( \alpha^i \) over all BPMs.
and arrive at the orbit amplification factor for the ring under consideration which we define by

$$\alpha_Q = \frac{1}{N_{\text{BPM}}} \sum_{i=1}^{N_{\text{BPM}}} (\alpha_Q')^2, \quad \alpha_B = \frac{1}{N_{\text{BPM}}} \sum_{i=1}^{N_{\text{BPM}}} (\alpha_B')^2. \quad (9)$$

Moreover, the rms orbit due to quadrupole misalignment is proportional to $\sum_{ik} (\tilde{C}_{ik})^2 N_{\text{BPM}}$. We can thus define a damping factor $\rho$ to describe the efficiency of the correction as the ratio of the uncorrected rms orbit to the corrected rms orbit which reads

$$\rho = \frac{1}{\alpha_Q} \sqrt{\sum_{ik} (\tilde{C}_{ik})^2 / N_{\text{BPM}}}. \quad (10)$$

Apart from the global parameters $\kappa, \alpha$, and $\rho$ which characterize a beam line we can utilize the matrices $A, B, F$, and $K$ to calculate the required corrector strengths and resulting orbits for given quadrupole misalignments very fast and efficiently by simple matrix multiplication. In this way many (typically 1000) different quadrupole misalignments are generated and histograms for the peak and rms orbits produced. In this way a good feel can be obtained about the achievability of small peak or rms orbits for given misalignments. The rms and
peak orbit histograms are characterized by their average, width and their 70% and 85% margins, i.e. 70% or 85% of the orbits have an rms or peak smaller than the number quoted in the second line in each of the histograms.

We now have a few characteristic numbers (κ, α, ρ) and an efficient means to generate many samples that allow us to judge the correctability of a beam line and the corrector strength needed to achieve a reasonable orbit for a given misalignment of BPMs or quadrupoles. Before applying the presented method to LHC we wish to generalize it to the treatment of two beams, that share a common piece of beam line and how to incorporate imperfections, such as BPM scale errors in the algorithm.

3 Two-Beam Correction

In the interaction regions of the LHC there are quadrupoles and orbit correctors that affect both beams as shown in Fig. 4 which depicts the topology of the interaction regions. Correctors and quadrupoles in section A and C affect beam 1 only, those in section D and E beam 2 only and section B affects both beams. It turns out the only place the presented method needs attention is the definition of the response matrices C and Ĉ, because we now have BPMs for beam 1 and for beam 2 and we have 3 kind of correctors and quads. Namely, those either affecting beam 1 (section A and C) or beam 2 (section D and E) only, and those affecting both beams (section B). Consequently the matrices C and Ĉ have the following
where the $X$ represent the corresponding $C$ or $\tilde{C}$ matrix elements that relate the corresponding BPMs and correctors or quadrupoles. The minus signs in brackets applies to the $C$ matrix for the correctors. In appendix B we digress on the details of the sign conventions. Having redefined the matrices $C$ and $\tilde{C}$ we can proceed along the lines laid out in the previous section to calculate the correctabilities $\kappa$ and the orbit amplification factors $\alpha$ for the interaction regions.

The method discussed so far is easily extendable to coupled beam lines, as is the case if the possibly very strong solenoid field of the detector must be included in the correction algorithm. Again, only the response matrices $C$ and $\tilde{C}$ have to be redefined. On the left hand side of eq. 1 we need to include horizontal and vertical BPMs and on the right hand side we have to consider both horizontal and vertical correctors and quadrupoles. Of course the matrices then contain the cross-plane response coefficients $C_{14}^{ij}, C_{32}^{ij}, \tilde{C}_{13}^{ik}$, and $\tilde{C}_{31}^{ik}$ in the appropriate places.

### 4 Imperfections and Iteration

In this section we discuss the treatment of imperfections, i.e. defect BPMs and BPMs with linearity scale errors. As already indicated in section 2 we can model defect BPMs by increasing their weight to a very large number. In that case they do not contribute in the fit any more, i.e. they are effectively neglected. We use this way to study how configurations with different defect BPMs affect the correctability and the residual achievable orbits. In particular, by switching off one BPM at a time and recalculating the correctabilities $\kappa_Q$ each time we can determine the most critical BPMs.

Scale errors $s$ in BPMs can have a common factor $s_0$, identical for all BPMs which could stem from calibration errors and a random component $\hat{s}$ with zero mean that varies from BPM to BPM. We can thus write

$$ s = s_0 + \hat{s} $$

(12)

where $\hat{s}$ can be realized on a computer by a random number generator. The major effect of these usually unknown errors is that there is a mismatch between the storage ring in the tunnel and the model on the computer that is used in the correction algorithm. We treat these scale errors in the following sections by multiplying the rows of the matrix $\tilde{C}$ by the appropriate scale factor, but leave the matrix $C$ unaffected.

We did not investigate the result of including a modelling error of the quadrupole strengths in the calculation of the response matrices, but suspect that the effect would
be qualitatively similar to those of BPM scale errors. We suspect the same to be true for corrector scale errors.

The just described BPM errors lead to a non-optimal orbit after correction that can usually be fixed by iterating the correction process. To this end we recall eqs. 2 and 6 can be used to relate the orbit after \( n-1 \) iterations \( y^{(n-1)} \) to the corrector excitations in the \( n \)-th iteration \( \varepsilon^{(n)} \) and the orbit after the \( n \)-th iteration \( y^{(n)} \)

\[
\varepsilon^{(n)} = -(C^T C)^{-1} C^T \Lambda y^{(n-1)}
\]
\[
y^{(n)} = (1 - C(C^T C)^{-1} C^T \Lambda) y^{(n-1)} = (1 - P_+ \Lambda) y^{(n-1)}
\]

where we introduce the abbreviation \( P_+ = C(C^T C)^{-1} C^T \) and the diagonal matrix \( \Lambda \) which contains the BPM scale factors on its diagonal.

In the absence of scale errors, e.g. \( \Lambda = 1 \) it is easy to show that \( P_+ \) and \( P_- = 1 - P_+ \) are projection operators, e.g. \( P_\pm^2 = P_\pm \) and \( P_\pm^T = P_\pm \) and therefore only have eigenvalues zero and unity. Consequently, iterating the correction process with ideal BPMs does not change the resulting orbit after the first correction and we have \( y^{(n)} = y^{(1)} \) and it is easy to see that \( \varepsilon^{(n)} = 0 \) for \( n \geq 2 \). We note that \( P_+ \) is the projection onto the subspace (of the space of BPM readings) that can be fixed, whereas \( P_- \) is the projection on subspace that can not be fixed. The eigenvectors of \( P_- \) to eigenvalue unity then are BPM reading patterns that can not be fixed by the correction process.

In the presence of a systematic scale error \( s_0 \) we have \( \Lambda = s_0 \cdot \mathbb{1} \). The eigenvalues of \( P_+ \Lambda \) will be \( s_0 \) and zero and those of \( 1 - P_+ \Lambda \) will be \( 1 - s_0 \) and unity. Since repeated application of the correction corresponds to repeated application of \( 1 - P_+ \Lambda \) to the initial orbit we conclude that the process converges to the same final orbit as in the case without systematic scale errors, provided \( 0 < s_0 < 2 \). Moreover, the closer \( s_0 \) is to unity, the faster the convergence will be.

In the presence of random scale errors \( \Lambda \) will be a general diagonal matrix, therefore no general statement about the eigenvalues of \( P_+ \Lambda \) can be made apart from the fact that they remain real, because both \( P_+ \) and \( \Lambda \) are symmetric. In particular the eigenvalues need not be of magnitude smaller or equal unity, as we convinced ourselves in numerical simulations. This implies that the correction may diverge. However, this divergence is weak, since in IR 1 reasonable random scale variations of 10%, truncated at two sigma, lead to eigenvalues that typically remain below 5%. Thus a diverging correction process hints at varying BPM scales (or, possibly, modelling errors).

The necessity for iterations can be easily estimated by the following consideration. In the presence of scale errors \( s \) the BPMs are affected by wrongly excited correctors on the order of \( \Delta x_B \equiv s \beta \varepsilon \) where \( \beta \) is the average beta functions at places where BPMs and correctors are placed. Since the corrector excitation in turn is determined by the quadrupole misalignment we can estimate the rms erroneous BPM reading by \( \sigma_B \approx s \beta \sigma_\varepsilon \approx s \beta \sigma_Q / \kappa_Q \). We see that the magnitude of the perturbation is given by \( s \beta / \kappa_Q \). Assuming \( s = 0.05 \) it turns out that in the LHC arcs this parameter is on the order of 0.2 and in the interaction regions it is about one order of magnitude bigger. Iterating the correction process will therefore be needed mainly in the interaction regions.
Figure 5: Orbit before correction in arc 1 from a quadrupole misalignment of 0.3 mm, truncated at three sigma. The small graphs show the quadrupole misalignment pattern with a scale given in mm.

5 LHC Arcs

In the arcs of the LHC an identical number of BPMs and correctors are foreseen in (BPH-QF-XCOR) and (BPV-QD-YCOR) groups with the exception of a single cell in the middle of the arc, where a single corrector is missing at a place where a sector valve is foreseen (vertical corrector in the odd arcs and horizontal corrector in the even arcs). With all correctors in place the arcs can be perfectly corrected in the sense that there is one corrector for every BPM which can then be steered to zero. Consequently, in this configuration, the orbit amplification factors $\alpha_Q$ and $\alpha_B$ are zero and the damping factors $\rho$ are infinite. However it turns out that the missing mid-arc corrector increases the quadrupole orbit amplification factors $\alpha_Q$ in the mid-arc region from zero to above three. It seems advisable to keep the corrector at the mid-arc location, unless an rms orbit excursion on the order of 1 mm is tolerable (assuming 0.3 mm rms quadrupole displacements).

Assuming all correctors in place the quadrupole correctabilities $\kappa_Q$ both in the horizontal and vertical direction are on the order of 30 m. Equation 5 and fig. 3 then guarantee that a maximum corrector strength of 50 $\mu$rad is sufficient to correct quadrupole misalignments of 0.3 mm except in 0.01 % of the cases. Both horizontal and vertical BPM correctabilities
Figure 6: Arc-orbit after correction, under the assumption that one horizontal and one vertical BPM are defect. Note the perfect orbit correction except at the two defect BPMs. The small graphs, labelled x/y COR/QUAD show the quadrupole misalignment pattern and the corrector excitation pattern calculated with the aid of eq. 2.

$\kappa_B$ are on the order of 100 m which guarantees in the light of eq. 3 that correctors are only weakly fooled into assuming large values by wrong BPM offsets or BPM readouts.

We already discussed the effect of a missing corrector in the previous paragraph. Here we deal with the effect of missing BPMs or BPMs with scale errors. In the following analysis we usually use quadrupole displacements with an rms of 0.3 mm, truncated at 3 sigma. This is probably a conservative assumption in the light of a recent LEP survey [3] which achieved 0.3 mm hard edge alignment along a design orbit, smoothed on a scale of a few betatron wavelengths. Figure 5 shows a sample orbit before correction and fig. 6 shows the same orbit after correction where one horizontal and one vertical BPM is assumed to be missing. Note that using the algorithm discussed in Appendix C does not take the orbit at the defect BPMs into account, but fixes the orbit everywhere else perfectly.

Figure 7 shows the distribution of rms and peak orbits from 20 different BPM configurations with one defect BPM in the horizontal and the vertical direction used to correct 50 different quadrupole misalignment configurations. Thus it represents the average over 1000 plots such as fig. 6. We see that the 85% margin for the rms orbits lie around 0.3 mm, but more importantly the 85% margin for the peak orbits lie at 1.5 mm. Since only one BPM is
Figure 7: Distribution of rms and peak generated from 1000 different configurations (20 different defect BPMs × 50 different quadrupole configurations) in arc 1.

assumed defect the rms and peak values are entirely determined by the orbit at that BPM which may lead to large (unnoticed) orbit excursions there. These excursions could e.g. be avoided in a real machine by remembering the last good readout of that BPM in a database and using that value with an increased BPM error and consequently a reduced weight in the correction algorithm. Thus, by including the suspected orbit at the defect BPM as a soft constraint, reasonable orbit excursions can be maintained.

The lack of robustness of the correction system in the presence of defect BPMs is caused mainly by the fact that the system is marginally determined (one BPM per corrector). We therefore also analyze a configuration in which all 48 BPMs (two per cell) in the arcs read both horizontal and vertical positions instead of only one or the other. The results for this BPM configuration are shown in the following tables as the second numbers.

In order to investigate the effect of more than one defect BPM we repeat the analysis that leads to fig. 7 with zero through five defect BPMs. Since the horizontal and vertical lattice is very similar we only report their averages in table 1 which shows the 70 and 85% margins for the rms and peak orbit distributions and the average quadrupole correctability $\kappa Q$ as a function of the number of defect BPMs.
<table>
<thead>
<tr>
<th># defect BPMs</th>
<th>rms(70%) [mm]</th>
<th>rms(85%) [mm]</th>
<th>peak(70%) [mm]</th>
<th>peak(85%) [mm]</th>
<th>$\kappa_Q$ [m]</th>
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<tr>
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<td>0.0/0.1</td>
<td>0.0/0.1</td>
<td>0.0/0.2</td>
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<td>30.9/31.0</td>
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<td>1.0/0.2</td>
<td>1.5/0.3</td>
<td>31.6/31.0</td>
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<td>1.8/0.4</td>
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</tr>
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<td>0.5/0.1</td>
<td>1.6/0.3</td>
<td>2.0/0.4</td>
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</tr>
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<td>2.0/0.4</td>
<td>2.3/0.6</td>
<td>33.9/29.6</td>
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Table 1: *The 70 and 85% margins for different numbers of defect BPMs in arc 1. The left numbers are for 24 BPMs in BPH-QF pairs and the right number is for 48 BPMs that read both horizontal and vertical positions.*

It is clearly visible that one or more defect BPMs in the 24 BPM configuration entail a fair chance of being unable to correct the peak orbit to better than 2 mm in the arcs using the plain algorithm discussed in appendix C. Using 48 BPMs will greatly alleviate this problem, even though no gain is made in the ideal machine with no defect BPMs. Increasing the number of BPMs does not increase the accuracy, but the *robustness*. However, if only 24 BPMs/arc are available we suggest to use the method of soft constraints, discussed above, which, however will only work once the BPMs have worked at least once. Some other strategy needs to be used at startup of LHC if there are defect BPMs at that time. Note that the correctabilities $\kappa_Q$ are only weakly affected by defect BPMs which implies that the correction strength will be sufficient with defect BPMs as well.

In order to determine the effect of the systematic scale errors, common to all BPMs we follow the path outlined in section 4. First we assume that all BPMs are operational, but have a common readout scale error. We include this scale error in the $\hat{C}$ matrices used in the correction algorithm, but not in the $C$ matrices. For a given scale error we generate 1000 different quadrupole misalignment configurations and correct them which takes about 10s CPU time on the CERN IBM mainframe. The 70 and 85% margins of the resulting rms and peak orbit distributions are reported in table 2.

<table>
<thead>
<tr>
<th>BPM scale systematic</th>
<th>rms(70%) [mm]</th>
<th>rms(85%) [mm]</th>
<th>peak(70%) [mm]</th>
<th>peak(85%) [mm]</th>
<th>$\kappa_Q$ [m]</th>
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<td>1.2/1.2</td>
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<td>0.0/0.1</td>
<td>0.0/0.1</td>
<td>0.0/0.2</td>
<td>0.0/0.2</td>
<td>30.8/31.2</td>
</tr>
<tr>
<td>1.05</td>
<td>0.2/0.2</td>
<td>0.2/0.2</td>
<td>0.3/0.3</td>
<td>0.4/0.4</td>
<td>29.4/29.7</td>
</tr>
<tr>
<td>1.10</td>
<td>0.4/0.3</td>
<td>0.4/0.3</td>
<td>0.7/0.7</td>
<td>0.8/0.7</td>
<td>28.2/28.3</td>
</tr>
<tr>
<td>1.15</td>
<td>0.5/0.4</td>
<td>0.6/0.5</td>
<td>1.0/1.0</td>
<td>1.2/1.2</td>
<td>27.1/27.1</td>
</tr>
</tbody>
</table>

Table 2: *The 70 and 85% margins for different systematic BPM scale errors in arc 1. The left number is for 24 BPMs and the right for 48 BPMs.*
We see that about 10% BPM systematic scale errors are acceptable if peak orbit excursions on the order of 0.7 mm are tolerable independent of the number of BPMs involved. Note also that table 2 shows the result of only a single iteration. As we argued in section 4 we can reach the best configuration by iterating the correction, thus the results in table 2 are too pessimistic. Note that the correctabilities $\kappa_Q$ decrease with increasing scale factors, because the BPM read larger values and the correctors are fooled into over-correcting the orbit.

Now we turn to the random BPM scale errors which vary from BPM to BPM. We assume a distribution of BPM scales with the sigma quoted, truncated at two sigma. Similar to the investigation of the effect of defect BPMs, we generate 20 different BPM configurations with different scales and use them to correct 50 different quadrupole misalignment configurations. The results for random scales with sigmas from zero to 0.15 are compiled in table 3.

<table>
<thead>
<tr>
<th>BPM scale random</th>
<th>rms(70%) [mm]</th>
<th>rms(85%) [mm]</th>
<th>peak(70%) [mm]</th>
<th>peak(85%) [mm]</th>
<th>$\kappa_Q$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0/0.1</td>
<td>0.0/0.1</td>
<td>0.0/3.2</td>
<td>0.0/0.2</td>
<td>30.9/30.9</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2/0.1</td>
<td>0.2/0.2</td>
<td>0.4/3.4</td>
<td>0.5/0.5</td>
<td>30.3/30.7</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3/0.2</td>
<td>0.4/0.3</td>
<td>0.9/3.8</td>
<td>1.1/1.0</td>
<td>29.3/29.9</td>
</tr>
<tr>
<td>0.15</td>
<td>0.5/0.4</td>
<td>0.6/0.4</td>
<td>1.3/1.2</td>
<td>1.7/1.5</td>
<td>27.5/28.6</td>
</tr>
</tbody>
</table>

Table 3: The 70 and 85% margins for different random BPM scale errors in arc 1. The left number is for 24 BPMs and the right for 48 BPMs.

Similar to the investigation of systematic BPM scale errors we find that BPM to BPM fluctuations on the order of 10% are tolerable in arc 1. The results for the configuration with 48 BPMs per plane turn out to be slightly worse than those of the other configuration, mainly because there are more BPMs (with erroneous readings) that fool the correctors. Note, however, that iterations will improve on the resulting orbits such that the loss in accuracy using the larger number of BPMs is not crucial. Again the correctabilities $\kappa_Q$ are only weakly affected by the BPM errors and will not significantly change the required corrector strengths.

Finally, in order to estimate the effect that a rather sick beam line, in which three horizontal and three vertical BPMs are not operational and all other BPMs have a 10% systematic scale error and 5% random error. We generate 20 different BPM configurations and use them to correct 50 quadrupole misalignment configurations each. The result is shown in table 4.
INDIVIDUAL CORRECTABILITIES, BEAM 1

Figure 8: Correctabilities \( \kappa_Q \) and \( \kappa_B \) for IR 1 for beam 1.

<table>
<thead>
<tr>
<th>rms(70%) [mm]</th>
<th>rms(85%) [mm]</th>
<th>peak(70%) [mm]</th>
<th>peak(85%) [mm]</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.6</td>
<td>1.7</td>
<td>2.1</td>
<td>24 BPM per plane</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>0.9</td>
<td>1.1</td>
<td>48 BPM per plane</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>48 BPM per plane, 2 iterations</td>
</tr>
</tbody>
</table>

Table 4: The 70 \% and 85 \% margins for a configuration with 3 defect BPMs, systematic scale error of 10 \% and random scale errors of 5 \% in arc 1.

Clearly there is a large probability to find peak orbit excursions on the order of 2 mm if only 24 BPMs per plane are used. The configuration with 48 BPMs per plane shows to be considerably more robust and peak orbit excursions less than 1 mm are likely to achieve. Iterating the correction even makes peak orbit excursions of less than 0.5 mm likely.
6 LHC Interaction Regions

In this section we will pay particular attention to interaction region 1 (IR 1) where the mini-beta quadrupoles and a few correctors affect both beams. We therefore need the algorithm outlined in section 3. Apart from the regular orbit correctors which are placed near almost every quadrupole (in QF-XCOR, QD-YCOR pairs), we assume that those originally foreseen for IP orbit angle and separation control (named K1LO and K2LO in the official Version 2 MAD deck) are available for orbit control, because otherwise misalignments of Q4 turn out to be hard to correct. The orbit amplification factor $\alpha_0^Q$ for Q4 in that case reaches 1.5 compared to a typical value of 0.4. Moreover, near the triplets we place the correctors according to the following pattern

$(\text{YCOR, QD3, XCOR}) (\text{QF2B, QF2A, YCOR}) (\text{QD1, XCOR}) (\text{IP})$

$(\text{XCOR, QF1}) (\text{YCOR, QD2A, QD2B}) (\text{XCOR, QF3, YCOR})$

In this way both beams see the same pattern as they traverse the IR 1 region and the more effective correctors (XCOR near QFs, YCOR near QDs) are placed on the outgoing side. A
Figure 10: RMS orbit amplification factors for the quadrupoles in IR 1. The horizontal and vertical damping factors $\rho$ are 248 and 224, respectively.

The total number of 22 BPM are placed in the IR region, one in every quadrupole, where the doubled quads (Q2, Q5, Q6, Q7) only count as one.

We first calculate the correctabilities $\kappa_Q$ and $\kappa_B$ and display $\kappa_i$ for each corrector of both beams in fig. 8 and 9. The correctabilities for the entire beam line are shown on the right margin of figs. 8 and 9. We see that both quadrupole and BPM correctabilities are on the order of 20 m, which, consulting eq. 5, implies a less than 0.1% chance of needing a corrector with an excitation exceeding 50 $\mu$rad which corresponds to about 15 kGm at 8 TeV. However, there are a few correctors near the mini-beta quadrupoles with correctabilities on the order of 10 m. Consulting eq. 5 or fig. 3 we conclude that it is reasonable to increase the size of their power supplies such that they will be able to provide either 75 or 100 $\mu$rad. Furthermore note that the correctabilities of correctors near the separator D2 are large, which can be understood by noting that the main sources of orbit distortion are the misaligned mini-beta quadrupoles, which generate orbit kicks that can be easily compensated by correctors near D2 which, owing to the large beta functions are placed roughly at the same phases as the mini-beta quadrupoles.

Figure 10 shows the orbit amplification factors $\alpha_Q$ for IR 1. The averages, which are on the order of 0.4 are shown on the right margin. Their value implies that the correction algorithm reduces BPM offsets and quadrupole misalignments by a factor of 2.5. We also see
that the main sources of an rms orbit increase are the mini-beta quadrupoles, owing to their large gradients. In order to keep their influence within bounds we suggest to pay special attention to the initial alignment of the mini-beta section. Especially the internal alignment of the triplets will be crucial in order to avoid excessive generation of dispersion.

In fig. 11 and 12 we show the distribution of rms and peak orbits arising from quadrupole misalignments of 0.3 mm, truncated at three sigma with all BPMs intact and no scale errors involved from correcting 1000 different quadrupole misalignment configurations. Clearly the rms orbit can almost always be corrected to better than 0.2 mm and the peak to better than 0.6 mm. We conclude that under good conditions (no defect BPMs and no BPM scales) the orbit can always be well corrected to the values just stated.

The orbit amplification factor due to BPM misalignment \( \alpha_B \) turns out to be on the order of 0.7, meaning that a 0.2 mm rms BPM misalignment results in a 0.15 mm rms orbit increase. This is of the same order of magnitude as the effect of the quadrupole misalignment and implies that the alignment of the BPMs in the center of the beam pipe and especially the placement with respect to the magnetic center of the quadrupoles should be performed with proper care to avoid the corruption of the correction algorithm due to BPM offsets.

We now turn to the investigation of imperfections. First we will investigate the effect of defect BPMs on the achievable rms and peak orbits. Similar to the previous section we
Figure 12: Peak orbit distribution for IR 1.

generate 20 different configurations with a given number of defect BPMs and use them to correct 50 different quadrupole misalignment configurations, each. For a given number of defect BPMs we generate plots such as fig. 11 and 12 and display the 70 and 85% margins for rms and peak orbit in table 5 after one and two iterations, where the first line refers to fig. 11 and 12.
<table>
<thead>
<tr>
<th># defect BPMs</th>
<th>rms(70 %) [mm]</th>
<th>rms(85 %) [mm]</th>
<th>peak(70 %) [mm]</th>
<th>peak(85 %) [mm]</th>
<th>$\kappa$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1/0.1</td>
<td>0.1/0.1</td>
<td>0.3/0.3</td>
<td>0.4/0.4</td>
<td>20.0/20.0</td>
</tr>
<tr>
<td>1</td>
<td>0.1/0.1</td>
<td>0.2/0.1</td>
<td>0.3/0.3</td>
<td>0.5/0.4</td>
<td>19.4/19.3</td>
</tr>
<tr>
<td>2</td>
<td>0.1/0.1</td>
<td>0.2/0.2</td>
<td>0.3/0.4</td>
<td>0.5/0.8</td>
<td>15.3/15.5</td>
</tr>
<tr>
<td>3</td>
<td>0.2/0.2</td>
<td>0.7/0.3</td>
<td>0.5/0.5</td>
<td>2.6/0.9</td>
<td>14.6/15.0</td>
</tr>
<tr>
<td>4</td>
<td>0.1/0.2</td>
<td>0.3/0.2</td>
<td>0.4/0.5</td>
<td>0.7/0.9</td>
<td>12.7/14.8</td>
</tr>
<tr>
<td>5</td>
<td>0.2/0.2</td>
<td>0.9/0.3</td>
<td>0.7/0.6</td>
<td>2.5/1.1</td>
<td>13.3/14.0</td>
</tr>
<tr>
<td>6</td>
<td>0.2/0.2</td>
<td>0.8/0.5</td>
<td>0.8/0.7</td>
<td>2.5/1.6</td>
<td>12.4/12.1</td>
</tr>
<tr>
<td>7</td>
<td>0.3/0.3</td>
<td>1.2/0.7</td>
<td>1.0/1.1</td>
<td>4.4/2.6</td>
<td>10.7/11.1</td>
</tr>
<tr>
<td>8</td>
<td>0.4/0.4</td>
<td>1.7/0.8</td>
<td>1.3/1.5</td>
<td>6.3/3.4</td>
<td>11.2/9.0</td>
</tr>
</tbody>
</table>

Table 5: The 70 and 85 % margins for different numbers of defect BPMs in IR 1 after one and two iterations. The fly in line 4 is discussed in the text.

From table 5 we conclude that we can possibly tolerate 1 or 2 defect BPMs among the BPMs for both beams. If there are 3 defect BPMs a few very bad configurations showed up such that the 85 % margin is pushed up. We suspect the BPM near Q4 to be responsible for these bad configurations as will become clear in the next paragraph. Apparently these bad defect-BPM configurations did not show up in the investigation of 4 defect BPMs, but are likely to appear if more than 4 BPMs are defect. We conclude further that it is likely that the IR 1 peak orbit remains smaller than 1 mm even if 4 BPMs are broken.

By switching off one BPM at a time and recalculating $\kappa_Q$ anew for each configuration we can determine critical BPMs, i.e. those that decrease $\kappa_Q$ significantly, if switched off. We find that the BPM near Q4 on the outgoing side for both beams decreases the horizontal $\kappa_Q$ by more than a factor of five if switched off. We suspect that the reason for this behavior is the large drift space between the triplet and Q4 that makes the BPM near Q4 very susceptible to kicks arising from the triplet area. We suggest to add another BPM in the long drift as a backup for the event that the BPM near Q4 should malfunction.

The last column in table 5 shows the average quadrupole correctability $\kappa_Q$ which decreases with increasing number of defect BPMs. This implies that defect BPMs tend to fool correctors into assuming larger values and may possibly pose a problem for the corrector power supplies.

We now move on to see the effect of systematic scale errors in the BPMs. For each assumed scale error we generate 1000 quadrupole misalignment configurations and display the 70 and 85 % margins of the rms and peak orbit distributions in table 6. From the discussion in section 4 we already know that for large number of iterations the optimal correction level is approached. This is also verified in numerical simulations. In table 5, however, we show the achievable orbit after the first and after the second iteration level.
<table>
<thead>
<tr>
<th>BPM scale systematic</th>
<th>rms(70 %) [mm]</th>
<th>rms(85 %) [mm]</th>
<th>peak(70 %) [mm]</th>
<th>peak(85 %) [mm]</th>
<th>$\kappa$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.1/0.1</td>
<td>0.1/0.1</td>
<td>0.3/0.3</td>
<td>0.4/0.4</td>
<td>20.0/19.8</td>
</tr>
<tr>
<td>1.01</td>
<td>0.3/0.1</td>
<td>0.4/0.1</td>
<td>0.7/0.3</td>
<td>0.8/0.4</td>
<td>19.9/20.1</td>
</tr>
<tr>
<td>1.02</td>
<td>0.5/0.1</td>
<td>0.7/0.1</td>
<td>1.2/0.3</td>
<td>1.6/0.4</td>
<td>19.6/19.9</td>
</tr>
<tr>
<td>1.03</td>
<td>0.8/0.1</td>
<td>1.1/0.1</td>
<td>1.7/0.3</td>
<td>2.4/0.4</td>
<td>19.5/20.1</td>
</tr>
<tr>
<td>1.04</td>
<td>1.0/0.1</td>
<td>1.4/0.2</td>
<td>2.2/0.3</td>
<td>3.2/0.4</td>
<td>19.4/19.8</td>
</tr>
<tr>
<td>1.05</td>
<td>1.2/0.1</td>
<td>1.7/0.2</td>
<td>2.7/0.3</td>
<td>3.9/0.4</td>
<td>19.2/20.0</td>
</tr>
<tr>
<td>1.06</td>
<td>1.4/0.2</td>
<td>1.9/0.2</td>
<td>3.3/0.4</td>
<td>4.6/0.5</td>
<td>19.1/20.1</td>
</tr>
<tr>
<td>1.10</td>
<td>—/0.3</td>
<td>—/0.4</td>
<td>—/0.7</td>
<td>—/0.9</td>
<td>—/20.3</td>
</tr>
</tbody>
</table>

Table 6: The 70 and 85\% margins for different systematic scale errors in IR 1. The left numbers are after the first iteration and the right after the second.

We see that even small systematic scale errors increase the margins considerably in the first iteration. We attribute this to the fact that the beta functions in the IR regions are very large (up to 4000 m) as already discussed in section 4. However the next iteration already brings all orbit margins very close to the margins of the unperturbed system. Larger number of iterations will improve the situation even further. We conclude that systematic scale errors will not be a problem, because they can always be remedied by iterating the correction process. The fact that the $\kappa_Q$ after the second iteration are almost independent of the systematic BPM scale error implies that the required corrector strength is the same.

In table 7 we compile the results from an analysis in which we vary the BPM scales from BPM to BPM with the rms variation stated in the table, truncated at two standard deviations.

<table>
<thead>
<tr>
<th>BPM scale random, rms</th>
<th>rms(70 %) [mm]</th>
<th>rms(85 %) [mm]</th>
<th>peak(70 %) [mm]</th>
<th>peak(85 %) [mm]</th>
<th>$\kappa$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.1/0.1</td>
<td>0.1/0.1</td>
<td>0.3/0.3</td>
<td>0.4/0.4</td>
<td>20.0/20.2</td>
</tr>
<tr>
<td>0.01</td>
<td>0.2/0.1</td>
<td>0.3/0.1</td>
<td>0.5/0.3</td>
<td>0.7/0.4</td>
<td>17.0/20.0</td>
</tr>
<tr>
<td>0.02</td>
<td>0.4/0.1</td>
<td>0.5/0.1</td>
<td>1.0/0.3</td>
<td>1.5/0.4</td>
<td>12.0/20.1</td>
</tr>
<tr>
<td>0.03</td>
<td>0.5/0.1</td>
<td>0.8/0.1</td>
<td>1.4/0.3</td>
<td>2.0/0.4</td>
<td>9.6/20.1</td>
</tr>
<tr>
<td>0.04</td>
<td>0.7/0.1</td>
<td>1.0/0.1</td>
<td>1.9/0.3</td>
<td>2.8/0.4</td>
<td>7.9/19.9</td>
</tr>
<tr>
<td>0.05</td>
<td>0.8/0.1</td>
<td>1.2/0.2</td>
<td>2.2/0.3</td>
<td>3.3/0.4</td>
<td>7.1/19.9</td>
</tr>
<tr>
<td>0.06</td>
<td>1.1/0.1</td>
<td>1.5/0.2</td>
<td>2.8/0.4</td>
<td>4.1/0.5</td>
<td>6.2/19.8</td>
</tr>
<tr>
<td>0.10</td>
<td>—/0.2</td>
<td>—/0.3</td>
<td>—/0.6</td>
<td>—/0.9</td>
<td>4.7/17.1</td>
</tr>
<tr>
<td>0.15</td>
<td>—/0.4</td>
<td>—/0.6</td>
<td>—/1.3</td>
<td>—/2.0</td>
<td>—/12.7</td>
</tr>
</tbody>
</table>

Table 7: The 70 and 85\% margins for different random scale errors in IR 1. The left numbers are after the first iteration and the right after the second.

Comparing this table with the previous one we see that the random variation of the BPM scales from BPM to BPM is similar to a systematic scale error for all BPMs is. Clearly
iterations help to desensitize the correction process, as can be seen from comparing the two numbers in every column in table 7. The possibility indicated in section 4 that the correction process may become divergent did not show up. We also investigated larger random BPM scales (up to 50%), but in all cases the orbit could be fixed by iterating the correction. We conclude that also random BPM scales do not pose a serious problem for convergence of the correction process.

Note that in the first iteration the correctabilities \( \kappa_Q \) decrease rapidly with increasing scale errors which may push the power supplies to their limit, whereas in the second iteration the \( \kappa_Q \) are almost independent of the scale errors implying that the quadrupole misalignment configuration is correctable with the given maximum corrector excitation, but that intermediate configurations may require larger corrector excitations. Since the beam line will certainly be first corrected at lower energy after commissioning where the correctors are more powerful this will not pose a problem.

Now we turn to other interaction regions and calculate the quadrupole correctabilities \( \kappa_Q \), the orbit amplification factors \( \alpha_Q \) and the 85% margin for the peak orbit distribution for all interaction regions of LHC. The calculations are performed for the ideal lattice, i.e. imperfections are not taken into account. The results are shown in table 8.

<table>
<thead>
<tr>
<th>IR</th>
<th>( \kappa_Q(x/y) ) [m]</th>
<th>( \alpha_Q(x/y) )</th>
<th>peak(85%) [mm]</th>
<th>( \rho(x/y) )</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.5/21.3</td>
<td>0.38/0.35</td>
<td>0.4</td>
<td>248/224</td>
<td>experimental</td>
</tr>
<tr>
<td>2</td>
<td>21.4/20.6</td>
<td>0.32/0.36</td>
<td>0.4</td>
<td>297/220</td>
<td>experimental</td>
</tr>
<tr>
<td>3</td>
<td>25.1/25.1</td>
<td>0.21/0.22</td>
<td>0.2</td>
<td>53/44</td>
<td>cleaning</td>
</tr>
<tr>
<td>4</td>
<td>21.7/20.9</td>
<td>0.30/0.37</td>
<td>0.4</td>
<td>42/31</td>
<td>detuned</td>
</tr>
<tr>
<td>5</td>
<td>23.0/23.0</td>
<td>0.21/0.22</td>
<td>0.2</td>
<td>58/51</td>
<td>dump</td>
</tr>
<tr>
<td>6</td>
<td>21.7/20.9</td>
<td>0.30/0.37</td>
<td>0.4</td>
<td>42/31</td>
<td>same as IR 4</td>
</tr>
<tr>
<td>7</td>
<td>20.3/21.3</td>
<td>0.40/0.34</td>
<td>0.4</td>
<td>233/229</td>
<td>experimental + RF</td>
</tr>
<tr>
<td>8</td>
<td>21.7/20.9</td>
<td>0.30/0.37</td>
<td>0.4</td>
<td>42/31</td>
<td>same as IR 4</td>
</tr>
<tr>
<td>-</td>
<td>31.4/32.0</td>
<td>0.00/0.00</td>
<td>0.0</td>
<td>( \infty )</td>
<td>arc</td>
</tr>
</tbody>
</table>

Table 8: The quadrupole correctabilities \( \kappa_Q \), the orbit amplification factors \( \alpha_Q \) and the 85% margin for the peak orbit distribution for all interaction regions of LHC. All calculations are performed for the ideal lattice. For comparison the figures for the arc are provided in the last line.

We observe that all interaction regions except the dump and cleaning insertion behave very similar. The reason for the different behavior of the dump and cleaning insertion are more relaxed quadrupole excitations. We also see that the correctabilities for the interaction region \( \kappa_Q \) are smaller than those of the arcs and that the orbit amplification factors \( \alpha_Q \) are smaller. This is due to larger beta functions in the interaction regions which are, however, compensated to some extent by a larger number of correctors.
7 Conclusions

In this report we developed methods to characterize the correctability of beam lines. The algorithm is applicable to simple beam lines which are traversed by one beam only, and to interaction regions where two beams share a common piece of beam line with common quadrupoles and orbit correctors. Two of the main relevant characterizing parameters are the orbit amplification factors \( \alpha_Q \), which relate the quadrupole misalignment to the residual rms orbit after correction, and the correctabilities \( \kappa_Q \), which determine the corrector excitation needed to minimize the rms orbit. The latter quantity allows to estimate the probability that correctors exceed their maximum possible excitation. A third relevant quantity, the 85\% margin for the peak orbits was determined by calculating the peak orbit excursions over typically 1000 different quadrupole misalignment configurations and finding that peak orbit which is larger than the peak orbit in 85\% of the cases. Furthermore, we determined the effects of defect BPMs, and BPM scale errors.

The method was applied to the arcs and the interaction regions of the LHC. In the arcs we found correctabilities of about 30 m, which result in a less than 0.01 \% chance that excessive corrector excitations are needed if the quadrupoles are misaligned by 0.3 mm rms. Due to the fact that there are equal numbers of BPM and correctors in the arcs the orbit amplification factors \( \alpha \) are zero, provided that a corrector at the mid-arc location near the sector valve is included in the analysis. Removing that corrector will result in drastically increased rms-orbit amplification factors for a few quadrupoles near the mid-arc. Furthermore, we found that even one or two defect BPMs can lead to peak orbits on the order of 2 mm, however, BPM scale errors, both systematic and varying from BPM to BPM can be tolerated up to 10\% without degrading the peak orbit excursions to more than 0.7 mm. Doubling the number of BPM per plane from 24 to 48 per arc improves the robustness with respect to defect BPMs of the correction dramatically such that even under demanding conditions orbit peak orbit excursions less than 1 mm are likely to achieve.

The eight interaction regions behave very similar and we concentrated on the study of IR 1. The correctabilities turned out to be on the order of 20 m which lead to a less than 0.1\% chance to require excessive corrector excitations if the rms quadrupole misalignment is 0.3 mm. The 4 horizontal and 4 vertical correctors in the shared piece of beam line have individual correctabilities of 10 m, we therefore suggest to increase their strength by 50\% which brings the probability down to 1\% from 10\%. The orbit amplification factors turned out to be about 0.4 with peaks at the triplet quadrupoles around unity which suggests to spend great effort to carefully align them beforehand. The analysis of imperfections showed that the BPM near Q4 on the outgoing side is very critical for a reliable alignment which suggests to provide for a backup of that BPM. Otherwise, the peak orbits remain around 0.5 mm even if two BPMs are defect and there remains a good chance to correct it, even if 4 BPMs are defect. BPM scale errors of up to 10\% rms turn out to pose no problems since they can be easily fixed by iterating the correction process.

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References


8. L. Vos, CERN, private communication.

Appendix A: Response Coefficients

In beam transfer lines the response of the orbit at a given position $i$ to an upstream orbit corrector excitation or a quadrupole displacement is given by transfer matrix elements. In a circular accelerator, e.g. a storage ring, the situation is slightly more complicated, because of the condition that the orbit has to be closed must be observed. In order to investigate this closed orbit condition we assume that the beam receives a generalized localized kick $\vec{\varepsilon} = (\Delta x, \Delta x', \Delta y, \Delta y')$ at a position labelled $j$. The closed orbit condition for the 4-vector just after the perturbing element $\vec{x}_i$ is then given by

$$\vec{x}_i = R^{ij} \vec{x}_j + \vec{\varepsilon}$$  \hspace{1cm} (14)

where $R^{ij}$ is the $4 \times 4$ (possibly coupled) linear one-turn-map starting at position $j$. Equation 14 can now be solved for the closed orbit $\vec{x}_j$, yielding

$$\vec{x}_j = (1 - R^{ij})^{-1} \vec{\varepsilon}.$$  \hspace{1cm} (15)

The closed orbit at position $j$ is then easily propagated to position $i$, where e.g. a BPM is situated, with the aid of the transfer matrix $R^{ij}$. We obtain for the closed orbit $\vec{x}_i$ at position $i$

$$\vec{x}_i = R^{ij}(1 - R^{ij})^{-1} \vec{\varepsilon} = C^{ij} \vec{\varepsilon}$$  \hspace{1cm} (16)

where we defined the $4 \times 4$ response matrix of an observation at position $i$ with respect to a perturbation at position $j$ by $C^{ij} = R^{ij}(1 - R^{ij})^{-1}$. In particular the $C^{ij}_{12}$ matrix element
describes the effect of a horizontal corrector at position \( j \) to the horizontal orbit at position \( i \). In what follows superscripts denote positions in the beam line and subscripts label \( x, x', y, y' \). Mostly, subscripts will be 12 or 34, and we usually suppress them in formulas.

We now turn to the effect of a quadrupole displaced by \( \delta \vec{x}^j = (\Delta x, 0, \Delta y, 0) \) at position \( j \) on the orbit at position \( i \). The orbit after the displaced quadrupole \( \vec{x}_{\text{out}} \) is given in terms of the incoming orbit \( \vec{x}_{\text{in}} \) and the quadrupole displacement \( \delta \vec{x}^j \)

\[
\vec{x}_{\text{out}} = -\delta \vec{x}^j + Q(\vec{x}_{\text{in}} + \delta \vec{x}^j) = Q\vec{x}_{\text{in}} + (Q - 1)\delta \vec{x}^j
\]

where \( Q \) denotes the transfer matrix through the quadrupole. Note that in eq. 17 positive \( \delta \vec{x} \) corresponds to the position of the beam with respect to the quadrupole. In other word, the quadrupole is displaced to the negative direction. We see that the outgoing orbit consists of the unperturbed propagated incoming orbit and a perturbation

\[
\vec{x}^j = (Q - 1)\delta \vec{x}^j.
\]

Equation 18 can now be used in conjunction with eq. 16 to obtain the response of the orbit at position \( i \) to a quadrupole displacement at position \( j \) as

\[
\vec{x}^i = R^{ij}(1 - R^{ij})^{-1}(Q - 1)\delta \vec{x}^j = \tilde{C}^{ij}\delta \vec{x}^j
\]

where we defined the \( 4 \times 4 \) quadrupole response matrix \( \tilde{C}^{ij} = R^{ij}(1 - R^{ij})^{-1}(Q - 1) \). In particular the \( \tilde{C}^{ij}_{11} \) element determines the effect of a horizontally displaced quadrupole at position \( j \) on the horizontal orbit at position \( i \).

Finally note that the argument presented in the previous paragraph is not limited to displaced quadrupoles only. The entire argument is equally valid if a section with transfer matrix \( Q \) is displaced. If the length of this section is denoted by \( L \), we can also calculate the response of the orbit at position \( i \) to an angle misalignment (in the \( x-z \) or \( y-z \) planes) around the center of the section by rewriting the \( \alpha 2 \) and \( \alpha 4 \) elements of \( \tilde{C} \) as

\[
\tilde{C}^{ij}_{\alpha 2, \alpha 4} \rightarrow \tilde{C}^{ij}_{\alpha 2, \alpha 4} - \frac{L}{2} \tilde{C}^{ij}_{\alpha 1, \alpha 3} \quad \text{for} \quad \alpha = 1, \ldots, 4.
\]

In this way the effect of e.g. arbitrarily misaligned girders can be investigated.

Due to the large number of BPMs, orbit correctors and quadrupoles in LHC an efficient and fast evaluation of the response coefficients \( C \) and \( \tilde{C} \) is warranted. We solve this problem by evaluating the one-turn-map only once and storing all intermediate transfer matrices from the start of the beam line to a given element \( R^{k1} \) in a (very large) array. The transfer matrix from position \( j \) to position \( i \) is then given by

\[
R^{ij} = R^{i1}(R^{i-1,j})^{-1}.
\]

Noting that all matrices \( R^{k1} \) are symplectic, their inverse is equal to their symplectic conjugate which can be evaluated very fast by reshuffling the matrix elements \([4,5]\) according to

\[
\begin{pmatrix}
  r_{11} & r_{12} & r_{13} & r_{14} \\
  r_{21} & r_{22} & r_{23} & r_{24} \\
  r_{31} & r_{32} & r_{33} & r_{34} \\
  r_{41} & r_{42} & r_{43} & r_{44}
\end{pmatrix}^{-1} =
\begin{pmatrix}
  r_{22} & -r_{12} & r_{42} & -r_{32} \\
  -r_{21} & r_{11} & -r_{41} & r_{31} \\
  r_{24} & -r_{14} & r_{44} & -r_{34} \\
  -r_{23} & r_{13} & -r_{43} & r_{33}
\end{pmatrix}.
\]
Furthermore, the $C^{ij}$ defined in eq. 16 can be expressed in terms of $R^{k1}$ and thereby saving a few matrix inversions.

$$C^{ij} = \begin{cases} R^{i1} \left[ R^{i1} - R^{i1} R^{11} \right]^{-1} & \text{if } i \geq j \\ R^{i1} R^{11} \left[ R^{11} - R^{i1} R^{i1} \right]^{-1} & \text{if } i < j \end{cases}$$

(23)

where $R^{11}$ denotes the one-turn-map starting at the first element.

Appendix B: The Beam Line for the Other Beam

For the two-beam correction algorithm used in the interaction regions we need to know response coefficients for the second beam. They are obtained from the lattice for beam 2 which is constructed from that of beam 1 according to the following procedure:

1. Reverse the sequence in which the beam line is traversed, because beam 2 is rotating in the opposite direction of beam 1.
2. Reverse the signs of all magnets, because the particles in beam 2 have the same charge as those in beam 1, but are travelling in the opposite directions. Focussing quadrupoles for beam 1 are defocussing for beam 2.
3. Exchange the absolute values of the main quadrupoles QF and QD in the arcs, because the sequence in which the odd and even arcs are traversed is different for the two beams.
4. Exchange horizontal and vertical orbit correctors, because they are usually located near horizontally focussing and defocussing quadrupoles, respectively and the sign change in item 2 exchanged the quadrupoles. Now the correctors must follow.

Following these guidelines guarantees that the tunes for both beams turn out to be the same.

The above procedure does not take care of the signs of the corrector excitations, because they are not explicitly stated in the lattice file, but only appear indirectly in the response coefficients. We therefore have to switch the sign of beam 1 correctors if they act on beam 2 as is the case near the interaction points. This is the reason for the negative sign in the second column in lower half of the matrix in eq. 11.

Beyond the changes just discussed one has to consider the convention that positive $x$-axes for both beams point towards the outside of the rings, thus making the coordinate system of clock-wise rotating beam 1 right-handed and that of beam 2 left-handed. This coordinate change can be accommodated in the analysis by changing the coordinates of beam 2 by the transformation matrix $\text{diag}(-1,-1,1,1)$. Note that in an uncoupled machine the response coefficients do not change under this transformation.

Appendix C: Pseudo Inverses

In beam lines with more BPMs than correctors the problem of finding the best corrector configuration is over determined, i.e. there are more constraints than there are correctors to fulfil them. We therefore require that the following $\chi^2$

$$\chi^2 = \sum_{i=1}^{N_{\text{BPM}}} \left( x^i + \sum_{j=1}^{N_{\text{COR}}} C^{ij} \varepsilon^j \right)^2$$

(24)
is minimal. Differentiating both sides of eq. 24 with respect to $\varepsilon^j$, in order to obtain the extremum, we obtain a linear system of equations for the minimizing $\varepsilon^j$, which in turn can be solved to yield

$$
\varepsilon^j = -\left(\left(C^T C\right)^{-1} C^T\right)^{ii} x^i.
$$

(25)

In the main body of this report the expression $(C^T C)^{-1} C^T$ is called $\chi^2$—inverse of $C$.

In the arcs of LHC the number of orbit correctors equals the number of BPMs which allows optimum steering by using each corrector to fix the orbit at an associated BPM and thus steering the rms orbit to zero. If, however, a BPM is defect the number of correctors exceeds the number of BPMs, leading to the task of solving an under determined linear system. In other words: we cannot simply invert the matrix $C$, featured prominently in eq. 1 in order to obtain the corrector strengths from the BPM readings, because $C$ is rank deficient. We therefore need an algorithm to invert a rank deficient matrix.

It turns out that a method based on Singular Value Decomposition (SVD) [6] of the matrix $C$ allows to do just this with the pleasant side effect of minimizing the corrector excitations at the same time [7]. The idea of SVD is to decompose the matrix $C$ into a product of two orthogonal matrices $U$ and $V$ and a diagonal matrix $W$ according to

$$
C = U W V^T
$$

(26)

where we assume that $C$ is made square by adding zeros in the appropriate places. The inverse of $C$ is then readily calculated because for orthogonal matrices their inverse is equal to their transpose. We thus obtain for the inverse of $C$

$$
C^{-1} = V W^{-1} U^T.
$$

(27)

The only problem arising from eq. 27 is the inverse of the diagonal matrix $W^{-1}$, because due to the rank deficiency of $C$ it contains one or more zeros on the diagonal. This problem can be overcome by noting that the orthogonal matrices $V$ and $W$ are made up of eigenvectors into the subspaces with the eigenvalues that appear on the diagonal of $W$. The eigenvectors corresponding to zero eigenvalues thus span the null space [6]. Consequently, by applying the rule $1/w_i = 0$ if $w_i = 0$ effectively removes any projection onto the nullspace and entails the minimization of the solution vector that results in applying $C^{-1}$ to some vector.

The algorithm presented in the previous paragraph uses the freedom gained – by not having to fit the orbit to a particular, defect BPM – to minimize the rms corrector excitation. However, then the defect BPM is not used in the correction algorithm and the orbit on that BPM can attain large values. Remedies to that adverse effect are dealt with in the main body of this paper.

**Appendix D: Comparison with MICADO**

In the main body of this paper we always use all available correctors in order to correct a given orbit due to quadrupole and BPM misalignment. In practice it is often desirable to use only a small number of correctors to fix a given orbit, because BPM and corrector scale factors can corrupt the correction process, especially if a large number of correctors is involved. To remedy the deterioration of the correction process we utilize the MICADO
algorithm to determine the most effective correctors and use only those. On the other hand, using a small number of correctors, the required correction strengths may increase. In this appendix we therefore investigate how the required rms corrector strength and the achievable rms and peak orbit vary as a function of the number of used correctors. We perform this investigation for IR 1 and for arc 1.

For this analysis we correct 200 quadrupole misalignment seeds with a rms quadrupole misalignment of $\sigma_Q = 0.3$ mm, truncated at three standard deviations, using the MICADO algorithm with a given fixed number of correctors. We then produce histograms of the required corrector strengths and the achievable rms and peak orbits which are characterized by the 70 and 85% margins similar to the analysis in sections 5 and 6. With only a few used correctors the histograms of the corrector strength show a pronounced double hump structure rather than a gaussian. This can clearly be attributed to the fact that MICADO chooses the most effective - and thereby stronger - correctors, which are consequently represented more prominently in the histogram, resulting in the afore-mentioned double hump structure. Nevertheless we characterize the width of the corrector distribution by its rms, which results in a pessimistic estimate of the width, because the center of the distribution is under-represented. This rms is now used to calculate an effective correctability $\kappa_{Q,\mathrm{eff}} = \sigma_Q / \sigma_e$ which in turn may
now be used in eq. 5 to calculate the probability of finding a required corrector strengths of larger than 50 \( \mu \text{rad} \). The results for IR 1 are shown in Fig. 15 where we display the 70 and 85\% margins of the achievable rms and peak orbits, the effective correctability, and the probability to exceed 50 \( \mu \text{rad} \).

We clearly see that all quantities are rather smooth functions of the number of used correctors and that using about 10 correctors in IR 1 (out of 22 independent correctors in both rings) allows to reduce the rms orbit to below 0.3 mm and the peak orbit below 1 mm. The probability of finding a required corrector strength exceeding 50 \( \mu \text{rad} \) is below 1\%.

Performing the same analysis for arc 1 we found the same smooth behaviour. We conclude that the statements about correctability and achievable rms and peak orbits in the IRs and the arcs made in the main body of this report also qualitatively apply, even if only the most effective correctors (MICADO) are used in the correction process.