TUNE SHIFTS FROM SELF-FIELDS AND IMAGES

A. Hofmann
CERN, Geneva, Switzerland

Abstract
The self-fields created by the particles in the beam are modified by the boundary conditions imposed by the beam surroundings. They act back on the beam and influence its motion and change the betatron frequencies. Part of this effect is caused by the direct Coulomb force between the particles. For a moving beam this Coulomb field is accompanied by a magnetic field. Since the forces due to these two fields have opposite signs there is a partial cancellation which becomes more perfect as the particle speed approaches that of light. This leads to the $1/\gamma^2$ dependence of the direct space-charge force. For a smooth and perfectly conducting vacuum chamber the boundary conditions for the field can be satisfied by introducing image charges or currents. Their fields act on the beam and create an indirect space-charge effect. Since the electric and magnetic fields have quite different boundary conditions the cancellation between the electric and magnetic forces is disturbed. The indirect space-charge force does therefore not decrease with $1/\gamma^2$ and can be dominant at high energy. If the beam as a whole does not oscillate the space-charge forces are stationary and affect only the focusing of the individual particles leading to the incoherent tune shifts. For a beam executing a center-of-mass motion the source of the space-charge fields changes resulting in a coherent tune shift which can be different from the incoherent one. This coherent effect is only important for the indirect space-charge fields since the direct effect is caused by internal forces which move with the beam. For a realistic beam the space-charge tune shift of an individual particle depends on its betatron amplitude and its longitudinal position along the bunch. As a consequence the space-charge forces lead to a tune spread. The beam occupies therefore a sizable area in the tune diagram leaving a limited space between the resonances.

1. INTRODUCTION

The motion of a single particle in an accelerator is determined by the external guide fields, i.e. the dipole and quadrupole magnets, the RF system, etc. A beam containing many particles represents a sizable charge and current which create fields of their own. They are modified by the boundary conditions imposed by the beam surroundings represented by the vacuum chamber, cavities etc. and act back on the beam. This influences the motion of the individual particles (incoherent effect) as well as of the beam as a whole (coherent effect).
We concentrate here on effects which influence the transverse particle motion and change the betatron frequencies. We also restrict ourself to simple cases which can be calculated without problems. More detailed investigations and more complicated cases can be found in some review papers [1, 2, 3]. Furthermore we assume here that the space-charge effect is weak and can be treated as a perturbation. Other methods must be used to describe space-charge dominated beams [4].

2. DIRECT INCOHERENT TUNE SHIFTS

2.1 Fields of a coasting beam with circular cross section

We consider now a coasting, i.e. unbunched beam with a circular cross section of radius $a$ and uniform charge density $\eta$ moving with constant velocity $\nu = \beta c$ in the $s$-direction, Fig. 1. This beam represents a charge per unit length $\lambda = \pi a^2 \eta$, a current density $J \beta c \eta$ and a total current $I = \beta c \lambda = \pi a^2 J = \pi a^2 \beta c \eta$. The charge produces an electric field $E$ and the current a magnetic field $B$ which are determined by the following relations:

$$\text{div } E = \eta / \varepsilon_0$$
$$|E| = E_r$$
$$\iint \text{div } E \, dV = \oint E \, dS \text{ (Gauss)}$$

$$\text{curl } B = \mu_0 J$$
$$|B| = B_\phi = \mu_0 \eta \beta c$$
$$\int B \cdot ds = \iint \text{curl } B \cdot dS \text{ (Stokes)}.$$ 

Here $dV$ is the volume, $dS$ the surface and $ds$ the line element of the integration. We consider a cylinder of radius $r$ and length $\ell$ as shown in Fig. 1 and calculate the fields inside ($r \leq a$) and outside ($r > a$) the beam.

$$r \geq a$$

$$\pi \ell r^2 \eta / \varepsilon_0 = 2 \pi \ell r E_r$$
$$E_r = \frac{\eta}{2 \varepsilon_0} r = \frac{\lambda}{2 \pi \varepsilon_0 a^2} = \frac{I}{2 \pi \varepsilon_0 \beta c a^2}$$

$$2 \pi r B_\phi = \mu_0 \pi r^2 J_z = \mu_0 \beta \eta c$$
$$B_\phi = \frac{\beta \eta}{2 \varepsilon_0 c} \frac{r}{2 \pi \varepsilon_0 a^2}$$

$$r \leq a$$

$$\pi \ell a^2 \eta / \varepsilon_0 = 2 \pi \ell E_r$$
$$E_r = \frac{\eta}{2 \varepsilon_0} a^2 = \frac{\lambda}{2 \pi \varepsilon_0 \beta c} = \frac{I}{2 \pi \varepsilon_0 \beta c r}$$

$$2 \pi r B_\phi = \mu_0 \pi r^2 J_z = \mu_0 c \pi \beta \eta$$
$$B_\phi = \frac{\beta \eta}{2 \varepsilon_0 c} \frac{a^2}{2 \pi \varepsilon_0 a^2} = \frac{I}{2 \pi \varepsilon_0 a^2}$$

We made the replacement $\mu_0 = 1 / (\varepsilon_0 c^2)$ and used the formal relation $\lambda = I / \beta c$ to express the line charge density $\lambda$ by the current $I$ which can be more easily measured. One should however remember that the electric field is created by charge and not by the current.
We calculate now the direct space-charge effect which is caused by the internal forces in the beam and apply the obtained results to calculate the space-charge forces on a particle with charge $\epsilon$ inside the beam. From the fields

$$E_r = \frac{\eta}{2\epsilon_0} r, \quad B_\phi = \frac{\beta \eta}{c 2\epsilon_e} r$$

we get the force

$$\mathbf{F} = \epsilon (\mathbf{E} + [\mathbf{v} \times \mathbf{B}]) = \frac{\epsilon \eta}{2\epsilon_0} (1 - \beta^2) r = \frac{\epsilon \eta}{2\epsilon_0 \gamma^2} r = \frac{eI}{2\pi \epsilon_0 \beta \gamma^2} r$$

where we introduced the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$ 

The force is transverse and has the components

$$F_x = \frac{e \eta}{2\epsilon_0 \gamma^2} x, \quad F_y = \frac{e \eta}{2\epsilon_0 \gamma^2} y.$$

The electrostatic and the magnetic forces have opposite directions which leads to the partial cancellation between the two forces which becomes more perfect as the velocity of the particle approaches the one of light. This is one of the most important properties of the space-charge effects. For relativistic particles with $\gamma \gg 1$ the direct space-charge force is relatively small. However, we will later investigate several effects which disturb the cancellation between the two forces.

2.2 The incoherent betatron tune shift due to the direct space-charge force

2.2.1 Coasting beam uniform density and focusing

The cross section of a beam circulating in a realistic machine has a different cross section in the focusing and defocusing quadrupoles. However, to start with the most
simple case, we approximate for a uniform focusing where the beta functions are constant and given by the betatron tunes $Q_x$ and $Q_y$, i.e. the number of betatron oscillations per turn, and the average radius $R$ of the ring

$$\beta_x \approx \langle \beta_x \rangle \approx \frac{R}{Q_x}, \quad \beta_y \approx \langle \beta_y \rangle \approx \frac{R}{Q_y}.$$  

Furthermore we assume the beam to have a circular cross section of radius $a$, a uniform density and to be unbunched.

The motion of the particle in the beam in the absence of space-charge is determined by the linear focusing provided by the quadrupoles and is described by the equation

$$\frac{d^2 y}{dt^2} + \omega^2 y = \ddot{y} + Q_y^2 \omega_0^2 y = 0,$$

where $\omega_0$ is the betatron frequency and $\omega_0 = \beta c/R$ the revolution frequency. The solution is a betatron oscillation of the form

$$y(t) = y_0 \cos(Q_y \omega_0 t).$$

The presence of the space-charge force imposes an additional acceleration to the individual particles in the beam

$$\ddot{y}_{s.c.} = \frac{F_y}{m_\gamma} = \frac{e\eta}{2\epsilon_0 \gamma^2 m_0 \gamma^3} y,$$

which has to be included and leads to the equation of motion

$$\ddot{y} + Q_y^2 \omega_0^2 y = \frac{e\eta}{2\epsilon_0 m_0 \gamma^3},$$

or

$$\ddot{y} + \left( Q_y^2 \omega_0^2 - \frac{e\eta}{2\epsilon_0 m_0 \gamma^3} \right) y = 0.$$  

The term in the large brackets represents the square of the new betatron frequency which includes the space-charge force. We call the corresponding tune $Q_y$ which is related to the unperturbed tune $Q_{y0}$ by

$$Q_y^2 = Q_{y0}^2 \left( 1 - \frac{e\eta}{2\epsilon_0 Q_{y0}^2 \omega_0^2 m_0 \gamma^3} \right) y. = Q_{y0}^2 \left( 1 - \frac{eI}{2\pi \epsilon_0 Q_{y0}^2 \omega_0^2 a^2 m_0 \beta c \gamma^3} \right) y.$$  

Since the space-charge force is linear in $y$ like the external focusing force it just leads to a change $\Delta Q_y$ of the betatron tune of the individual particle. We assume now that this change is small, $\Delta Q_y \ll Q_{y0}$, such that the quadratic term in it can be neglected

$$Q_y^2 = (Q_{y0} + \Delta Q)^2 \approx Q_{y0}^2 + 2Q_{y0} \Delta Q$$

and get for the incoherent tune shift

$$\Delta Q_y \approx -\frac{e\eta}{4\epsilon_0 Q_{y0}^2 \omega_0^2 m_0 \gamma^3}. = -\frac{e\eta}{4\pi \epsilon_0 Q_{y0}^2 \omega_0^2 a^2 m_0 c^2 \beta \gamma^3}.$$  

332
We introduce the classical electron radius

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2} = 1.54 \cdot 10^{-18} \text{ m for protons}$$

$$2.82 \cdot 10^{-15} \text{ m for electrons}$$

and get

$$\Delta Q_y = -\frac{r_0 c I}{\varepsilon_0 \beta^3 \gamma^3}.$$

We approximated for uniform focusing with a beta function $\beta_y$ and an emittance $E_y$

$$\beta_y \approx \langle \beta_y \rangle \approx \frac{R}{Q_{y0}}, \quad E_y \approx \frac{a^2}{\beta_y} \approx \frac{Q_{y0}}{R} a^2.$$

With this we get an expression for the incoherent tune shift which uses more practical machine and beam parameters

$$\Delta Q_y = -\frac{r_0 c I}{\varepsilon c E_y \beta^3 \gamma^3}, \quad \Delta Q_x = -\frac{r_0 R I}{\varepsilon c E_x \beta^3 \gamma^3},$$

where we also give the corresponding expression for the horizontal plane.

In deriving these equation we made several assumptions like uniform focusing, circular beams, unbunched beams and uniform density. We will now have a look at these restrictions.

2.2.2 Non-uniform focusing

We can derive the space-charge tune shift differently by applying an expression commonly used in beam optics for the tune shift due to a localized weak lens

$$\Delta Q_x = \frac{K_x \ell \beta_y}{4\pi},$$

where $\ell$ is the length of the lens and $K_x$ the focusing parameter which is given by

$$K_x = \frac{e}{m_0 c \beta \gamma} \frac{dB_y}{dx}$$

for a magnetic quadrupole lens or by

$$K_x = -\frac{d(1/\rho)}{dx} \approx -\frac{1}{x} \frac{d^2 x}{ds^2}$$

for the general case with the curvature $1/\rho \approx d^2 x/ds^2$ of the trajectories in a linear lens. The space-charge force gives an acceleration

$$\ddot{x} = \frac{F_x}{m_0 \gamma} \approx \frac{d^2 x}{ds^2} \beta^2 c^2 = \frac{2r_0 c I}{e a^2 \beta \gamma^3} x$$

which gives for the focusing parameter

$$K_x = \frac{2r_0 I}{\pi e a^2 \beta^3 \gamma^3}.$$
We regard now a beam disk of length $ds$ as a short lens which produces a tune change
\[ d(\Delta Q_x) = \frac{\beta_x K_x}{4\pi} ds = -\frac{r_0 I \beta_x}{2\pi eca^2\beta^3\gamma^3} ds = -\frac{r_0 I}{2\pi ecE_x\beta^3\gamma^3} ds. \]

The emittance $E_x = a^2/\beta_x$ is invariant and we integrate around the ring and get for the total space-charge tune shift the same expression as before
\[ \Delta Q_x = -\int_{0}^{2\pi R} \frac{r_0 I}{2\pi ecE_x\beta^3\gamma^3} ds = -\frac{r_0 I R}{ecE_x\beta^3\gamma^3}. \]

The corresponding expression for the horizontal plane is of the same form since we assumed circular beams.

It is interesting to note that the direct space-charge tune shift depends on the emittance and not on the $\beta$-function (local focusing) or the tune $Q_x$ (global focusing). At places where the beta function is small the beam is small too which results in a strong space-charge force. On the other hand a given lens produces a smaller tune change if it is located at a small value of the beta function. However, we assumed here a beam being circular all around the ring which is not realistic and will be discussed later. The tune shift decreases with the third power of the Lorentz factor $\gamma$. The cancellation between the magnetic and electric forces contributes a power of two and the stiffness of the beam a power of one to this dependence. For a given current the space-charge effect increases with the radius $R$ of the machine. However, we can eliminate this dependence by expressing the current with the number of circulating particles $N_b$
\[ I = \frac{eN_b\omega_0}{2\pi} = \frac{eN_b\beta c}{2\pi R}, \]

which gives
\[ \Delta Q_x = -\frac{r_0 N_b}{2\pi E_x\beta^2\gamma^3} = -\frac{eq_b}{8\pi^2 c_0 E_x E\beta^2\gamma^2}, \]
where $e$ is the charge of a particle and $q_b = eN_b$ of the whole beam and $E = m_0c^2\gamma$ the particle's energy.

2.2.3 Uniform elliptic beam

We generalize the above result for a beam with elliptic cross section with half-axes $a$ and $b$ but still with uniform charge density $\eta$. The electric field inside such a beam is given by [5]
\[ E = \frac{I}{\pi e_0(a+b)\beta c} \left( \frac{x}{a}, \frac{y}{b}, 0 \right), \quad B = \frac{\mu_0 I}{\pi(a+b)} \left( -\frac{y}{b}, \frac{x}{a}, 0 \right) \]
which can be checked to be consistent with $\text{div} E = \eta/e_0$ and $\text{curl} B = \mu_0 J$. The space-charge force becomes
\[ F = e(E + [v \times B]) = \frac{I}{\pi e_0\beta c\gamma^2(a+b)} \left( \frac{x}{a}, \frac{y}{b}, 0 \right) \]
and the space-charge tune shifts

\[ \Delta Q_x = - \frac{2r_0 I_{,x} \beta_x}{2 \pi c \epsilon_0 \beta_3 \gamma^3 (a + b) a} \int ds = - \frac{2r_0 I}{ec \beta_3 \gamma^3 E_x} \left( \frac{a}{a + b} \right) \]

\[ = - \frac{r_0 IR}{ec \beta_3 \gamma^3 E_x} \left( \frac{2}{1 + \sqrt{\kappa \beta_y / \beta_x}} \right) \]

\[ \Delta Q_y = - \frac{2r_0 I_{,y} \beta_y}{2 \pi c \epsilon_0 \beta_3 \gamma^3 (a + b) b} \int ds = - \frac{2r_0 I}{ec \beta_3 \gamma^3 E_y} \left( \frac{b}{a + b} \right) \]

\[ = - \frac{r_0 IR}{ec \beta_3 \gamma^3 E_y} \left( \frac{2}{1 + \sqrt{\beta_x / (\kappa \beta_y)} \right), \]

where \( \kappa = E_y / E_x \) is the ratio between the vertical and horizontal emittances, also called coupling factor. We see that in the case of elliptic beams the space-charge tune shift depends somewhat on the beam optics of the ring.

### 2.2.4 Bunched beams

For most applications we deal with bunched beams and the current \( I(s - s_0) \) is a function of the longitudinal position \( s \) from the bunch center \( s_0 \). This leads to some longitudinal force and can make the space-charge effect more complicated. However, for relativistic beams a simple approximation can be used. The electric field of a particle moving with velocity \( v = \beta c \) is mainly transverse with an opening angle of order \( 1/\gamma \). The transverse forces seen by a particle in the beam are mostly caused by the part of the beam which is within a longitudinal distance \( \Delta s \leq a / \gamma \). If the bunch is sufficiently long that the current \( I(s) \) changes little over this distance \( \Delta s \) we get the direct space-charge tune shift of a bunched beam by replacing in the coasting beam expressions the beam current \( I \) by its local value \( I(s) \)

\[ \Delta Q_y(s - s_0) = - \frac{r_0 I_{,y}(s - s_0)}{ec E_y \beta_3 \gamma^3} \quad \text{and} \quad \Delta Q_x(s - s_0) = - \frac{r_0 I_{,x}(s - s_0)}{ec E_x \beta_3 \gamma^3}. \]

The space-charge tune shift of a particle depends now on its longitudinal position in the bunch. This leads to a tune spread in the bunch and, since the particle executes synchrotron oscillations, to a modulation of the tune.

### 2.2.5 Non-uniform charge density

So far we assumed that the charge density \( \eta \) in the beam is constant which gave a linear defocusing force. For a more general transverse charge distribution this force is no longer linear and the tune shift will depend on the betatron amplitude of a particle. We illustrate this for a circular beam with a Gaussian density distribution

\[ \eta(r) = \frac{I}{2 \pi \beta c \sigma^2} e^{-\frac{r^2}{2 \sigma^2}} \]

which leads to the field components

\[ E_r = \frac{I}{2 \pi \epsilon_0 \beta c} \frac{1}{r} \left( 1 - e^{-\frac{r^2}{2 \sigma^2}} \right) \quad \text{and} \quad B_\phi = \frac{I}{2 \pi \epsilon_0 c^2} \frac{1}{r} \left( 1 - e^{-\frac{r^2}{2 \sigma^2}} \right) \]
The presence of a perfectly conducting plate close to the beam imposes a boundary condition \( E_{\parallel} = 0 \) on its surface. This modifies the electric field of the beam. It is possible to solve the electrodynamic equations for the new boundary conditions. There is a relatively simple method to do this by introducing image charges. We consider now a beam which has at a distance \( h \) a horizontal infinite plate which is perfectly conducting. We introduce an image charge with the same magnitude of line density as the beam but of opposite sign, Fig. 2. On the surface of the plate the electric field \( E_i \) of the image is such that it cancels the parallel field component of the beam itself. The total field in front of the plate consists of the direct part \( E_d \) and the part created by the images.

\[
E = E_d + E_i, \quad E_{\parallel} = E_{d\parallel} + E_{i\parallel} = 0.
\]

The image charge is all behind the plate and its divergence in front of the plate vanishes

\[
\text{div} \, E = \text{div} \, E_d + \text{div} \, E_i = \text{div} \, E_d = \frac{\eta}{\varepsilon_0}.
\]
In front of the plate the total field fulfills therefore Maxwell's equations and the boundary condition.

We would now like to calculate the effect of the beam surroundings on the space-charge effect. We consider first a flat vacuum chamber and approximate it by two parallel conducting plates at the vertical distance $\pm h$ from the beam. To fulfill the boundary conditions we introduce first a negative line charge at the distance $h$ behind the upper and lower plate, i.e. at the distance $\pm 2h$ from the beam. However, the field created by the image of the upper plate is also influenced by the boundary condition of the lower plate. This can be satisfied by introducing a positive image line charge at $3h$ below the lower plate. To fulfill all these conditions we end up with negative images at the distances $\pm 2nh$ with $n = 1, 3, 5, \ldots$ and positive images at $\pm 2nh$ with $n = 2, 4, 6, \ldots$ as shown in Fig. 3.

We calculate the field of each image $\lambda_{in}$ as a function of the coordinates $x$ and $y$ in the vicinity of the beam. Since the images are relatively far away from the beam compared to its size, $h \gg a$, we neglect quadratic and higher terms in $x$ and $y$. The field of the $n$th image pair at the distance $\pm 2nh$ is

$$E_{in\gamma} = (-1)^n \frac{\lambda}{2\pi \varepsilon_0} \left( \frac{1}{2nh + y} - \frac{1}{2nh - y} \right) = (-1)^n \frac{\lambda}{2\pi \varepsilon_0 (2nh)^2} - 2y = -\frac{\lambda y}{2\pi \varepsilon_0 h^2} \frac{(-1)^n}{n^2}. $$

$$E_{in\gamma} = -\frac{\lambda y}{4\pi \varepsilon_0 h^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\lambda y}{4\pi \varepsilon_0 h^2} \frac{\pi^2}{12}. $$
To get the field of all charges we summed over \( n \) and used the expression
\[
\sum_{1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}
\]
which can be found in standard text books. The horizontal field could be evaluated the same way but we can get it directly from
\[
\text{div} \mathbf{E}_i = \frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} = 0
\]
from which one gets
\[
\frac{\partial E_{ix}}{\partial x} = -\frac{\partial E_{iy}}{\partial y} = -\frac{\lambda}{4\pi \varepsilon_0 h^2} \frac{\pi^2}{12}
\]
or
\[
E_{ix} = -\frac{\lambda y}{4\pi \varepsilon_0 h^2} \frac{\pi^2}{12}.
\]

The boundary condition imposed by the conducting plates does not affect the magnetic field. As a consequence there is no cancellation effect for the forces due to the images. The total force is therefore
\[
F_x = \frac{2e\lambda y}{2\pi \varepsilon} \left( \frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2} \right)
\]
\[
F_y = \frac{2e\lambda y}{2\pi \varepsilon} \left( \frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2} \right)
\]
and the resulting tune shift
\[
\Delta Q_x = -\frac{2r_0 I R(\beta_x)}{ec\beta^3 \gamma} \left( \frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2} \right)
\]
\[
\Delta Q_y = -\frac{2r_0 I R(\beta_y)}{ec\beta^3 \gamma} \left( \frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2} \right),
\]
where we used again the formal substitution \( I = \lambda \beta c \).

The first term in the large brackets is the direct space-charge effect which decreases as \( 1/\gamma^2 \) due to the cancellation between the forces. It also depends on the beam size, i.e. on the charge density. The second term gives the indirect effect of the wall which does not have the cancellation and which does not depend on the beam size within our approximation \( h \gg a \) but on the distance \( h \) of the plate. In addition both effects decrease as \( 1/\gamma \) due to the particles becoming more rigid at higher energy.

### 3.2 Ferromagnetic boundary

We look now at ferromagnetic boundaries which affect the magnetic field and can be treated in a way similar to the conducting plates. They appear mostly in the form of the dipole magnet pole pieces and can be approximated by a horizontal ferromagnetic sheet at a distance \( g \). They impose the boundary condition \( B_{||} = 0 \) for the magnetic field. This can be satisfied by a positive image current at a distance \( g \) behind the sheet, i.e. at \( 2g \) from the beam; Fig. 4. The original and the image current produce fields on
the ferromagnetic surface with opposite parallel components. To approximate a realistic magnet we need a sheet above and below the beam which makes it necessary to introduce image currents at $\pm 2gh$ from the beam similarly to the conducting plates but here all image currents have the same sign as the beam itself. Using again only linear terms in $x$ and $y$ we get for the magnetic field produced by these image currents

$$B_{ix} = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left( \frac{1}{2ng-y} - \frac{1}{2ng+y} \right) = \frac{\mu_0 I_y}{4\pi g^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\mu_0 I_y \pi^2}{4\pi g^2 6},$$

where we used the expression

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Including this we get for the total vertical force

$$F_y = \frac{2e I_y}{2\pi e^2 \beta_c} \left( \frac{1}{2a^2 \gamma^2} + \frac{\pi^2}{48h^2} + \frac{\pi^2 \beta^2}{24g^2} \right),$$

and for the tune shift

$$\Delta Q_y = -\frac{2r_0 I R(\beta_y)}{e c \beta^3 \gamma} \left( \frac{1}{2a^2 \gamma^2} + \frac{\pi^2}{48h^2} + \frac{\pi^2 \beta^2}{24g^2} \right).$$

To get the horizontal tune shift due to this horizontal ferromagnetic sheet we use the same reasoning as before

$$\text{div} B = \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} = 0$$

from which we get the vertical magnetic field

$$B_{iy} = -\frac{\mu_0 I \pi^2}{4\pi g^2 6},$$
and the tune shift
\[ \Delta Q_z = -\frac{2r_0 I R(\beta_z)}{\epsilon c \beta^3 \gamma} \left( \frac{1}{2a^2 \gamma^2} - \frac{\pi^2}{48 h^2} - \frac{\pi^2 \beta^2}{24 g^2} \right). \]

The ferromagnetic sheet influences the magnetic field. However, realistic pole pieces are usually also conducting and can affect the electric field also. In most practical cases the conducting vacuum chamber is between the beam and the pole piece \((g > h)\) and not penetrated by the electric field.

4. **COHERENT TUNE SHIFTS**

4.1 Coherent transverse oscillations

So far we treated the effect of the space-charge on the incoherent oscillation of individual particles inside the beam. In this case the center of mass is at rest, i.e., the source of the space-charge fields does not move. Now we investigate the effect on coherent oscillations in which the beam as a whole executes a betatron oscillation such that all particles move together in phase. The space-charge fields are now modulated with this motion and act back on the beam with a well-determined phase. For this reason the coherent tune shift can be quite different from the incoherent one.

4.2 Coherent tune shift in free space

In the absence of any walls there are only the direct space-charge forces acting on the beam. If the beam executes a coherent oscillation these space-charge forces are internal and move with the beam. They do not affect the motion of the beam as a whole and the coherent tune is not changed. However, the internal motion of the individual particles with respect to the center of mass is influenced by the direct space-charge force in the same way as for a beam at rest. Sometimes one calls the coherent tune shift the difference between the coherent and the incoherent tune. Since the coherent tune does not change we find for the free space case

\[ Q_{coh.} = Q_0, \quad Q_{incoh.} = Q_0 + \Delta Q, \quad \delta Q_{coh.} = Q_{coh.} - Q_{incoh.} = -\Delta Q. \]

4.3 Coherent tune shifts for conducting parallel plates

We come back to the approximation of the vacuum chamber by conducting plates treated before but we assume now that the beam makes a coherent motion and has at present a vertical deviation \(\tilde{y}\). Since the beam as a whole is displaced the images move also. We calculate the field produced by the first image pair at the position \(y = \tilde{y}\) of the beam. The upper image is at the nominal distance \(2h\). If the beam is displaced upwards by \(\tilde{y}\) the image moves towards the beam by the same amount such that the distance from the beam to the upper image becomes \(2h - 2\tilde{y}\), and to the lower image \(2h + 2\tilde{y}\) giving a field at the beam, Fig. 5.
The field of all image charges is

\[ E_{cl_1} = \frac{-\lambda}{2\pi\epsilon_0} \left( \frac{1}{2h + \tilde{y} + \tilde{y}} - \frac{1}{2h - \tilde{y} - \tilde{y}} \right). \]

The second upper-image charge, which is really an image of the first lower-image charge, moves in the same direction as the beam such that its distance from the beam does not change. The field due to the second image pair can be expressed as

\[ E_{cl_2} = \frac{\lambda}{2\pi\epsilon_0} \left( \frac{1}{2h + \tilde{y} - \tilde{y}} - \frac{1}{2h - \tilde{y} + \tilde{y}} \right). \]

Generalizing this we get the field of the \( n \)-th image pair at the location of the displaced beam

\[ E_{cn_1} = \frac{(-1)^n\lambda}{2\pi\epsilon_0} \left( \frac{1}{2h + \tilde{y}(1 - (-1)^n)} - \frac{1}{2h - \tilde{y}(1 - (-1)^n)} \right) \]

\[ = \frac{(-1)^n\lambda\tilde{y}}{4\pi\epsilon_0h^2} \left( \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right). \]

The field of all image charges is

\[ E_{cn_1} = \frac{\lambda}{2\pi\epsilon_0h^2} \left( \sum_{1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{1}^{\infty} \frac{1}{n^2} \right) = \frac{\lambda}{2\pi\epsilon_0h^2} \left( \frac{\pi^2}{12} + \frac{\pi^2}{6} \right), \]

and the coherent force

\[ F_{cv} = \frac{\epsilon\lambda\tilde{y}}{\pi\epsilon_0h^2} \frac{\pi^2}{16}. \]
This gives for the coherent tune

\[ Q_{\text{coh.}} = Q_0 - \frac{\pi^2 2r_0 I R(\beta_\gamma)}{16 e c B^2 \gamma^2 h^2}. \]

The difference between coherent and incoherent tune becomes

\[ \delta Q_{\text{coh.}} = Q_{\text{coh.}} - Q_{\text{inc.}} = \frac{\pi^2 2r_0 I R(\beta_\gamma)}{16 e c B^2 \gamma^2} \left( \frac{1}{2a^2 \gamma^2} \frac{\pi^2}{48h^2} - \frac{\pi^2}{16h^2} \right). \]

For the horizontal conducting plates considered here a coherent horizontal motion is not influenced and there is no coherent horizontal tune change \( Q_{\text{coh.}} = Q_0 \). For a more realistic boundary condition like the one imposed by an elliptic chamber this is no longer the case.

4.4 Circular chamber

We consider now a beam surrounded by a conducting circular cylinder of radius \( \rho \). For the equilibrium case the beam is assumed to go along the \( z \)-axis. As a result a uniform surface charge of \( -\lambda \) is induced on the inside of this cylinder with a total charge per unit length. The field of this symmetrically distributed charge as well as its derivative vanish on the axis. Therefore the presence of the conducting cylinder does not affect the incoherent space-charge tune shift of the centered beam.

Next we assume the beam to be displaced in the horizontal direction by \( \bar{x} \). The induced surface charges are no longer uniformly distributed and will produce a field at the beam. We try to satisfy the boundary condition on the surface by introducing an image line charge at the distance \( b \) from the axis. For symmetry reason it must lie in the \( x - z \) plane as the displaced beam. Using cylindrical coordinates \( \rho, \phi, z \) we get for the boundary condition

\[ E_\parallel = E_\phi = 0. \]

According to the upper part of Fig. 6 the beam produces such a field component

\[ E_{\phi 1} = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r_1} \sin \alpha_1, \]

with

\[ r_1^2 = \rho^2 + \bar{x}^2 - 2\rho \bar{x} \cos \phi, \quad \frac{\sin \alpha_1}{\sin \phi} = \frac{\bar{x}}{r_1}. \]

we get

\[ E_{\phi 1} = \frac{\lambda}{2\pi \epsilon_0} \frac{\bar{x} \sin \phi}{\rho^2 + \bar{x}^2 - 2\rho \bar{x} \cos \phi}. \]

The field due to the image is obtained with help of the lower part of Fig. 7.

\[ E_{\phi i} = -\frac{\lambda}{2\pi \epsilon_0} \frac{1}{r_2} \sin \alpha_2, \]

with

\[ r_1^2 = \rho^2 + b^2 - 2\rho b \cos \phi, \quad \frac{\sin \alpha_2}{\sin \phi} = \frac{b}{r_2}. \]
The image field at the beam has only an \( x \) component

\[
E_{ix}(\bar{x}) = \frac{\lambda}{2\pi\varepsilon_0 \rho^2} \frac{\bar{x}}{\bar{x}}
\]
which leads to a coherent tune shift

\[ Q_{\text{coh.}} - Q_0 = \frac{1}{2} \frac{2r_0 I R^2 (\beta_x)}{ec^2 \beta^3 \gamma^2 \rho^2} . \]

Due to circular symmetry one gets the same tune shift in the vertical plane.

4.5 The Laslett coefficients

We have calculated the indirect space-charge effect for the cases of a conducting and a ferromagnetic plate only. For more general beam surroundings the expressions have a similar form but with different numerical factors. We can write the incoherent tune shift in the form

\[ \Delta Q_v = - \frac{2r_0 I R (\beta_y)}{ec^2 \beta^3 \gamma^2} \left( \frac{1}{2a^2 \gamma^2} + \varepsilon_1^V \frac{1}{h^2} + \varepsilon_2^V \frac{1}{g^2} \right) \]

where \( \varepsilon_1^V \) and \( \varepsilon_2^V \) are the electric and magnetic Laslett space-charge coefficients for the vertical plane. The coherent shift can be written in a similar form with the coefficients \( \xi_1^V \) and \( \xi_2^H \).

We get for the conducting plates at a distance \( h \)

\[ \varepsilon_1^V = \frac{\pi^2}{48}, \quad \varepsilon_1^H = -\frac{\pi^2}{48} \]
\[ \varepsilon_2^V = \frac{\pi^2}{24}, \quad \varepsilon_2^H = -\frac{\pi^2}{24} \]
\[ \xi_1^V = \frac{\pi^2}{16}, \quad \xi_2^H = 0 \]

and for a circular chamber of radius \( h \):

\[ \varepsilon_1^V = \xi_1^H = 0 \]
\[ \xi_1^V = \xi_2^H = 0.5. \]

5. COMMENTS

5.1 Incoherent tune shifts

- The direct space-charge changes the incoherent betatron tunes. The cancellation between the electric and magnetic forces is very important for this effect and results in it decreasing like \( 1/\gamma^2 \) with energy. Another factor of \( 1/\gamma \) is due to the relativistic mass increase. For unbunched beams with a circular cross section of radius \( a \) and uniform density we find

\[ \Delta Q = -\frac{r_0 I R (\beta_y)}{ec^2 \beta^3 \gamma^2 a^2}. \]

This result is obtained by assuming the space-charge defocusing is weak and can be treated as a first-order perturbation. For the case of beams which are dominated by space-charge other methods have to be used.
For bunched beam the space-charge force depends on the longitudinal distance from the bunch center. This leads to a modulation of the tune for a particle executing a synchrotron oscillation and to a tune spread. In a beam with a non-uniform density the space-charge tune shift depends on the amplitude of the betatron oscillation.

The beam surroundings impose some boundary conditions on the space-charge fields. Since they are in general different for the electric and magnetic fields the cancellation between the two forces is perturbed. This can lead to important tune shifts even at high energies.

Beams can get partially neutralized by ions (electron and antiproton beams) or by electrons (proton beams). In this case the electric space-charge field is reduced and the cancellation between the forces perturbed leading to important tune shifts even for relativistic electron beams.

In colliding beams the electric and magnetic forces produced by the other beam have the same direction and do not cancel. This leads to the well known beam-beam tune shift which has otherwise properties similar to the direct space-charge tune shift in a single beam.

5.2 Coherent effects

A betatron oscillation of the beam as a whole represents a coherent motion. This has little effect on the direct space charge effects which are determined by the internal forces inside the beam. However, the image charges and currents induced in the wall are modulated by the coherent beam oscillation. This leads to coherent tune shifts which can be quite different from the incoherent ones.

5.3 Consequences and problems caused by space-charge tune shifts

In the presence of optical machine imperfections the values of the betatron tunes should not be on or close to a rational number of low order otherwise some resonances are excited. The space-charge force can produce a large tune spread and it is more difficult to find a place in the tune diagram sufficiently far from the resonance.

The separation between the coherent and incoherent betatron frequency reduces the space available in the tune diagram where neither of the two tunes is close to a resonance. Furthermore the separation between the tunes can reduce the beneficial effect of Landau damping which is usually present for beams with a frequency spread.

5.4 Illustration of a difference between coherent and incoherent frequencies

The space-charge force produces a condition in which the coherent and incoherent betatron frequencies are different. Since this is a situation which is difficult to visualize we try to illustrate it by a simple example. We consider a swing with several children playing as shown in Fig. 7. If the oscillations of the individual swings are arbitrary they represent an incoherent motion and the frame of the swing does not move; top picture. The oscillation frequency is then given by \( \omega_{x0} = \sqrt{g/L} \), where \( g \) is the gravitational acceleration and \( L \) is the length of the swing.
If now all the children oscillate their swings in phase we have a coherent oscillation which applies a force on the frame. If this frame is not rigid it will move slightly with the coherent oscillation and change the frequency. This example is different from the cases of space-charge in so far as there are no direct forces between the individual swings. They are only coupled together via the frame which takes the role of the wall in the space-charge case.

![Swings example](Image)

Figure 7: Example of a situation with different coherent and incoherent frequencies

6. PRACTICAL EXAMPLES

We have a brief look at two practical examples of space-charge effects in accelerators and their consequences.

First the direct space-charge effect in the PS-booster [6]. This machine is a synchrotron which accelerates proton bunches from 50 to 800 MeV in about 0.6 s. The tunes occupied by the particles are indicated in the diagram of Fig. 8 by the shaded areas. At injection ($t = 0$ ms) the space-charge effect is strongest as indicated by the lightly shaded area which covers nearly a tune span of 0.5. As time goes on the en-

---

1Drawing by Regula Hofmann, Brooklyn, NY 11231, U.S.A
nergy increases and the space-charge tune spread gets smaller covering at $t = 100$ ms the tune area shown by the darker shading. The point of highest tune corresponds to the particles which are least affected by space-charge. This point should not move much during acceleration. However, the external focusing is adjusted such that the reduced tune spread lies in about the center of the region of the tune diagram which has not to many resonances. Finally the small dark area shows the situation at $t = 600$ ms during the acceleration when the beam has reached the top energy of 800 MeV. The reduction of the tune spread is weaker than expected from the $1/\gamma^3$ dependence since the bunch dimensions decrease during acceleration.

![Figure 8: Direct space-charge tune shift in the PS-booster [6]](image)

The second example shows the tune shift due to the indirect space-charge effect in the ISR [7]. The beam is unbunched and has an energy of about 26 GeV where the direct space-charge is already small due to the $1/\gamma^2$ dependence. This beam has a large energy spread which manifests itself by the large horizontal space of nearly $\pm 40$ mm it occupies as indicated in Fig. 9. The calculated horizontal and vertical tune shifts are shown for the different parts of the beam. They have opposite signs as expected for the indirect space-charge effect. The space occupied by this beam in the tune diagram has the form of a thin line. It can be reduced in length and straightened by some multipole magnets.
Figure 9: Indirect space-charge tune shift in the ISR [7]

References