FORMULAE FOR ANTIPROTON PRODUCTION OPTICS

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INTRODUCTION

An antiproton source consists essentially of a target in which the antiprotons are produced and a special focusing lens (e.g. magnetic horn, Li lens, plasma lens); a special prefocusing lens for the proton beam hitting the target may eventually be added. This note summarizes the principal relations between the main optical parameters of these devices: focus size and position, focal length of the lens, aperture, etc. for which it seems very practical to use the betatron function formalism as far as possible, not only to ease the matching with the proton and antiproton transfer lines but also inside the target itself to find its own matching conditions. The formulae which are identical for protons and antiprotons (which differ only by their momentum) are given only once.

1. PASSIVE TARGET (i.e. WITHOUT CURRENT) OR DRIFT SPACE BETATRON FUNCTION1

(for small angles)

Notations used in this chapter (see Fig. 1):

- $\beta$ = betatron (or amplitude or envelope) function (m)
- $\beta'$ = $\beta$ slope = $d\beta/ds$ (m)
- $s$ = distance from the focus (waist) (m)
- $z$ = abscissa from the target origin (m)
- $\beta_0$ = $\beta$ at focus (m)
- $z_0$ = abscissa of the focus (m)

For $x$ and $y$ ($\beta$ may be different):

$$\beta = \beta_0 + \frac{s^2}{\beta_0} = \beta_0 + \frac{(z - z_0)^2}{\beta_0}.$$ (1.1)

$\beta$ doubles at a distance $\beta_0$ from the focus.

$$\beta' = \frac{2s}{\beta_0} = \frac{2(z - z_0)}{\beta_0}.$$ (1.2)

Inversely,

$$\beta_0 = \frac{\beta}{1 + \beta'^2/4}$$ (1.3)

$$s = z - z_0 = \frac{\beta'\beta_0}{2} = \frac{\beta'\beta}{2(1 + \beta'^2/4)}.$$ (1.4)

These last two formulae give the size and position of a virtual focus (if any) in an active optical element.

2. CONDUCTING TARGET OR LINEAR LENS BETATRON FUNCTIONS

(with uniform current density and for small angles)

New notations used hereafter:

- $I$ = total current (A),
- $j$ = current density (A/m²),
K = g/(B₀) (different for p and ¯p) (m⁻²),
g = gradient of magnetic induction (T/m),
B₀ = particle rigidity. B₀ = p/e = momentum/charge (T.m),
xₑ = target radius (m),
ℓ = target length (m),
{βₑ = β at entry, βₑ at exit (m),
{βₑ = β' at entry, βₑ at exit

2.1 Defocusing (e.g. for protons in a conducting target): for x and y: cf [1]

\[ \beta = \beta_e \cosh^2(\sqrt{K} z) + \frac{\beta_e^2}{2K} \sinh[2\sqrt{K} z] + \frac{1 + (1 + \beta_e^2/4)}{KB_e^2} \sinh^2[\sqrt{K} z] \]

(2.1)

\[ \beta' = \beta_e' \cosh[2\sqrt{K} z] + \frac{1 + \beta_e^2/4}{\sqrt{K} \beta_e} \sinh[2\sqrt{K} z] \]

(2.2)

2.2 Focusing (e.g. for antiprotons): for x and y (cf. [1])

\[ \beta = \beta_\ell \cos^2[\sqrt{K}(z - \ell)] + \frac{\beta_\ell^2}{2\sqrt{K}} \sin[2\sqrt{K}(z - \ell)] + \frac{1 + \beta_\ell^2/4}{KB_\ell^2} \sin^2[\sqrt{K}(z - \ell)] \]

(2.3)

\[ \beta' = \beta_\ell' \cos[2\sqrt{K}(z - \ell)] + \frac{1 + \beta_\ell^2/4}{\sqrt{K} \beta_\ell} \sin[2\sqrt{K}(z - \ell)] \]

(2.4)

If the boundary betatron values are given at entry, replace (z - ℓ) by z and βₑ°βₑ° by βₑ°βₑ°. N.B.: The betatron function formalism does not apply for the outer trajectories where the field is decreasing as 1/r (cf. chapter 5).

3. SIMPLE RELATIONS BETWEEN FOCUS AND LENS EXIT
(for small angles)

β' is zero for the focus and the lens exit plane.

Symbols used in this chapter: cf. Fig. 2.

β, K, α = betatron function, beam radius and divergence at focus
β, K, α = same at lens exit plane (m, m, rad)
A = acceptance/π corresponding to K and α (m.rad)
f = focal length of the lens (m)
d = focus to lens distance (m)
L = lens length (m)
3.1 Lagrange-Helmoltz theorem (i.e. Liouville)

\[ \bar{\alpha} = \bar{\sigma} = \Lambda \]  
(3.1)

3.2 Focal length

\[ f = \bar{\sigma} = \frac{1}{\sin(\bar{f} L)} \]  
(3.2)

\[ \bar{f} \bar{L} = f \Lambda \]  
(3.3)

\[ \sigma \bar{\sigma} = \Lambda / f \]  
(3.4)

\[ K = \frac{g}{(B_0)} = \frac{1}{f^2 - q^2} = \frac{\mu_0 I}{2 \pi R^2 (B_0)} = \frac{\mu_0 I}{2 (B_0)} \]  
(3.5)

\[ L = \frac{1}{f} \sin^{-1}(f/K)^{-1} = \frac{1}{f \Lambda} \tan^{-1}(d/K)^{-1} \]  
(3.6)

3.3 Betatron functions

\[ \bar{R} = \sqrt{\Lambda \bar{\beta}} \]  
(3.7)

\[ \bar{\alpha} = \sqrt{\Lambda / \beta} \]  
(3.8)

\[ \beta \bar{\beta} = f^2 \]  
(3.9)

\[ \bar{R} = \bar{\sigma} = f \]  
(3.10)

3.4 Other relations

As will be seen in chapter 5 the maximum angle \( \hat{\alpha} \) for a given current is found for an immersed focus \((d = 0)\). This angle is \( \hat{\theta} \) and only depends on the total current.

\[ \hat{\alpha}_{\text{max(immersed focus)}} = \hat{\theta} = \sqrt{\frac{\mu_0 I}{2 \pi (B_0)}} \]  
(3.11)

One can then express \( f, \bar{R}, \bar{L} \) with only \( \hat{\alpha}, \hat{\theta} \) and \( d \):

\[ f = \frac{1}{d \sqrt{1 - \hat{\alpha}^2 / \hat{\theta}^2}} \]  
(3.12)

\[ \bar{R} = \frac{\hat{\alpha}}{d \sqrt{1 - \hat{\alpha}^2 / \hat{\theta}^2}} \]  
(3.13)

\[ \bar{L} = \frac{\sqrt{\hat{\alpha}^2 / \hat{\theta}^2}}{d \sqrt{1 - \hat{\alpha}^2 / \hat{\theta}^2}} \tan^{-1} \left( \frac{f \hat{\alpha}}{d \hat{\theta}} \right) = \frac{\sqrt{\hat{\alpha}^2 / \hat{\theta}^2}}{d \sqrt{1 - \hat{\alpha}^2 / \hat{\theta}^2}} \tan^{-1} \left( \frac{\hat{\alpha}}{\sqrt{1 - \hat{\alpha}^2 / \hat{\theta}^2}} \right) \]  
(3.14)
\[ \frac{L}{\theta} = \frac{1}{\theta} \tan^{-1} \left( \frac{\theta}{\delta} \right) = \frac{1}{8} \sin^{-1} \left( \frac{\theta}{\delta} \right) \quad \frac{\theta + \delta}{\theta} \frac{\pi}{2} \] (3.15)

\[ \frac{L}{\theta} = \frac{\delta}{\theta} \sin^{-1} \left( \frac{\theta}{\delta} \right) \quad \frac{\delta + \theta}{2} \frac{\pi}{2} \] (3.16)

4. BEAM DISTRIBUTION2,3

4.1 Proton Beam

The normal (gaussian) distributions in \( x, x' \) (and \( y, y' \)) of a matched beam whose parameters vary along \( z \), around the focus, are given by the betatron functions.

Notations used in this chapter:

\( f(x,x') \) = bivariate distribution density,
\( E_n \) = proton beam emittance for \( n_0, n_0' \),
\( \sigma, (\sigma_x, \sigma_{x'}, \ldots) \) = standard deviation (of corresponding parameters...),
\( \kappa_{xx'}, \kappa_{yy'} \) = correlation (joint moment of 2nd order),
\( \kappa_{xx'} \) = correlation coefficient = \( \kappa_{xx'}/\sigma_x\sigma_{x'} \),
\( \alpha, \beta, \gamma \) = Twiss betatron functions.

For \( x, x' \) one has:

\[ \sigma_x = \frac{1}{n} \sqrt{E_n} \beta \] (4.1)

\[ \sigma_{x'} = \frac{1}{n} \sqrt{E_n} \gamma = \frac{1}{n} \sqrt{E_n} \left( \frac{1 + \alpha^2}{\beta} \right) = \frac{1}{n} \sqrt{E_n} \left( \frac{1 + \beta^2/4}{\beta} \right) \] (4.2)

\[ \rho_{xx'} = \frac{\beta'}{2\sqrt{\beta\gamma}} = \frac{\beta'}{2\sqrt{(1 + \alpha^2)}} = \frac{\beta'}{\sqrt{4 + \beta^2}} \] (4.3)

\[ E_n = n^2 \sigma_x \sigma_{x'} \sqrt{1 - \rho_{xx'}^2} = n^2 \sqrt{\sigma_x^2 \sigma_{x'}^2 - \kappa_{xx'}^2} \] (4.4)

\[ f(x,x') = \left( 2\pi \sigma_x \sigma_{x'} \sqrt{1 - \rho_{xx'}^2} \right)^{-1} \exp \left[ -\frac{x^2/\sigma_x^2 + x'^2/\sigma_{x'}^2 - 2\rho_{xx'}xx'/\sigma_x \sigma_{x'}}{2(1 - \rho_{xx'})} \right] \] (4.5)

By integrating over \( x' \):

\[ f(x) = \left( \frac{2\pi}{\sigma_x} \right)^{-1} \exp \left[ -x^2/2\sigma_x^2 \right] \] (4.6)

Same equations hold for \( y, y' \).

If multiple Coulomb scattering is present, its standard deviations in \( x \) and \( x' \) have to be added (quadratically) to the corresponding standard deviations.
4.2 Antiproton Beam

The distribution of antiprotons is not so simple, not because of the angular production distribution, which can be assumed normal, but because the reabsorption in a long narrow target itself influences strongly the resulting angular distribution, an effect which can be calculated by numerical computing only. 

5. Trajectories In and Out of Target, Lens and Horn

For optical devices where a circular symmetry exists, and for a field varying linearly with the radius inside the cylinder of the target or lens, the two coordinates x and y can be treated independently for paraxial trajectories (i.e. when the projected trajectories do small angles with the axis) with the known techniques (e.g. the matrix formalism). This is no longer true for trajectories in a field varying as the inverse radius as in the outer space of conducting targets and horns. In the following the constant θ is used instead of the constant K because it has a physical meaning for several particular cases of interest.

Symbols used in this chapter: cf. Fig. 3

\[ B = \text{field induction (T),} \]
\[ B_x, B_y = \text{field components (T),} \]
\[ r = \text{radius (m),} \]
\[ r_{\text{max}} = \text{max. radius of the trajectory at distance } r_0 \text{ from the origin,} \]
\[ \alpha = \text{angle of the trajectory with a parallel to the axis,} \]
\[ \theta = \text{characteristic constant of the current } I \text{ in a cylinder of radius } r_C. \]

5.1 Linearly Varying Field

\[ B = \frac{\mu_0 I}{2\pi r_C^2} x^2 + y^2 \quad \text{with} \quad B = \frac{\mu_0 I}{2\pi r_C^2} \]

\[ B_x = -\frac{\mu_0 I}{2\pi r_C^3} y \]

\[ B_y = \frac{\mu_0 I}{2\pi r_C^3} x \]

\[ g^2 = \frac{\mu_0 I}{2\pi (B_0)} = K r_C \quad \text{with} \quad B_C = \frac{B}{(B_0)} \]

In the case of focusing (i.e. I < 0 for \( \mathbf{p} \)) the differential equations of the trajectories are:
These equations are decoupled as soon as \( V_z \) is assumed to be constant. There is focusing for \( x \) and \( y \) together.

The meridian trajectories (in a plane containing the axis) are given by the condition \( V_x/V_y = x/y \); the solution of the differential equation is then for \( x \) and \( y \) a sum of \( \cos(Bs/rc) \) and \( \sin(Bs/rc) \) for which the wavelength is \( 2\pi r_c/\theta \). (5.7)

Helical trajectories are possible in the two directions (helical trajectories and meridian trajectories are not possible in a common quadrupole) as soon as the condition \( V_x + V_y = 0 \) is fulfilled. The helix angle is

\[
\alpha = \theta r/r_c. \quad (5.8)
\]

Its pitch is \( 2\pi r_c/\theta \) and is equal to the wavelength of the meridian sinusoidal trajectories.

The angle \( \alpha \) of a meridian trajectory issuing from the axis with an angle \( \alpha_0 \) is given by:

\[
\cos\alpha - \cos\alpha_0 = \frac{g^2}{2} \left( \frac{r}{r_c} \right)^2, \quad (5.9)
\]

which, for \( r = r_c \), gives the exit angle \( \alpha_S \):

\[
\cos\alpha_S - \cos\alpha_0 = \frac{g^2}{2}, \quad (5.10)
\]

which, for small angles, reduces to:

\[
\alpha_0^2 - \alpha_S^2 = g^2. \quad (5.11)
\]

The trajectories just tangent to the outer surface have angles on the axis:

\[
\alpha_0 = \theta. \quad (5.12)
\]

5.2 Field Varying as \( 1/r \). Cf. [3], [4]

\[
B = \frac{\mu_0 I}{2\pi \sqrt{x^2 + y^2}} \quad (5.13)
\]

\[
B_x = \frac{\mu_0 I}{2\pi} \frac{y}{(x^2 + y^2)} \quad (5.14)
\]

\[
B_y = \frac{\mu_0 I}{2\pi} \frac{x}{(x^2 + y^2)} \quad (5.15)
\]
With the same \( \theta \) constant (5.5), the differential equations of the (generally skew) trajectories in the case of focusing (e.g. \( I < 0 \) for \( p \)) are:

\[
\begin{align*}
V_x &= \frac{dx}{ds} \\
\frac{dv_x}{ds} &= -g^2v_z\left(\frac{x}{x^2 + y^2}\right) \\
V_y &= \frac{dy}{ds} \\
\frac{dv_y}{ds} &= -g^2v_z\left(\frac{y}{x^2 + y^2}\right) \\
V_z &= \frac{dz}{ds} \\
\frac{dv_z}{ds} &= +g^2\left(\frac{v_x + v_y}{x^2 + y^2}\right).
\end{align*}
\]  

(5.16)-(5.18)

These equations being coupled cannot be treated independently but can easily be solved with a computer by the Euler or Runge-Kutta techniques.

Meridian trajectories are again given by the \( \frac{V_x}{V_y} = \frac{x}{y} \) condition which yields:

\[
\begin{align*}
\rho &= r_{\text{max}}\exp\left(-\frac{1 - \cos \alpha}{g^2}\right) \\
z &= \frac{r_{\text{max}}}{\theta}\int \exp\left(-\frac{1 - \cos \alpha}{g^2}\right) \cos \alpha \, ds,
\end{align*}
\]

which can be approximated by:

\[
\begin{align*}
\rho &= r_{\text{max}}\exp\left(-\frac{\alpha^2}{2g^2}\right) \\
z &= \frac{r_{\text{max}}}{\theta}\sqrt{\pi} \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}g}\right)
\end{align*}
\]

There are helical trajectories in the two directions if the condition \( V_{x}x + V_{y}y = 0 \) is fulfilled. Their angle with a parallel to the axis is given by:

\[
\cos \alpha = -g^2 + \sqrt{g^4 + 4}
\]

which is correctly approximated by \( \alpha > g \).

The pitch of the helix is:

\[
\frac{2\pi}{\tan \alpha}, \quad \frac{2\pi}{\theta}
\]

which, for \( r = r_c \) is equal to the pitch of the inner helical trajectory and to the wavelength of the sinusoidal meridian inner trajectory.

6. HORN_SHAPES\(^4-8\)

The horn shapes found by means of the equations given in the preceding section 5.2 are given in the parametric form and in the paraxial approximation (\( \alpha \) small). 

New notations used (Fig. 4):

\( R_H, Z_H \) = horn radius and distance from focus (m),
\( R_S, Z_S \) = horn radius and distance from focus (m),
\( \alpha \) = angle of a meridian trajectory with the axis (rad),
\( R_{\text{max}}, Z_{\text{max}} \) = max. horn radius and max. distance from focus (m).

6.1 Simple Horn (one cone)

\[
\begin{align*}
R_H &= \frac{Z_{\text{max}} \alpha}{1 + \alpha \exp(\alpha^2/2\sigma^2) \sqrt{\pi/2} \text{erf}(\alpha/\sqrt{2}\sigma)/\sigma} \\
Z_H &= R_H/\alpha
\end{align*}
\]

6.2 Horn with a Straight Exit Cone of Angle \( \alpha_c \) (\( \alpha C = c \))

\[
\begin{align*}
R_H &= \frac{(Z_{\text{max}} - R_{\text{max}}/c) \alpha}{1 + \alpha \exp(\alpha^2/2\sigma^2) \sqrt{\pi/2} \text{erf}(\alpha/\sqrt{2}\sigma)/\sigma} \\
Z_H &= R_H/\alpha
\end{align*}
\]

6.3 Horn with a Constant Focal Length \( f \) (not dependent on \( \alpha \))

\[
\begin{align*}
R_H &= R_S \exp(-\alpha^2/2\sigma^2) \\
Z_H &= R_H/\alpha \\
R_S &= f\alpha \\
Z_S &= Z_H + R_S \sqrt{\pi/2} \text{erf}(\alpha/\sqrt{2}\sigma)/\sigma
\end{align*}
\]

All these shapes exhibit a zero radius for \( \alpha = 0 \): this is not possible and is replaced by a neck. The choice of the shape can be made according to the constraint of being not re-entrant; the maximum angle accepted by the horn is then \( \alpha = 0 \) and only depends on the current in the horn for a given particle momentum.

REFERENCES
1. G. Guignard, Selection of Formulae Concerning Proton Storage Rings, CERN 77-10, 6 June 1977.

7. E. Jones, \( \bar{p} \) Production and Collection, CERN/PS-AA/83-46, October 1983.

