Validation of the LNE51 AEGIS Transfer Line Optics and Electrostatic Deflector

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Abstract

The Antihydrogen Experiment: Gravity, Interferometry, Spectroscopy (AEGIS) will move from its current location to another area within the Antiproton Decelerator (AD) hall. Beam line LNE51 will provide AEGIS with 100 keV antiprotons from the ELENA decelerator ring. This beam line uses quadrupoles and electrostatic deflectors to guide and focus the antiproton beam from the decelerator to the experiment. The validation of the new 37.7 degree standalone electrostatic deflector (ZDSD) design is described, and the element’s transfer matrix is numerically determined.

1 Introduction

This note will describe the process used to determine the optical transfer matrix for the ZDSD deflector that will be installed in transfer line LNE51 from the Elena decelerator to the future location of AEGIS in the AD hall [4]. The deflector’s design is based on the ZDSB deflector [3]. The transfer matrix will be used to parameterise the optical properties of the deflector in the MAD-X [8] model of the transfer lines. The standalone deflector has semi-spherical electrodes made of AISI 316L stainless steel, a nominal bending radius of 600 mm with a 65 mm gap, and an axial (vertical) radius of curvature of 400 mm (see Fig. 1).

2 Particle tracking

In order to determine the transfer matrix, antiprotons were tracked through an electric field map generated in OPERA[6] using the 3D geometry of the deflector. A field map containing the x-,y-, and z-components of the electric field at 2 mm intervals was used. After defining the initial conditions of the incoming antiproton according to the design reference trajectory, as defined in shown schematically in Figure 2, the path through the deflector could be determined by numerically integrating the equation of motion. A 100 mm drift is placed on either side of the ideal hard-edge deflection described by an arc of 37.7 degrees and bend radius of 600 mm, giving a reference length of 394.7 mm through the deflector.
Figure 1: Two views of the 37.7 degree standalone ZDSD deflector.

Figure 2: Reference case. 100 mm drift areas are in black, 37.7 degree sector is in red. The coordinate system of the deflector in the x-z plane is also shown.

The coordinate system of the field map placed the origin at the center of the design reference trajectory through the deflector, with the x-axis pointed radially outward, the z-axis directed upstream at half the nominal bend angle, and the y-axis pointing upward. The particle momentum was tracked numerically by integrating along the z-direction, finding the force, momentum, velocity, and position at each integration step and iterating stepwise. The electric field was scaled in order to minimize the x-momentum where the z-coordinate was zero, i.e. the particle makes zero angle with respect to the z-direction at the symmetry point of the deflector. As a result of the fringing field at either end of the deflector, the antiproton starts its deflection before the design trajectory specified in Figure 2 giving an offset of -1.55 mm at the centre of the deflector.
2.1 Electric field interpolation

The first implementation of the tracking algorithm found the closest point in the field map to the particle and used the electric field at that location in the equations of motion. The final version of the algorithm used a first-order Taylor series approximation to interpolate the electric field at the antiproton’s location between the nearest points in the field map. This resulted in a much smoother change in the field acting upon the antiproton between iterations in the z-direction, as shown in Fig. 3.

2.2 Convergence tests

Convergence tests were carried out to determine what integration step size would be sufficiently small to produce accurate results without being unnecessarily computationally intensive. To this end, the reference antiproton was tracked through the deflector with integration steps running from 5 mm to 0.02 mm in 0.02 mm increments. The final net momentum and computation time were recorded at the end of each pass. The results of the convergence study are shown in Fig. 4 where the difference between the final and initial particle momentum is plotted as a function of the inverse of the step size of integration. The residual momentum converged to -73.60 eV/c, determined from the average of the last five terms (0.02 mm ≤ dz ≤ 0.1 mm). Given that the initial momentum is 13.699 MeV/c, the residual constitutes a precision on the order of 10^{-6}. The non-zero convergence can be attributed to limits in the density of the field map, the precision of the finite element solver, the first-order interpolation of the electric field data, as well as the fact that the tracking had finite initial and final positions inside the potential of the deflector. The level of precision is satisfactory for our requirements. Without linear interpolation in the field map the solution converged to 76.97 eV/c.

Using the results of the convergence tests, an integration step size of 0.08 mm was selected for further studies. This value ensures that the resulting data should be sufficiently accurate whilst at the same time keeping computation time low (approximately 1.15 seconds per pass).

3 Transfer matrix

The transfer matrix approach is a common way to map the dynamics of a charged particle through a linear system, where the final position and divergence of a particle in the x- and y-directions are mapped from initial conditions with respect to a given reference. It is important to note that the reference system used to compute the transfer matrix is different from the coordinate system of the ZDSD deflector and represents that of the reference particle. The direction of the longitudinal axis (Z-axis) used in determining the transfer matrix is represented by the direction of the momentum vector of the reference particle i.e. tangential to the reference particle’s trajectory, see Figure 5. The Y-axis of the reference particle is the same as that of the electrostatic deflector, and the reference X-axis follows the right-handed convention perpendicular to the Z-direction. Uppercase letters are used to differentiate the reference coordinate system from the coordinate frame of the field map. The format of the
(a) The reference particle’s trajectory through the deflector.

(b) Electric field components experienced by antiproton before introducing an interpolating function

(c) Electric field components after introducing interpolating function

Figure 3: Particle tracking and electric field components. A scale factor of 0.966 is applied to the field map in all cases.
Figure 4: Results of convergence studies using an electric field scale factor of 0.966 and $0.2 \text{mm} \leq \text{dz} \leq 5.0 \text{mm}$.

Figure 5: Example of a reference trajectory of a particle. The uppercase axes indicate the reference coordinates used in defining the transfer matrix, while the lowercase axes are those of the field map. Also shown is the horizontal radial displacement $X'$ of an offset particle whose trajectory and momentum are in blue.

The transfer matrix is as follows:

\[
\begin{pmatrix}
X \\
X' \\
Y \\
Y' \\
l \\
\Delta P
\end{pmatrix}
= \begin{pmatrix}
\text{horizontal displacement} \\
\text{horizontal radial displacement} \\
\text{vertical displacement} \\
\text{vertical radial displacement} \\
\text{path length difference} \\
\text{momentum deviation}
\end{pmatrix}.
\]

Each matrix element has the form $R_{ab}$, where $a$ refers to the output parameter and $b$ refers to the input parameter and with $X = 1$ and $\Delta P = 6$. For example, $R_{12}$ is the matrix element that provides insight into how the final horizontal position $X_{\text{out}}$ depends on the initial horizontal angular displacement $X'_{\text{in}}$. As stated above, the transfer matrix includes the ZDSD deflector as well as a drift length of 100 mm up and downstream of the deflector. This was done consistent with the other deflectors already implemented into the MAD-X
model of the transfer line.

The process of computing the transfer matrix elements consisted of varying the initial conditions (X, X’, Y, Y’ or ∆P) of an antiproton, numerically tracking it through the system and comparing the exit coordinates to those of the reference particle. The impact on bunch length and was neglected. The final coordinates of the offset antiproton are plotted against the initial conditions, as shown in Figure 6.

\[
\begin{align*}
\text{R}_{12} & : \text{linear slope} = 0.550070974923 \\
& : \text{cubic term} = 0.1661045641705 \\
& : \text{quadratic term} = -0.1028431207871 \\
& : \text{linear term} = 0.549892412516 \\

\text{R}_{43} & : \text{linear slope} = -1.37389642923 \\
& : \text{cubic term} = 6.435674618060e+01 \\
& : \text{quadratic term} = -4.407780361262e-02 \\
& : \text{linear term} = -1.379016229194 \\
\end{align*}
\]

Figure 6: (a) Plot of final horizontal displacement versus initial horizontal radial displacement, element \( R_{12} \) of the transfer matrix. Note that for large values of \( X' \), the relationship becomes nonlinear. The black data points are from the TRACK code and show good agreement.

(b) Plot of final vertical radial displacement versus initial vertical displacement, element \( R_{43} \). Here, there is a 5% disagreement between the two code’s numeric results.

The result is linear over a wide variation of input and the slope of the resulting linear fit for each element gives the value for the matrix element (see Fig. 6). Elements \( R_{ij} \) for \( 1 \leq i, j \leq 4 \), \( R_{16}, R_{26}, \) and \( R_{66} \) were computed. All other matrix terms are zero (or one in the case of \( R_{55} \)). The resulting numerically-derived transfer matrix is given by

\[
\begin{align*}
R_n = \begin{pmatrix}
0.838548 & 0.550071 & -0.000207308 & -0.000118750 & 0 & 0.381342 \\
-0.546538 & 0.830891 & -0.000199134 & -0.000137548 & 0 & 1.27182 \\
0.000170601 & 0.0000192792 & 0.597963 & 0.466300 & 0 & 0 \\
0.0006199220 & 0.000285962 & -1.37289 & 0.602254 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.998773
\end{pmatrix}
\end{align*}
\]

From this matrix, one can determine that the 37.7 degree ZDSD deflector provides focusing in both the X- and Y-directions. It is also worthwhile to note that the \( R_{66} \) element is not
exactly one, which reflects the imperfections of the numerical tracking model; the field is not perfectly conservative as discussed in Section 2.2. The results for all matrix elements are collected in Appendix A.

3.0.1 Comparison with TRACK

As an additional check, a different program using TRACK [7] performed the same computations and good agreement was observed using the same field map. The greatest disagreement was about 5% between the two numeric models for the \( R_{34} \) and \( R_{43} \) elements. Appendix A contains plots with the two sets of data for the \( R_n \) elements, with the TRACK results plotted in black. TRACK data points for the \( R_{12} \) and \( R_{43} \) elements are also included in Figure 6.

3.1 Analytic approach

After the transfer matrix was numerically derived, it was analytically computed to check the agreement with a standard result. The formulae used to derive the relevant matrix elements are described by Hinterberger [5] and re-printed below for reference:

\[
\begin{align*}
R_{11} &= R_{22} = \cos(\sqrt{k_x}L) \\
R_{21} &= -\sqrt{k_x} \sin(\sqrt{k_x}L) \\
R_{16} &= \frac{2 - \beta_0^2}{\rho_0 k_x} (1 - \cos(\sqrt{k_x}L)) \\
R_{34} &= \frac{\sin(\sqrt{k_y}L)}{\sqrt{k_y}} \\
R_{33} &= R_{44} = \cos(\sqrt{k_y}L) \\
R_{26} &= \frac{2 - \beta_0^2}{\rho_0 \sqrt{k_x}} \sin(\sqrt{k_x}L) \\
R_{43} &= -\sqrt{k_y} \sin(\sqrt{k_y}L).
\end{align*}
\]

The \( k_x \) and \( k_y \) terms are computed with the following formulae:

\[
\eta_E = 1 + \frac{\rho_0}{r_0}, \quad k_x = \frac{3 - \eta_E - \beta_0^2}{\rho_0^2}, \quad \text{and} \quad k_y = \frac{\eta_E - 1}{\rho_0^2},
\]

where \( \rho_0 \) is the bending radius of the reference particle and \( L \) is the effective length given by the product of the bending radius \( \rho_0 \) and the bend angle \( \alpha \). The axial radius of curvature is represented by \( r_0 \) and \( \beta_0 \) is the unitless velocity of the reference particle.

Recall that the bending radius \( (\rho_0) \) is 600 mm while the vertical radius of curvature of the electrodes \( (r_0) \) is 400 mm. The effective length of the deflector is given by \( L = 600 \text{ mm} \cdot 0.6579 \text{ rad} = 394.7 \text{ mm} \). The analytic matrix for the deflector was found to be

\[
M = \begin{pmatrix}
0.893746 & 0.380709 & 0 & 0 & 0 & 0.255092 \\
-0.528537 & 0.893746 & 0 & 0 & 0 & 1.26889 \\
0 & 0 & 0.692485 & 0.353428 & 0 & 0 \\
0 & 0 & -1.47262 & 0.692485 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
This matrix only accounts for the deflector, not the 100 mm drift before and after it. To fairly compare the numeric and analytic models, the drift matrix \((D)\) was introduced as follows:

\[
D = \begin{pmatrix}
1 & 0.100 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0.100 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Our final analytic transfer matrix is given by \(R_a = D \cdot M \cdot D\), which becomes,

\[
R_a = \begin{pmatrix}
0.840892 & 0.554173 & 0 & 0 & 0 & 0.381982 \\
-0.528537 & 0.840892 & 0 & 0 & 0 & 1.26889 \\
0 & 0 & 0.545229 & 0.477199 & 0 & 0 \\
0 & 0 & -1.47262 & 0.545229 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

### 3.2 Model benchmarking

Comparing the \(R_n\) and \(R_a\) reveals some differences. Analytically, the initial \(Y\) and \(Y'\) values should have no impact on the final \(X\) and \(X'\) values and vice versa, however, \(R_n\) shows a very slight dependence on the order of \(10^{-3}\). The disagreement between models is small and not surprising given the imperfections of the numerical integration. The largest disagreement between the first two rows of the matrices is a difference of 3.3\% between the \(R_{n12}\) and \(R_{a12}\) elements. There is greater disagreement between the third and fourth columns; the greatest discrepancy being a 9.5\% difference between the two \(R_{44}\) elements. A complete overview of the disagreements between elements (excluding terms where the analytic element is zero) is given in Table 1. The differences between the two models is much more significant for the third and fourth rows, possibly indicating imperfections in the \(y\)-components of the field map. In light of the likely sources of these disagreements, the numeric model can be used as is, with future improvements available as needed.

<table>
<thead>
<tr>
<th>Element</th>
<th>% disagreement</th>
<th>Element</th>
<th>% disagreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_{11})</td>
<td>0.279531</td>
<td>(R_{33})</td>
<td>8.081894</td>
</tr>
<tr>
<td>(R_{12})</td>
<td>0.745722</td>
<td>(R_{34})</td>
<td>2.33734</td>
</tr>
<tr>
<td>(R_{21})</td>
<td>3.29364</td>
<td>(R_{43})</td>
<td>7.26424</td>
</tr>
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<td>(R_{22})</td>
<td>1.20365</td>
<td>(R_{44})</td>
<td>9.46859</td>
</tr>
<tr>
<td>(R_{16})</td>
<td>0.167828</td>
<td>(R_{66})</td>
<td>0.0122851</td>
</tr>
<tr>
<td>(R_{26})</td>
<td>0.230379</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Figure 7: MAD-X output of LNE51 showing optics before and after the transfer matrix for the ZDSD deflector was included. The placeholder matrix had been for a 33.16 degree deflector. One can see that the effect of the change propagates through the line, beginning around two meters, where the deflector is located.

3.3 Implementing the matrix into LNE51

Once the numeric matrix was determined and checked, the matrix was entered into MAD-X [8] scripts that model the beam moving through the line. These scripts record $\beta_x$, $\beta_y$, and the dispersion in the x and y directions $D_x$, $D_y$ as a function of position along the beam line. The transfer matrix for a similar 33.16 degree electrostatic deflector was previously used as a placeholder in the location of the 37.7 degree ZDSD element. A plot comparing $\beta_x$, $\beta_y$, $D_x$, and $D_y$ for the two deflectors in the same location was made (see Fig.7), and in all cases the four terms approach zero at the end of the line, as expected for the small difference in the deflector models.

4 Conclusion

The beam dynamics of antiprotons through the 37.7 degree ZDSD deflector has been studied and characterised. Convergence studies have shown that the tracking algorithm is able to track antiprotons through the deflector with precision to $10^{-6}$. A transfer matrix for the ZDSD deflector and a drift length of 100mm on either side of the deflector was numerically derived and verified with an analytic model and another numerical tracking code. The LNE51 beam line MAD-X model has since been updated using this matrix.
References


A Numeric Model Plots

Below are the plots used to determine the transfer matrix numerically. The linear fit’s slope was used for all elements. The black points are data from a similar program using TRACK [7]. The two programs show good agreement.
R43

linear slope = -1.37289390479

cubic term = 7.116295333757e+01

quadratic term = -1.695730308789e-02

linear term = -1.379540524632

R44

linear slope = 0.602229643923

cubic term = 2.048137625746e+00

quadratic term = -5.762326763730e-03

linear term = 0.600027895976

R66

linear slope = 0.998772758209

cubic term = 1.123547496911e+01

quadratic term = -2.307931976420e-01

linear term = 0.998674271997

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