Lower limit on the entropy of black holes as inferred from gravitational wave observations

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Abstract

Black hole (BH) thermodynamics was established by Bekenstein and Hawking, who made abstract theoretical arguments about the second law of thermodynamics and quantum theory in curved spacetime respectively. Testing these ideas experimentally has, so far, been impractical because the putative flux of Hawking radiation from astrophysical BHs is too small to be distinguished from the rest of the hot environment. Here, it is proposed that the spectrum of emitted gravitational waves (GWs) after the merger of two BHs, in particular the spectrum of GW150914, can be used to infer a lower limit on the magnitude of the entropy of the post-merger BH. This lower bound is significant as it is the same order as the Bekenstein-Hawking entropy. To infer this limit, we first assume that the result of the merger is an ultracompact object with an external geometry which is Schwarzschild or Kerr, but with an outer surface which is capable of reflecting in-falling GWs rather than fully absorbing them. Because of the absence of deviations from the predictions of general relativity in detected GW signals, we then obtain a bound on the minimal redshift factor of GWs that emerge from the vicinity of the object’s surface. The lack of deviations also means that the merger remnant essentially needs to have an absorbing surface, and thus it must effectively be a BH. Finally, a relationship between the minimal redshift factor and the BH entropy, which was first proposed by ’t Hooft, is used to set a lower bound on the entropy of the post-merger BH.
1 Introduction

It has long been thought that the picture of a black hole (BH) from classical general relativity (GR) accurately depicts the end point of gravitational collapse, perhaps with a suitable regularization scheme for taming the infamous BH singularity. This belief has recently been questioned due to the realization that, even as a matter of principle, the paradoxical nature of BH evaporation has no simple remedy along the lines of observer complementarity [1, 2, 3, 4, 5, 6, 7]. One then concludes that the synthesis of classical BHs with the quantum process of Hawking radiation is untenable as currently understood. A self-consistent picture requires discarding one, at the very least, of several cherished tenets: unitary evolution, locality, causality, the strong subadditivity of entropy (or monogamy of entanglement), Einstein’s principle of equivalence and/or effective field theory in its domain of applicability.

A different route is to assert that the traditional model of a BH is what needs to be changed and then argue on behalf of an exotic alternative. This idea has led to models for ultracompact objects that resemble classical BHs in some ways yet differ in others. Proposals of this nature include fuzzballs [2, 8], firewalls [3, 5], wormholes [9], gravastars [10], anti-de Sitter bubbles [11], boson stars [12], graviton condensates [13], strongly anisotropic stars [14, 15, 16, 17] and, as promoted by the current authors, collapsed polymers [18]. A partial catalogue of sorts was recently presented in [19]. To avoid the paradoxical dilemmas of an evaporating BH, an ultracompact object cannot have a “true” horizon, although its outer surface may still display some horizon-like characteristics. As such, it lacks a surface where the Tolman redshift factor, $\sqrt{|g_{tt}|}$ vanishes locally: a surface of infinite blueshift.
The Tolman redshift factor is related to the standard redshift parameter $z$ as $\sqrt{|g_{tt}|} = (1 + z)^{-1}$. The minimal value of the Tolman factor or, equivalently, the maximal value of $z$ is an important parameter that may be used to distinguish among the different models.

The BH entropy, on the other hand, can be interpreted as a geometrical quantity or as a measure of entanglement across the horizon. The entropy has, at times, also been attributed to the number of internal microstates, the amount of information that is shielded by the horizon and so forth (see [20] for further discussion along these lines). We would argue that all these are different descriptions of the same entropy, as distinct observers could well have conflicting perspectives on its origin.

For an asymptotic observer who is aware of the existence of quantum fields, a small value for the Tolman redshift factor implies that the near-horizon modes of these fields have very large energies and correspondingly small wavelengths. As a result, the observer expects many such modes to be closely squeezed together in the proximity of the horizon and form a “thermal atmosphere” around the BH horizon [21]. The density of states of the near-horizon modes increases as the Tolman redshift factor decreases and therefore, so does the entropy of the thermal atmosphere. In the case of our asymptotic observer — who, as a disciple of GR, only knows about what is outside the horizon and would assume a mostly empty interior — the entropy of the BH would have to be attributed to this thermal atmosphere. The asymptotic observer would then conclude that the magnitude of the BH entropy depends on the value of the minimal Tolman redshift factor. But a free-falling observer, for example, is unaware of any redshift effect and would
have to conclude differently.

Gravitational waves (GWs) that are emitted from BH mergers\cite{22,23} provide a means for constraining deviations away from GR\cite{24,25,26}. Our proposal is that such GW data, which can already be used to put an upper bound on the observed minimal value of the Tolman factor, can similarly be used to infer a lower bound on the entropy of the post-merger BH. This means that BH mergers, which can be viewed as cosmological “scattering experiments”, can be used to infer the entropy of astrophysical BHs. As far as we are aware, no similar proposal has been made in this context.

The more exotic compact objects allow for (at least) two other additional sources of emitted waves as compared to the BHs of GR. First, an ultracompact object containing a significant amount of matter could very well support modes that are analogous to the fluid modes of relativistic stars\cite{27}. For further discussion on these “fluid-like” modes, see\cite{28,29}. Another class of waves to consider are those that are sourced by the so-called echoes\cite{9,30,31,32,33,19}, which were discussed for rotating compact objects in\cite{34,35,36,37,38,39}. The echoes for a star-like ultracompact object are modes that reflect any number of times between the peak of the potential (or photosphere) and the outermost surface of the ultracompact object. We note that a reflection from the outermost surface is essentially equivalent to unhindered propagation through the object, as the waves would then be perceived as being totally reflected at the center of the object. The wormhole echoes\cite{30} are reflected between one potential peak and its mirror image. The reflected modes will eventually escape from this cavity or “echo chamber” outward through the gravitational barrier. The reason that these
other modes can be distinguished (at least in principle) from the standard BH modes is because of an additional length scale, \( L_I \) for the \( I^{th} \) exotic alternative. This scale will be made explicit in the section to follow.

The value of these additional length scale \( L_I \) does not necessarily fix the precise type of ultracompact object. Some models are endowed with a natural choice of length scale but then some are not. For instance, in the case of the collapsed-polymer model, \( L_I \) is related to the string length. The string length must be greater than Planck length \( l_P \), but its precise value is otherwise difficult to pinpoint. In comparison, \( L_I \) is a completely free parameter for the wormhole model whereas, for the gravastar model, it is related to the width of an outer shell which is supposed to be about twenty orders of magnitude larger than \( l_P \). [40]

We do support the point of view that GR is only an approximate classical theory and there should, as such, be corrections to the spectra of both the ringdown modes and the emitted GWs which follow after a BH merger. On general grounds, and as argued in [29], these excitations should have lower frequencies and longer lifetimes in comparison to the standard GR modes. But, since we expect that the boundary conditions at the surface are, to a good approximation, totally absorbing boundary conditions, it follows that the amplitude of the additional modes has to be substantially suppressed as compared to the GR modes. The same reasoning would apply to the aforementioned fluid-like modes. Hence, for the purposes of the current discussion, GR-corrected and fluid-like modes can be discounted.

The echo modes, on the other hand, represent the complete opposite of our expectations, as these obey totally reflecting boundary conditions and
serve as a means to quantify the deviations away from GR in this extreme region of parameter space. The simplest way to have reflecting boundary conditions is if, for some unknown reason, the GWs pass undisturbed through the interior of the BH (as in the gravastar model). Such waves would indeed be viewed from the exterior as being totally reflected. One could, of course, invent intermediate models; for example, by modifying the gravitational potential by hand. It does, however, seem difficult to justify such impromptu modifications. In any event, the resulting spectrum of emitted GWs would still be expected to be qualitatively similar to the case of total reflection, provided that the reflection is not parametrically small. This point is clarified later.

The rest of the paper starts by establishing the quantitative relationship between the entropy and the minimal redshift factor for the BHs of GR and then, similarly, for the exotic alternatives. This is followed by a numerical analysis that determines the current bounds on the redshift factor and, with it, the current bounds on the entropy. The paper concludes with a brief overview.

2 Entropy and redshift

Here, we will elaborate on the significance of the redshift factor, with the focus being initially on the BHs of GR and, following this, on the exotic alternatives. We will, as a starting point, assume non-rotating and electrically neutral compact objects in $3+1$ dimensions, so that the object’s exterior geometry is accurately described by the Schwarzschild metric $g_{ab}$. 
for which \(-g_{tt} = g^{rr} = 1 - \frac{R_S}{r}\) where \(R_S\) is the Schwarzschild radius. Rotating BHs will, however, be eventually considered; in which case, the exterior geometry is described by the Kerr metric with
\[-g_{tt} = 1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\]
and
\[g^{rr} = \frac{r^2 - 2Mr + a^2}{r^2 + a^2 \cos^2 \theta} = \frac{(r-r_+)(r-r_-)}{r^2 + a^2 \cos^2 \theta} .\]
Here, \(M\) is the mass, \(a = J/M\) is the dimensional spin parameter and \(r_{\pm} = M \pm M\sqrt{1-a^2/M^2}\) is the location of the outer/inner horizon (or would-be horizons for the alternative models).

### 2.1 Minimal redshift for black holes and exotic alternatives

Even in the idealized case of GR BHs, the notion of a vanishing redshift factor is not realistic. Smaller distances than \(l_P\) are impossible to probe without turning the probing instrument itself into a BH. And so an observer cannot have better resolution than \(l_P\) when attempting to locate a BH horizon.

The following calculation reveals that a proper distance \(\Delta r\) of one Planck length away from the horizon describes a radial distance in Schwarzschild coordinates of about \(d = l_P^2/R_S\):

\[
\Delta r = \int_{Rs}^{Rs+d} dx \sqrt{g_{rr}(x)} \simeq \sqrt{d} R_S .
\] (1)

This, in turn, implies that the surface has a local (Tolman) redshift factor of

\[
F \equiv \sqrt{|g_{tt}|} \simeq \frac{\Delta r}{R_S} \geq \frac{l_P}{R_S} .
\] (2)

The GR value for the minimal redshift factor is then \(F_{\text{BH}} = l_P/R_S\). (See also [41, 42, 43] for a different perspective leading to the same conclusion.)

For a solar mass BH, the minimal redshift factor is extremely small, \(F_{\text{BH}} = 5 \times 10^{-38}\).
The Bekenstein–Hawking value of the BH entropy, \( S_{\text{BH}} = A/4l_P^2 = \pi R_S^2/l_P^2 \), can now be expressed as \( S_{\text{BH}} = \pi F_{\text{BH}}^{-2} \). As mentioned in the Introduction and explained in detail in the next subsection, this reasoning is based on ascribing the BH entropy to the thermal atmosphere of the closely surrounding quantum field modes.

### 2.2 Entropy and minimal redshift

The properties of the thermal atmosphere were first calculated by 't Hooft \[21\] by counting the number \( n(E) \) of excited quantum field modes having energy \( E \) and satisfying the wave equation \( g^{rr}k^2 = -g^{tt}E^2 + \cdots \). This was done by using the Bohr quantization condition to determine \( n(E), \pi n(E) = \int dr k(E), \) where \( k(E) \) is the wave number. Since \( k(E) \sim \sqrt{-g^{tt}g_{rr}}E \sim E/|g_{tt}| \), it follows that \( n(E) \), just like \( k(E) \), is inversely proportional to the square of the minimal redshift factor. Using this estimate, 't Hooft proceeded to show that the free energy and therefore the energy and entropy grow in inverse proportion to the square of the minimal redshift factor. (See \[41\] for further details.) Actually, the estimates of 't Hooft are only valid for cases in which the minimal redshift factor \( F_{\text{min}} \) is parametrically small, \( F_{\text{min}} \ll 1 \). This limitation will be important later when we derive bounds on \( F_{\text{min}} \).

Since the entropy formally diverges when the minimal redshift factor vanishes, 't Hooft regularized the result by placing a hard cutoff (a “brick wall”) at a small distance away from the horizon, which he later identifies as a proper length on the order of \( l_P \). A specific choice of cutoff reproduces the expected value of the BH entropy \( S_{\text{BH}} \); however, 't Hooft could just as well have followed our prescription for a minimal redshift factor to obtain the
correct value. This provides a formal explanation as to why a BH can have a parametrically much larger entropy than a run-in-the-mill star of the same mass. And there is no “species problem” in this regard because the renormalized value of $l_p$ depends on the number of species $\mathcal{N}$, and it does so in just the right way to cancel $\mathcal{N}$ out of the brick-wall calculation.

The case of rotating BHs was discussed in [49] and the result is very similar to that of the Schwarzschild case. The area of the BH horizon is now given by $A = 2\pi Mr_+$ and the entropy is again inversely proportional to the square of the minimal redshift factor, $S \propto F_{\text{BH}}^{-2}$.

Let us now consider the other ultracompact objects, the exotic alternatives. Each of these can be expected to have a characteristic length scale that represents the displacement of the object’s outer surface from its Schwarzschild radius and, with it, an associated minimal redshift factor. Adapting a convention from [19], we define $\epsilon_I = (\Delta r)_I/R_S$ as a dimensionless form of the characteristic length for the $I^{th}$ model. Since $\epsilon_I$ is also the near-horizon redshift factor, as made clear by Eqs. (1) and (2), it follows that the object has a minimal redshift factor of $F_{I,\text{min}} = \epsilon_I$ and then a thermal atmosphere entropy of $S_I \propto 1/\epsilon_I^2$. We can absorb the constant of proportionality between $S_I$ and $1/\epsilon_I^2$ into the definition of $(\Delta r)_I$ and then write

$$S_I = 1/\epsilon_I^2,$$

which is the adopted form of entropy for the remainder of the paper.

$^1$Although BHs and compact stars can still have a similar entropy if it is rescaled by the object’s mass and temperature [44 45].
2.3 A bound on redshift and entropy

The basic idea that enables the bounding of the minimal redshift factor $\epsilon_{I, \text{min}}$ will now be explained. We begin this discussion with non-rotating BHs from GR and later move on to include rotation and/or alternative models.

In the classical-BH case, any emitted GWs can be attributed to damped ringdown or quasinormal modes (QNMs) which ensue when a perturbed BH — such as one formed from the merger of a BH binary — settles down to its equilibrium state.\textsuperscript{2} To understand the properties of these QNMs, one can consider waves propagating in from infinity that are scattered by the gravitational potential barrier at $r \simeq \frac{3}{2} R_S$. There are then two possibilities: the wave is either reflected off the barrier or passes right through it. The first possibility describes the emitted GWs, whereas the second class of waves are never heard from again because a classical BH is a perfect absorber\textsuperscript{3}. For future reference, the emitted waves have an angular frequency $\omega_{\text{BH}}$ of about $R_S \omega_{\text{BH}} \sim 1$ and the longest-living modes have an inverse damping time $\tau_{\text{BH}}^{-1}$ of similar magnitude, $R_S \tau_{\text{BH}}^{-1} \sim 1$. These estimates reflect the fact that a Schwarzschild BH has only one length scale $R_S$. For the exact numerical values of $f_{\text{BH}}$ and $\tau_{\text{BH}}$, see \cite{50, 51, 52}.

Moving on to the exotic alternatives, let us recall that the other types of modes can be distinguished from the standard BH modes by an additional length scale. Here, rather than the scale $\epsilon_I$, it is more appropriate to consider the distance $L_I$ from the object’s outermost surface to its photosphere but

\textsuperscript{2}There are many excellent reviews on the QNMs of BHs; see, e.g., \cite{50, 51, 52}.

\textsuperscript{3}Quantum mechanically, some minimal amount of reflection is expected based on the Hawking rate of emission.
in terms of the tortoise coordinate $r_\ast = \int \frac{dr}{1 - \frac{R_S}{r}}$. This coordinate choice is because the near-horizon geometry then looks similar to a flat geometry. And so the relevant length scale for the $I^{th}$ model is given by (the photosphere is placed at $r = (3/2)R_S$ for simplicity)

$$L_I = \int_{(\Delta r)_I}^{R_S/2} \frac{d(r - R_S)}{1 - \frac{R_S}{r}} \simeq R_S \ln \left[ \frac{R_S}{2(\Delta r)_I} \right] = R_S |\ln \epsilon_I|,$$  \hspace{1cm} (4)

where we have assumed that $\epsilon_I$ is small. In the rotating case, $r_\ast = \int \frac{dr}{r^2 + a^2 - 2Mr}$ and the expression for $L_I$ is similarly obtained, $L_I = r_+ |\ln \epsilon_I|$, valid for $\epsilon_I \ll 1$.

The frequency of an echo mode, in the case of a non-rotating BH, is determined by the inverse of the total length of its trip (i.e., the length of the cavity times the number of reflections). Hence, the highest-lying angular frequency of the echo modes would go as

$$R_S \omega_I \sim \frac{R_S}{L_I} = \frac{1}{|\ln \epsilon_I|}.$$

As $\epsilon_I \ll 1$ for most models of interest, the frequency of even the most rapidly oscillating modes of the echoes is always suppressed relative to $\omega_{\text{BH}}$.

When the compact object is rotating, the frequency in Eq. (5) is modified by the rotation in accordance with [53, 54] (also see [39])

$$\omega_I = -\frac{\pi q}{2|r_0^0|} + m\Omega,$$  \hspace{1cm} (6)

where

$$r_0^0 \sim M[1 + (1 - a^2/M^2)^{-1/2}] |\ln \epsilon_I|,$$  \hspace{1cm} (7)

$$\Omega = \frac{a/M}{2r_+},$$  \hspace{1cm} (8)
and $q$ is a positive integer denoting the overtone number, for which we will always choose the dominant one, $q = 1$. Because $\omega_I - m\Omega$ is negative for $\epsilon_I \ll 1$ and $a \lesssim M$, the typical solution is formally unstable, just like the $r$-mode solutions for neutron stars (e.g., [55]). However, the instability is very weak and we will, conservatively, ignore the possible growth of the echo modes due to this effect.

The inverse of the damping time is also suppressed, but even more so because of the very small cost in wave amplitude for any given trip through the cavity (however, see the second caveat below). A calculation reveals that the (inverse) decay time for the most rapidly oscillating mode would be given by [19]

$$\frac{1}{\tau_I} \sim -\frac{2\beta_{s\ell}M}{|r_0^2|} \frac{r_+}{r_+ - r_-} [\omega_I(r_+ - r_-)]^{2\ell+1}(\omega_I - m\Omega),$$

with $s = 2$ and $(\ell, m)$ representing the spin and harmonic mode, respectively, of the graviton, and $\beta_{s\ell}$ is a spin/mode-dependent number. The mode will be chosen as $(2,2)$ and, with this choice, $\beta_{22} = 1/225$.

For the case of interest, $|\ln \epsilon_I| \gg 1$ and $s = \ell = m = 2$,

$$\omega_I \simeq m\Omega = \frac{a/M}{r_+}$$

and

$$\tau_I \approx \frac{225}{\pi} \frac{1}{M} |r_0^2|(a/M)^{-5} \left(\frac{r_+ - r_-}{r_+}\right)^{-4}$$

$$\quad = \frac{225}{\pi} r_+ |\ln \epsilon_I|^2 \frac{1 + \sqrt{1 - a^2/M^2}}{1 - a^2/M^2} (a/M)^{-5} \left(\frac{r_+ - r_-}{r_+}\right)^{-4}.$$  

Notice that the decay time of the echoes is fortuitously long because of the factor of $|\ln \epsilon_I|^2$. This huge enhancement in the lifetime of the additional modes is essential for their detection.
To get a better idea about the magnitude of the numerical coefficients, one can evaluate $\tau_I$ for the case $a = 0.68M$, which corresponds to the remnant BH of the famous detection GW150914 \[22\],

$$\tau_I = 3100 r_+ |\ln \epsilon_I|^2. \quad (12)$$

So that, in addition to the enhancement due to the large value of $|\ln \epsilon_I|^2$, one finds an equally large numerical coefficient. Indeed, 3100 is the same order of magnitude as what $|\ln \epsilon_I|^2$ would be for the inverse of the square root of the Bekenstein–Hawking entropy, $\epsilon_{Bek-Hawk} = 1/\sqrt{S_{bh}} \sim 10^{-40}$.

One can understand, heuristically, the scaling of the lifetime of the echoes in terms of ideas that were introduced in \[56\]. First recall from Eq. (6) that the “intrinsic” angular frequency is $\omega_I \sim \frac{1}{r_+} \frac{1}{|\ln \epsilon_I|}$ (which is further modified by the external rotation $m\Omega$). This means that the GR external observer, who is unaware of the origin of the echo modes, would conclude that the wavelength of the radiation at distances far away from the horizon is $\lambda_I \sim r_+ |\ln \epsilon_I|$. On the other hand, the same observer attributes the source with having an area of about $Mr_+$. She then just needs to know that the transmission cross-section for such long wavelength modes through a proportionally smaller surface of area $A$ is determined by the ratio $A/\lambda^2$, which translates into $Mr_+ / \lambda_I^2 \sim \frac{1}{|\ln \epsilon_I|^2}$. The lifetime is inversely proportional to the transmission rate and therefore scales as $\tau_I \sim |\ln \epsilon_I|^2$. This estimate is based on the assumption that most of the mode’s energy is being emitted in the form of coherent waves rather than dissipating as heat (see the third caveat below).

The above heuristic argument can also be used to discuss a compact object whose reflection at the would-be horizon is not so perfect. Our contention
is that, unless the reflection is parametrically small (in which case, the absorption is essentially complete and then the object effectively has a horizon), the mode’s lifetime scales in the same manner as it does for the case of total reflection. The key point is that partial absorption occurs at the surface of the compact object. Then, given the above logic, the additional energy absorption through that surface would similarly scale as $A/\lambda_I^2 \sim \frac{1}{|\ln \epsilon_I|^2}$. But, for small $\epsilon$, the area of the object’s surface scales the same as the area of the gravitational potential barrier. It then follows that the energy absorption through the object’s surface scales in the same way as its energy leakage through the potential barrier, given that the wavelengths of the leaking modes are similarly set by the minimal redshift (as argued in [56]). And, since it is the leakage through the barrier that determines the lifetime of the emitted waves, the conclusion is that, up to some numerical factor, Eq. (11) applies just as well for the case of a finite absorber.

Observational data from the BH merger event which led to GW150914 [22, 24] can be used to support an upper limit on the minimal value of $\epsilon_I$ (see Section 3 for the precise statement and details),

$$\epsilon_{I,\text{min}} \lesssim 10^{-40};$$

(13)

which can, according to our proposal, be used to bound the entropy from below,

$$S = \frac{1}{\epsilon_{I,\text{min}}^2} \gtrsim 10^{80}.$$

(14)

This bound provides the first experimental indication that an astrophysical BH possesses extremely large entropy, on the order of that predicted by the Bekenstein–Hawking area law. And given that it has both entropy and energy (or mass), a temperature for the BH must then follow.
2.4 Caveats

Like in any complicated physical situation, the above picture has caveats. Let us touch upon just a few:

1. For the BHs of GR, the spectral properties of the QNMs depend strongly on both the external geometry and the boundary conditions at the horizon (those of in-falling waves only). And so, whether or not there are spacetime modes akin to the standard modes of a GR BH will depend on the boundary conditions at the outermost surface of the model in question. It is quite possible that the absence or modification of the conventional spacetime modes is already enough to rule out some otherwise-plausible compact objects. The exterior geometry of a non-rotating exotic compact object is expected to be a Schwarzschild geometry due to Birkhoffs theorem, if GR is assumed to be valid. On the other hand, the exterior geometry of a rotating object could differ noticeably from that of Kerr; but this correction is expected to be small when $\epsilon_I \ll 1$. A notable exception is the polymer model because, as explained in [29], it behaves just like a perfect absorber in the limit that the string coupling (equivalently, $\hbar$) goes to zero. Hence, the polymer BHs are emitters of both conventional spacetime modes and fluid modes; the latter of which exhibit some similarities with the scaling properties of the echo modes thanks to the previous barrier-leakage argument [56].

2. It is sometimes assumed that the echoes have a suppression in amplitude, relative to the initial burst, which is in addition to that of
Eq. (11). This suppression is supposed to grow stronger with the number of reflections; in other words, there is a further cost in amplitude for each trip through the cavity [19]. The degree of suppression per trip is a factor that must be put in by hand.

Such a suppression in the amplitude of GWs means that they are partially absorbed by the compact object. Let us recall that total reflection is really no different than unhindered propagation through the object, as the waves would then be perceived as being totally reflected at \( r = 0 \). Meanwhile, the intermediary case of partial absorption requires an extremely strong interaction. Such interactions would be expected to cause dissipation rather than coherent motion (see below). They are also not particularly realistic since standard matter interacts extremely weakly with GWs; in fact, the whole universe is transparent to them. One could try to model the partial absorption with an ad-hoc modification of the gravitational potential, as done for example in [38], but the justification for such an arbitrary manipulation is lacking.

Nevertheless, as long as the absorption is not close to perfect, the lifetimes of the echo modes will not be significantly modified, as argued above. This means that our bounds on the entropy can be expected to remain intact.

3. The echo modes are also subject, in principle, to intrinsic dissipation. The analysis to follow is premised on the assumption that such dissipation is negligible, which is consistent with total reflection as the echoing process requires essentially no interactions between the emitted GWs.
the interior of the compact object.

4. Formulating the full reflection of GWs from a surface is a rather daunting challenge and it is even unclear if a consistent description exists for the case of rotating objects [32]. It is likely though that, for a slow-enough rotation, various approximations should be applicable with a reasonable degree of accuracy.

3 Bound on entropy from gravitational-wave observations

In this section, closely following our own methodology from [29], we derive bounds on the entropy for models of BH alternatives. This analysis assumes that GW signals from the remnant for GW150914 [22, 24] are consistent with the QNMs of a BH.

We will be making use of Eqs. (6) and (9) (and also [36, 19]) and only be considering the dominant $s = \ell = m = 2$ mode of the gravitons. In what follows, a subscript of $I$ indicates a quantity that is relevant to the QN echo modes for the $I^{th}$ model, whereas a subscript of $BH$ is reserved for the QNMs of a classical BH. A spectral frequency $f = \omega/2\pi$ carries no such subscript.

3.1 Gravitational wave spectra

Let us begin by using the knowledge that QNMs are described by damped sinusoids. In fact, the Fourier waveform of the QNMs for a BH alternative
can be expressed as follows \[58, 29\]:

\[
\tilde{h}(f) = e^{2\pi f I} A_I \tau_I \frac{2 f_I^2 Q_I \cos \phi_I - f_I (f_I - 2 i f Q_I) \sin \phi_I}{f_I^2 - 4 i f f_I Q_I + 4 (f_I^2 - f^2) Q_I^2}, \tag{15}
\]

for which the square of the root-mean-square (RMS) magnitude goes as

\[
|\tilde{h}|^2 = \frac{4 A_I^2 f_Q^2 Q_I^4}{2 \pi^2 \left\{ 16 f_Q^2 f_Q^2 Q_I^2 + [4 Q_I^2 (f_Q^2 - f^2) + f_Q^2]^2 \right\}}, \tag{16}
\]

where \(A_I\) is the amplitude of this time-domain waveform, \(f_I = \omega_I/2\pi\) is the frequency of the echo mode, \(Q_I\) is defined by \(Q_I = \pi f_I \tau_I\), \(t_I\) is the time delay of an echo mode relative to a typical BH mode and \(\phi_I\) is a constant phase which will be set to zero for simplicity. It should be kept in mind that, for the case of interest, \(|\ln \epsilon_I| \gg 1\), \(f_I \propto m \Omega\), \(\tau_I \propto |\ln \epsilon_I|^2\) and so \(Q_I \gg 1\).

The QNM amplitude \(A_I\) will be given by \(A_I = \alpha A_{\text{bh}}\) where \(\alpha\) is a parameter that is smaller than 1 but is of order unity. For simplicity, we choose \(\alpha = 1\). This choice that amounts to neglecting a small correction which is due to the reflections of the echo modes and possibly some absorption (see the second caveat at the end of the last section). This assumption can easily be relaxed. The BH QNM amplitude is given by

\[
A_{\text{bh}} = \frac{M_{\text{bh}}}{r} \mathcal{F} \sqrt{\frac{8 \varepsilon_{\text{RD}}}{M_{\text{bh}} Q_{\text{bh}} f_{\text{bh}}}}, \tag{17}
\]

where \(r\) is the distance from the observers to the object in question and \(M_{\text{bh}}\) (more generally, \(M_I\)) denotes its mass. Also, \(\varepsilon_{\text{RD}} \approx 0.44 \eta^2\) is the ringdown efficiency parameter, with \(\eta\) representing the symmetric mass ratio of the binary whose merger led to the detected GWs. Finally, \(\mathcal{F}\) is a numerical factor that depends on the position of the source in the sky. We determine
this value by insisting that the signal-to-noise ratio (SNR) of the ringdown signal for GW150914 is 7 \cite{24}.

The essential point is that, for the relevant case of \( Q_I \gg 1 \), the maximum of \( |\hat{h}|^2 \) at \( f = f_I \) is proportional to \( Q_I^2 \propto \tau_I^2 \) (cf, Eq. (16)), and so the RMS value of \( \hat{h}(f_I) \) scales as \( A_I \tau_I \). This is a very large enhancement factor as compared to the standard BH signal.

### 3.2 Current bounds from GW150914

When a horizonless compact object is formed as a result of a merger, additional modes are expected as discussed in the previous sections. The fact that such modes are absent for GW150914 allows us to place bounds on \( \epsilon_I \) and, hence, on the entropy of the remnant BH. The GW spectra for such additional modes are shown in Fig. 1 together with the noise spectral density of Advanced LIGO (aLIGO) in the O1 run. Roughly speaking, the ratio between the GW spectra and noise corresponds to the signal-to-noise ratio of the putative additional modes. Though unlike, e.g., Fig. 13 of \cite{59}, we show \( 2|\hat{h}|/\sqrt{\tau} \) instead of \( 2|\hat{h}|/\sqrt{f} \). This is because \( 1/\tau \) is much smaller than \( f \) and the ratio between \( 2|\hat{h}|/\sqrt{\tau} \) and \( \sqrt{S_n} \) provides a closer estimate of the actual SNR. We also note that the search for echoes by the LIGO and Virgo Collaborations \cite{33} were carried out up to 2048 seconds after the merger, which should be enough to detect such modes if they are present.

As can be seen in Fig. 1 as well as in Eq. (6), the frequency of the additional modes increases with decreasing \( \epsilon_I \) and reaches a constant value of \( 2\Omega/2\pi \approx 200 \) Hz for very small values of \( \epsilon_I \). Because the limiting frequency is in the LIGO sensitivity band, we are able to place very stringent bounds
Figure 1: The GW spectra of additional QNMs for various $\epsilon_I$ (solid, colored), assuming that the remnant of GW150914 is an exotic alternative to a BH. For reference, we also present the spectrum for a BH QNM (black, solid) and the noise spectral density for aLIGO during its O1 run (brown, dashed). Observe that the absence of such modes with GW150914 already allows us to probe the Bekenstein–Hawking entropy (red, thick solid). Note that we have scaled $|\tilde{h}|$ with $1/\tau^{1/2}$, since $1/\tau \ll f$.

on the existence of additional modes.

Let us now derive bounds on $\epsilon_I$ using Fig. 1. This figure clearly shows that aLIGO would have detected the additional echo modes if $10^{-40} \lesssim \epsilon_I \lesssim 0.01$. The lower bound comes from our prior belief that it does not make sense to consider $\epsilon_I$ smaller than $l_P/r_+$ (see below for more details). Therefore, the absence of such additional modes allows us to formally exclude values of $\epsilon_I$ in this range. Meaning that, formally, we are able to deduce that either $S_I \gtrsim 10^{80}$ or $S_I \lesssim 10^4$.  

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However, values of $\epsilon_I \gtrsim 0.01$ are outside the domain of validity of our analysis because, as previously emphasized, our framework requires $\epsilon_I \ll 1$. So that, formally, the range $\epsilon_I \gtrsim 0.01$ cannot be ruled out by our analysis and, therefore, we cannot argue conclusively against very small entropies. It is, however, highly unlikely that an astrophysical compact object, which represents the final state of collapsing matter, could have an entropy as low as $10^4$. Moreover, if $\epsilon_I$ is too large, the inspiral part of the waveform becomes inconsistent with that of binary black holes due to non-vanishing tidal deformabilities [60].

On the other hand, since it does not make sense to consider $\epsilon_I$ any smaller than the effective limit of $l_P/r_+$, we are left with a “range” of $\epsilon_I \sim l_P/r_+ \sim 10^{-40}$. This range, as mentioned above, corresponds to an entropy of $S_I \sim 10^{80}$, a value which is of the order of the Bekenstein–Hawking entropy for a BH of this same mass, $S_{\text{Bek–Hawk}} \simeq 3 \times 10^{80}$.

Let us suppose, for the sake of discussion, that the amplitude of the echoes $A_I$ is much smaller than the BH amplitude $A_{BH}$. In this hypothetical case, one might argue that the bound from the non-observation of the echoes should be interpreted as a bound on $A_I$ rather than on $\epsilon_I$. But this scenario is highly unlikely, as we now argue. To begin, one can notice that roughly the same flux of energy is flowing inwards towards the surface of the compact object as flowing outwards in the form of gravitational waves [62]. Therefore, for $A_I$ to be parametrically smaller than $A_{BH}$, the reflection coefficient $R$ has to be parametrically smaller than unity. However, for any regular, compact object without an horizon, no matter how exotic it might otherwise be, the
coefficient $R$ has to be of order unity. This is an expected result, as relatively smaller-entropy objects, such as a regular compact object, should have a smaller absorption rate than that of higher-entropy objects like a BH.

To summarize, we can conclude that the GW150914 merger event most likely produced an object with an entropy on the order of the Bekenstein–Hawking value!

4 Conclusion

We have proposed that BH mergers should be viewed as cosmological scattering experiments that provide an opportunity to infer, or put bounds on, the entropy of astrophysical BHs. Our main idea is that a lower bound on the entropy of a classical BH or any other exotic, ultracompact, astrophysical object, can be deduced from a bound on the minimal (Tolman) redshift factor of waves emerging from the vicinity of its outer surface. This minimal redshift factor can be, and already has been, constrained by observational data. More specifically, we have argued that the entropy of any ultracompact, astrophysical object can be expected to scale with the inverse of the square of this minimal redshift factor.

Our argument closely follows logic that was first used by ’t Hooft to show that the entropy of a BH does indeed agree with the Bekenstein–Hawking area law, subject to the assumption that the exterior metric is Schwarzschild regardless of the boundary conditions at the surface. We used the scaling

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$^4$The coefficient can be evaluated by estimating the reflection from the gravitational barrier near the surface of the object for a momentum transfer corresponding to twice the frequency of the waves.
relation between the entropy and the minimal redshift factor to show that a large range of entropy values has already been excluded by data from the GW150914 event. The main result of our analysis is that astrophysical BHs most likely have an extremely large entropy; on the order of the Bekenstein–Hawking entropy $S \sim 10^{80}$ for the remnant of GW150914.

Formally, our analysis cannot exclude very low entropy values on the order of $10^4$. These values are not excluded because our framework is only valid for values of the minimal redshift factor that are parametrically small. It is, however, highly unlikely that an astrophysical compact object, which represents the final state of collapsing matter, could have an entropy as low as $10^4$.

Our bounds are based on the assumption that any given exotic alternative has a reflecting outer surface. Nevertheless, we have also argued that the degree of reflection is not so important for the validity of the bounds. Even if the reflection is not complete, the bounds would still be strong because, for exotic compact objects with small enough values of $\epsilon_I$, the frequencies of the additional modes lie in the LIGO sensitivity band and their amplitudes are much larger than those of the standard BH modes.

To reach a precise quantitative lower bound on the remnant entropy, a more sophisticated analysis is required \cite{33, 63, 64}. Besides using actual data, the analysis should account for the possibility of absorption in a quantitatively precise way and likewise for the possible instabilities. However, our results do indicate that such an analysis would also yield bounds of the order of the Bekenstein–Hawking entropy.\footnote{The uncertainty principle of quantum gravity may set a fundamental resolution limit}
interferometers such as the Einstein Telescope [65] and the Cosmic Explorer [66] would be able to provide much more accurate results.

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to GW observations that is well above the Planckian scale [61]. Our estimate is different from [61] as we do not measure any Planckian effect and the lower bound on the entropy partially comes from a theoretical consideration.


