Probing the Scale of New Physics
in the $ZZ\gamma$ Coupling at $e^+e^-$ Colliders

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Abstract:
The $ZZ\gamma$ and $Z\gamma\gamma$ triple neutral gauge couplings are absent in the Standard Model (SM) at the tree level. They receive no contributions from effective dimension-6 operators, but can arise from effective operators of dimension-8. We study the scale of new physics associated with such dimension-8 operators that can be probed via measurements of the $e^+e^-\rightarrow Z\gamma$ process at future $e^+e^-$ colliders including the ILC, CEPC, FCC-ee and CLIC. We show how final-state angular distributions can play a key rôlê in suppressing SM backgrounds. We find that the dimension-8 new physics scale can be probed up to the multi-TeV region at such lepton colliders.

1. Introduction

At the time of writing, there is no confirmed evidence for phenomena in accelerator experiments that require new physics beyond the Standard Model (SM) [1], pending clarifications of the apparent discrepancy between the SM prediction and the experimental value of the anomalous magnetic moment of the muon, and of the apparent anomalies in b-hadron decays into strange and charmed particles. It is therefore plausible to assume that the SM particles have the same dimension-4 interactions as in the SM, and seek to characterize possible deviations from SM predictions in terms of higher-dimensional effective operators constructed out of SM fields, whose contributions are suppressed by some power of an underlying new physics scale \( \Lambda \gg 100\text{ GeV} \) [2].

This Standard Model Effective Field Theory (SMEFT) approach has mainly been applied with the assumption that only dimension-6 SMEFT operators [3] contribute to the experimental observables under study [4]. With this restriction, global SMEFT analyses [5] have been made of the available data from the LHC and other accelerators, and the sensitivities of experiments at possible future accelerators to the scales of new
physics in dimension-6 operators have also been estimated [5–7]. However, there are some instances in which dimension-6 contributions are absent, and the first SMEFT operators to which experimental measurements are sensitive are those of higher dimensions [8]. Examples where dimension-8 operators dominate include light-by-light scattering [9], $\gamma\gamma \to \gamma\gamma$, which has recently been measured for the first time in heavy-ion collisions at the LHC [10] and $gg \to \gamma\gamma$ scattering [11], which is constrained by ATLAS measurements of events with isolated diphotons in pp collisions at the LHC [12]. We note also that the effect of dimension-8 operators on Higgs observables was discussed in [13].

Another promising way to probe directly dimension-8 operators is via the $ZZ\gamma$ and $Z\gamma\gamma$ triple neutral gauge couplings [14, 15]. These couplings are absent in the SM and receive no dimension-6 contributions [2, 3]. Within the SMEFT approach, the first contributions arise from effective operators of dimension 8. We study here how these dimension-8 operators can be probed via the $e^+e^-\to Z\gamma$ process at future $e^+e^-$ colliders including the ILC [16], CEPC [17], FCC-ee [18] and CLIC [19], offering a rare direct window on the new physics at dimension-8.

Our analysis framework is described in Section 2. We first discuss in Section 2.1 how the neutral triple-gauge couplings $ZV\gamma$ ($V = Z, \gamma$) can be generated by effective dimension-8 operators, and then present cross sections for $e^+e^-\to Z\gamma$ production in the different $Z$ polarization states $Z_{T,L}$ in Section 2.2. Since the SM produces $Z_T\gamma$ final states copiously, with the vector bosons emerging preferentially in the forward and backward directions, one can make use of angular distributions in the $e^+e^-$ centre-of-mass frame and $Z$ decay frame to separate the SM contribution to $Z\gamma$ final states and distinguish $Z_L$ from $Z_T$ via their decays into $\bar{f}f$ pairs. We study angular observables in Section 3, where the angular distributions are presented in Section 3.1 and their uses for isolating and analyzing new physics contributions are discussed in Section 3.2, with the focus on $O(\Lambda^{-4})$ contributions in Section 3.2.1 and on $O(\Lambda^{-8})$ contributions in Section 3.2.2. A systematical analysis of the sensitivities to $\Lambda$ by measurements at different $e^+e^-$ collider energies $\sqrt{s}$ from 250 GeV to 5 TeV is presented in Section 3.3. Finally, we summarize our conclusions in Section 4. The $5\sigma$ sensitivity to $\Lambda$ may reach into the multi-TeV range, depending on the $e^+e^-$ collision energy, even after taking into account the fact that in many new physics scenarios the SMEFT approach may be valid only when $\Lambda \gtrsim \sqrt{s}$, or at least $\Lambda \gtrsim \sqrt{s}/2$. Thus, the $e^+e^-\to Z\gamma$ process may provide a unique and interesting probe of new physics in $e^+e^-$ collisions.

2. Neutral Triple-Gauge Couplings and $e^+e^-\to Z\gamma$ Production

In this Section, we first discuss the neutral triple-gauge couplings $ZV\gamma$ ($V = Z, \gamma$), and the corresponding dimension-8 effective operators as their unique lowest-order gauge-invariant formulations in the SMEFT. We then analyze the scattering amplitudes for
\[ e^+ e^- \rightarrow Z \gamma, \] considering separately transversely and longitudinally polarized final-state Z bosons.

2.1. \(Z \gamma\) Coupling from Dimension-8 Operator

The neutral triple gauge couplings (nTGCs) \(Z \gamma (V = Z, \gamma)\) vanish at tree level in the SM and do not receive contributions from any dimension-6 effective operators. However, at the dimension-8 level there are four CP-conserving effective operators that can contribute to the nTGCs \[15\],

\[ \Delta L \text{(dim-8)} = \sum_{j=1}^{4} \frac{c_j}{\Lambda^4} O_j = \sum_{j=1}^{4} \frac{\text{sign}(c_j)}{\Lambda^4} O_j, \] (2.1)

where the dimensionless coefficients \(c_j\) may be \(O(1)\), with signs \(\text{sign}(c_j) = \pm\), and the corresponding cutoff scales \(\Lambda_j \equiv \Lambda/|c_j|^{1/4}\). The four dimension-8 CP-even effective operators \(O_j\) contributing to the nTGCs may be written as

\[ O_{\tilde{B}W} = i H^\dagger \tilde{B}_{\mu\nu} W^{\mu\rho} \{ D_\rho, D_\nu \} H + \text{h.c.,} \] (2.2a)
\[ O_{\tilde{B}W} = i H^\dagger B_{\mu\nu} \tilde{W}^{\mu\rho} \{ D_\rho, D_\nu \} H + \text{h.c.,} \] (2.2b)
\[ O_{\tilde{W}W} = i H^\dagger \tilde{W}_{\mu\nu} W^{\mu\rho} \{ D_\rho, D_\nu \} H + \text{h.c.,} \] (2.2c)
\[ O_{\tilde{B}B} = i H^\dagger \tilde{B}_{\mu\nu} B^{\mu\rho} \{ D_\rho, D_\nu \} H + \text{h.c.,} \] (2.2d)

where \(H\) denotes the SM Higgs doublet. The dual field strengths \(\tilde{B}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta}\) and \(\tilde{W}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta} W^{\alpha\beta}\), and \(W_{\mu\nu} \equiv W^3_{\mu\nu}\) corresponds to the third component of the weak gauge group \(SU(2)_W\). Among these dimension-8 operators, one can use the equations of motion and integration by parts to show that \(O_{\tilde{B}W}\) is equivalent to \(O_{\tilde{B}W}\) up to operators with more currents, or more field-strength tensors, or with quartic gauge boson couplings. Moreover, the operators \(O_{\tilde{W}W}\) and \(O_{\tilde{B}B}\) do not contribute to \(Z \gamma\) coupling for on-shell \(Z\) and \(\gamma\). Thus, there is only one independent CP-conserving dimension-8 operator to be considered in our nTGC study. We choose \(O_{\tilde{B}W}\) for our analysis, and denote the corresponding cutoff scale by \(\Lambda_{\tilde{B}W} = \Lambda\), for simplicity.

The dimension-8 operator \(O_{\tilde{B}W}\) yields the following effective \(Z \gamma Z^*\) coupling in momentum space:

\[ i \Gamma_{Z\gamma Z^*}(q_1, q_2, q_3) = \text{sign}(c_j) \frac{v M_Z (q_3^2 - M_Z^2)}{\Lambda^4} \epsilon^{\mu\nu\alpha\beta} q_{2\beta}, \] (2.3)

where \(v\) is the Higgs vacuum expectation value. However, \(O_{\tilde{B}W}\) does not contribute to the \(Z \gamma \gamma^*\) coupling for on-shell gauge bosons \(Z\) and \(\gamma\).

2.2. \(Z \gamma\) Production at \(e^+ e^-\) Colliders

The SM contributes the production process \(e^- (p_1) e^+ (p_2) \rightarrow Z(q_1) \gamma(q_2)\), via \(t\)- and \(u\)-channel exchange diagrams at tree level. In general, the final-state Z boson may have
either longitudinal or transverse polarization.

Working in the centre-of-mass (c.m.) frame of the $e^+e^-$ collider and neglecting the electron mass, we define the momenta of the initial- and final-state particles as follows:

\begin{align}
  p_1 &= E_1(1, 0, 0, 1), \quad p_2 = E_1(1, 0, 0, -1), \\
  q_1 &= (E_Z, q \sin \theta, 0, q \cos \theta), \quad q_2 = q(1, -\sin \theta, 0, -\cos \theta),
\end{align}

where the electron (positron) energy $E_1 = \frac{1}{2}\sqrt{s}$ and the momentum $q = \frac{1}{2\sqrt{s}}(s-M_Z^2)$, and the $Z$ boson energy $E_Z = \sqrt{q^2+M_Z^2}$. The squared scattering amplitudes for the SM contributions to final states with the different $Z$ polarizations take the following forms:

\begin{align}
  |T_{\text{sm}}|^2[Z_L\gamma_T] &= e^4(8s_W^4 - 4s_W^2 + 1) \frac{M_Z^2 s}{c_W^2 s_W^2 (s-M_Z^2)^2}, \quad (2.5a) \\
  |T_{\text{sm}}|^2[Z_T\gamma_T] &= e^4(8s_W^4 - 4s_W^2 + 1) \frac{(1+\cos^2 \theta) (s^2+M_Z^4)}{2s_W^2 c_W^2 \sin^2 \theta (s-M_Z^2)^2}, \quad (2.5b)
\end{align}

where we have averaged over the initial-state spins, and used the notations $(s_W, c_W) = (\sin \theta_W, \cos \theta_W)$ with $\theta_W$ being the weak mixing angle. We have verified that the above formulae agree with the previous results in the literature [15].

We see from the above equations that the squared amplitude for a final-state longitudinal weak boson $Z_L$ is suppressed by $1/s$ in the high-energy region $s \gg M_Z^2$. This behaviour can be understood via the equivalence theorem [21], which connects the longitudinal scattering amplitude to the corresponding Goldstone boson amplitude at high energies,

\begin{equation}
  T[Z_L\gamma_T] = T[\pi^0\gamma_T] + O(M_Z/\sqrt{s}), \quad (2.6)
\end{equation}

where $\pi^0$ is the would-be Goldstone boson absorbed by the longitudinally polarized $Z$ via the Higgs mechanism of the SM. Since the SM does not contain any tree-level $ZV\gamma$ and $\pi^0V\gamma$ ($V = Z, \gamma$) triple couplings, at tree level the production processes $e^+e^- \rightarrow Z_L\gamma_T$ and $e^+e^- \rightarrow \pi^0\gamma_T$ must proceed through the $t$-channel electron-exchange process. Since the electron Yukawa coupling $y_e = \sqrt{2}m_e/v = O(10^{-6})$ is very small and can be neglected for practical purposes, we have for the SM contributions

\begin{equation}
  T_{\text{sm}}[\pi^0\gamma_T] \simeq 0, \quad |T_{\text{sm}}[Z_L\gamma_T]|^2 = O(M_Z^2/s). \quad (2.7)
\end{equation}

This explains the high-energy behavior of Eq.(2.5a).

We note also that for the final state with a transverse weak boson $Z_T$, Eq.(2.5b) exhibits a collinear divergence at $\theta = 0, \pi$ due to our neglect of the electron mass $m_e \simeq 0$. In the following analysis we implement a lower cut on the transverse momentum of the final state photon: $P_T^\gamma = q \sin \theta > P_T^{\gamma_0}$ to remove the collinear divergence, corresponding to a lower cut on the scattering angle $\theta > \delta = \arcsin(P_T^{\gamma_0}/q)$. For
\( \theta \neq 0, \pi \), Eq.(2.5b) gives the asymptotic behavior, \( \mathcal{T}_{\text{sm}}[Z_T\gamma_T] = O(s^0) \), in the high-energy regime \( s \gg M_Z^2 \), as expected. This completes the explanation why production of the transversely polarized final state \( Z_T\gamma_T \) dominates over that of the longitudinal final state \( Z_L\gamma_T \).

The contributions of the dimension-8 operator include \( \mathcal{O}(\Lambda^{-4}) \) and \( \mathcal{O}(\Lambda^{-8}) \) terms. The term of \( \mathcal{O}(\Lambda^{-4}) \) arises from the interference between the dimension-8 operator contribution and the SM contribution,

\[
2 \Re \left( \mathcal{T}_{\text{sm}} \mathcal{T}^{(8)} \right) [Z_L\gamma_T] = \pm \frac{e^2(1-4s_W^2)}{2s_W c_W} \frac{M_Z^2 s}{\Lambda^4}, \tag{2.8a}
\]

\[
2 \Re \left( \mathcal{T}_{\text{sm}} \mathcal{T}^{(8)} \right) [Z_T\gamma_T] = \pm \frac{e^2(1-4s_W^2)}{2s_W c_W} \frac{M_Z^4}{\Lambda^4}, \tag{2.8b}
\]

which is consistent with results in the literature [15]. [Here the \( \pm \) signs of the \( \mathcal{O}(\Lambda^{-4}) \) term correspond to the two possible signs of a given dimension-8 operator, \( \text{sign}(c_j) = \pm \), as shown in Eq.(2.1).] We see that the contribution to the \( Z_L\gamma_T \) production channel is enhanced relative to that of the \( Z_T\gamma_T \) production channel by a factor of \( s/M_Z^2 \) at \( \mathcal{O}(\Lambda^{-4}) \).

The \( \mathcal{O}(\Lambda^{-8}) \) term originates from the pure dimension-8 contribution,

\[
|\mathcal{T}^{(8)}[Z_L\gamma_T]| = \frac{(8s_W^4 - 4s_W^2 + 1)(\cos2\theta + 3)}{32} \frac{M_Z^2(s-M_Z^2)^2 s}{\Lambda^8}, \tag{2.9a}
\]

\[
|\mathcal{T}^{(8)}[Z_T\gamma_T]| = \frac{(8s_W^4 - 4s_W^2 + 1)\sin^2\theta}{8} \frac{M_Z^4(s-M_Z^2)^2}{\Lambda^8}. \tag{2.9b}
\]

The energy dependence in the above formulas can be directly understood by power counting,

\[
\mathcal{T}^{(8)}[Z_L\gamma_T] \sim \mathcal{T}^{(8)}[\gamma_0\gamma_T] \sim \frac{M_Z s^2}{\Lambda^4}, \tag{2.10a}
\]

\[
\mathcal{T}^{(8)}[Z_T\gamma_T] \sim \frac{M_Z^2 s}{\Lambda^4}, \tag{2.10b}
\]

which explains the asymptotic high-energy behaviors in Eq.(2.9) when \( s \gg M_Z^2 \). We see that at \( \mathcal{O}(\Lambda^{-8}) \) the \( Z_L\gamma_T \) production channel dominates over the \( Z_T\gamma_T \) production channel at high energies \( s \gg M_Z^2 \).

We can understand further the asymptotic behavior of the interference terms (2.8) for \( s \gg M_Z^2 \). In the case of the final state \( Z_L\gamma_T \), since we have \( \mathcal{T}_{\text{sm}}[Z_L\gamma_T] \sim \frac{M_Z s^2}{\sqrt{s}} \) [Eq.(2.7)] and \( \mathcal{T}^{(8)}[Z_L\gamma_T] \sim \frac{M_Z^3 s}{\Lambda^4} \) [Eq. (2.10a)], we find that their interference term behaves as \( \mathcal{T}_{\text{sm}} \mathcal{T}^{(8)}[Z_L\gamma_T] \sim \frac{M_Z^3 s}{\Lambda^4} \). This explains nicely the asymptotic behavior of Eq.(2.8a). However, for the final state \( Z_T\gamma_T \), using the naive power counting from Eqs.(2.5b) and (2.10b) we infer the asymptotic behaviors, \( \mathcal{T}_{\text{sm}}[Z_T\gamma_T] \sim s^0 \) and \( \mathcal{T}^{(8)}[Z_T\gamma_T] \sim \frac{M_Z^2 s}{\Lambda^4} \). Combining these would lead to the following behavior for their interference: \( \mathcal{T}_{\text{sm}} \mathcal{T}^{(8)}[Z_T\gamma_T] \sim \frac{M_Z^2 s}{\Lambda^4} \). However, this naive power counting contradicts
Eq. (2.8b), where we see that \( \overline{\mathcal{T}}_{\text{sm}} T_{(8)}[Z_T \gamma_T] \sim \frac{M_Z^2 s^0}{A^4} \). Naive power counting fails in this case for a nontrivial reason, which is connected to the special structure of the helicity amplitude \( T_{(8)}[Z_T \gamma_T] \). We see from Eqs. (A.5a) and (A.6) of Appendix A.1 that the off-diagonal helicity amplitudes \( T_{(8)}[Z_T \gamma_T] \) with \( \lambda \lambda' = +-, -+ \) vanish because of the antisymmetric tensor \( \epsilon^{\mu \nu \alpha \beta} \) contained in the \( Z \gamma Z \) vertex [Eq. (2.3)]. Hence, the energy dependence of \( \overline{\mathcal{T}}_{\text{sm}} T_{(8)}[Z_T \gamma_T] \) is determined by the diagonal helicity amplitudes with \( \lambda \lambda' = ++, -- \). The SM amplitude \( \overline{\mathcal{T}}_{\text{sm}} T_{(8)}[Z_T \gamma_T] \) has a negative power of energy \( \propto s^{-1} \) in its diagonal helicity amplitudes as shown in Eq. (A.4a). This explains neatly the high-energy behavior \( \overline{\mathcal{T}}_{\text{sm}} T_{(8)}[Z_T \gamma_T] \sim \frac{M_Z^2 s^0}{A^4} \), in agreement with Eq. (2.8b).

3. Probing New Physics in the \( ZV \gamma \) Coupling at \( e^+e^- \) Colliders

In this Section we first analyze the kinematical structure of the reaction \( e^+e^- \rightarrow Z\gamma \) followed by leptonic \( Z \) decays. We then propose suitable kinematical cuts to suppress effectively the SM backgrounds, and derive the optimal sensitivity reach for the scale of the new physics in the \( ZV \gamma \) coupling. In Section 3.1, we analyze the angular observables for \( Z\gamma \) production with \( Z \rightarrow \ell^+\ell^- \), and then study probes of the new physics contributions at \( \mathcal{O}(\Lambda^{-4}) \) in Section 3.2.1 and at \( \mathcal{O}(\Lambda^{-8}) \) in Section 3.2.2, making use of angular observables to suppress the SM backgrounds for the specific \( e^+e^- \) collision energy \( \sqrt{s} = 3 \text{ TeV} \). Finally, we extend the analysis to other collider energies \( \sqrt{s} = (250, 500, 1000, 5000) \text{ GeV} \) in Section 3.3, showing the increase in sensitivity obtainable from increasing the collider energy.

3.1. Analysis of Angular Observables

In this subsection, we analyze the kinematical observables for the reaction \( e^+e^- \rightarrow Z\gamma \) followed by the leptonic decays \( Z \rightarrow \ell^+\ell^- \). We illustrate the kinematics in Fig. 1, where the scattering plane is determined by the incident \( e^-e^+ \) and the outgoing \( Z\gamma \) in the collision frame (with scattering angle \( \theta \)), and the directions of the final state leptons \( \ell^-\ell^+ \) determine the decay plane. We denote the angle between the two planes as \( \phi \) in the laboratory frame (which is equal to \( \phi_* \) in the \( Z \) rest frame).

In order to study the leptonic final states \( Z(q_1) \rightarrow \ell^-(k_1)\ell^+(k_2) \), we denote the lepton momenta as follows in the \( Z \) rest frame:

\[
\begin{align*}
k_1 &= \frac{M_Z}{2} (1, \sin \theta_* \cos \phi_*, \sin \theta_* \sin \phi_*, \cos \theta_*), \\
k_2 &= \frac{M_Z}{2} (1, -\sin \theta_* \cos \phi_*, -\sin \theta_* \sin \phi_*, -\cos \theta_*). \tag{3.1a}
\end{align*}
\]

Here the positive \( z_* \) direction in the \( Z \) rest frame is chosen to be opposite to the final-state photon direction in the laboratory frame. The \( \theta_* \) denotes the angle between the positive \( z_* \) direction and the \( \ell^- \) moving direction in the \( Z \) rest frame. When boosted back to the \( e^-e^+ \) collision frame (laboratory frame), the angle \( \theta_* \) changes but
the azimuthal angle $\phi_*$ is invariant. This is why the angle $\phi_*$ is equal to the angle $\phi$ between the scattering plane (defined by the incoming $e^- e^+$ directions and the outgoing $Z \gamma$ directions) and $Z$ decay plane (defined by the outgoing $\ell^-$ and $\ell^+$ directions) in the $e^- e^+$ collision frame.

Imposing a lower cut on the scattering angle in the laboratory frame $\theta > \delta$ (where $\delta \ll 1$) will correspond to a lower cut on the transverse momentum of the final-state photon $P_T^\gamma > q \sin \delta$. With this lower cut, we find the following total cross section for $Z\gamma$ production:

$$
\sigma(Z\gamma) = \frac{e^4 (1 - 4 s_W^2 + 8 s_W^4) [- (s - M_Z^2)^2 - 2 (s^2 + M_Z^4) \ln(\sin^2 \frac{\delta}{2})]}{32 \pi s_W^2 c_W (s - M_Z^2) s^2} \\
\pm \frac{e^2 (1 - 4 s_W^2) M_Z^2 (s - M_Z^2) (s + M_Z^2)}{32 \pi s_W c_W \Lambda^4 s^2} (3.2)
$$

the two possible signs of a given dimension-8 operator, sign($c_j$) = ±, as shown in Eq.(2.1). We further compute the exact numerical cross sections of $e^+ e^- \rightarrow Z\gamma$,\footnote{Since the leptonic vector coupling of $Z$ boson is proportional to $(1 - 4 s_W^2)$, it is sensitive to the value of $s_W^2$. Here we use the $\overline{\text{MS}}$ value $s_W^2 = 0.23122 \pm 0.00003$ ($\mu = M_Z$) [20].} as a function of the new physics scale $\Lambda$ and for different collider energies.

In the following, we present these cross sections using a photon transverse momentum cut $P_T^\gamma = q \sin \delta$ with $\delta > 0.2$,
\( \sqrt{s} = 250 \text{GeV}, \quad \sigma(Z\gamma) = \left[ 7749 \pm 8.90 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^4 + 1.98 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \) (3.3a)

\( \sqrt{s} = 500 \text{GeV}, \quad \sigma(Z\gamma) = \left[ 1624 \pm 1.38 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 + 0.929 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \) (3.3b)

\( \sqrt{s} = 1\text{TeV}, \quad \sigma(Z\gamma) = \left[ 390 \pm 0.566 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 2.62 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \) (3.3c)

\( \sqrt{s} = 3\text{TeV}, \quad \sigma(Z\gamma) = \left[ 42.9 \pm 0.0354 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.843 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \) (3.3d)

\( \sqrt{s} = 5\text{TeV}, \quad \sigma(Z\gamma) = \left[ 15.4 \pm 0.0145 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 + 1.09 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}. \) (3.3e)

As we will show shortly in Sec. 3.2-3.3 (cf. Table 2), the sensitivity reach of \( \Lambda \) in each case is such that on the right-hand-side of the corresponding formula above, the ratio inside each \([\cdots]\) is \( \mathcal{O}(1) \). Thus, we see from Eq.(3.3), that for the relevant sensitivity reaches of \( \Lambda \), the contributions of the dimension-8 operator are always much smaller than the SM contribution, so the perturbation expansion is valid. Also, Eq.(3.3) shows that for \( \sqrt{s} < 1\text{ TeV} \), the \( \mathcal{O}(\Lambda^{-4}) \) contribution is dominant, whereas for \( \sqrt{s} \gtrsim 1\text{ TeV} \), the \( \mathcal{O}(\Lambda^{-8}) \) contribution becomes dominant. This is because the \( \mathcal{O}(\Lambda^{-8}) \) contributions have higher energy dependence than the \( \mathcal{O}(\Lambda^{-4}) \) contributions, as shown in Eqs.(2.8)-(2.9).

The total cross section for \( e^+e^- \rightarrow Z\gamma \rightarrow \ell^+\ell^-\gamma \) is given by the product

\[ \sigma(\ell^+\ell^-\gamma) = \sigma(Z\gamma) \times \text{Br}(\ell^+\ell^-). \] (3.4)

The differential cross section is a function of the three kinematical angles \((\theta, \theta^*, \phi^*)\), and will be computed from the helicity amplitudes (A.12)-(A.13) in Appendix-A.2. We define the normalized angular distribution function as

\[ f^j_\xi = \frac{d\sigma^j}{\sigma^j d\xi}, \] (3.5)

where \( \xi = \theta, \theta^*, \phi^* \), and \( \sigma^j \) (with \( j = 0, 1, 2 \)) represents the SM contribution \( (\sigma_0) \), the \( \mathcal{O}(\Lambda^{-4}) \) contribution \( (\sigma_1) \), and the \( \mathcal{O}(\Lambda^{-8}) \) contribution \( (\sigma_2) \), respectively.

We find the following normalized polar angular distribution functions \( f^j_\theta \) and \( f^j_{\theta^*} \),

\[ f^0_\theta = -\frac{\csc\theta \left[ 3s^2 + \cos2\theta (s - M^2_Z)^2 + 2M^2_Z s + 3M^4_Z \right]}{4 \left[ (s - M^2_Z)^2 + 2(s^2 + M^4_Z) \ln(\sin^2 \frac{\theta}{2}) \right]}, \] (3.6a)

\[ f^1_\theta = \frac{1}{2} \sin \theta, \] (3.6b)

\[ f^2_\theta = \frac{3 \sin \theta \left[ 3s + \cos2\theta (s - 2M^2_Z)^2 + 2M^2_Z \right]}{16 (s + M^2_Z)}; \] (3.6c)
Figure 2. Normalized angular distributions in the polar scattering angle $\theta$ in the laboratory frame for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ terms, respectively. We use a polar angle cut $\delta = 0.2$ for illustration.

and

$$f^0_{\phi_*} = \frac{3 \sin \theta_* (3 + \cos 2 \theta_*)}{16} + \frac{3 \sin \theta_* (1 + 3 \cos 2 \theta_*) M^2_Z s}{8 \left[ (s - M^2_Z)^2 + 2(s^2 + M^4_Z) \ln(\sin^2 \theta_*) \right]} + O(\delta),$$  \hspace{1cm} (3.7a)

$$f^1_{\phi_*} = \frac{3 \sin \theta_* \left[ 2s - \cos 2 \theta_* (2s - M^2_Z) + 3 M^2_Z \right]}{16(s + M^2_Z)} + O(\delta),$$  \hspace{1cm} (3.7b)

$$f^2_{\phi_*} = \frac{3 \sin \theta_* \left[ 2s - \cos 2 \theta_* (2s - M^2_Z) + 3 M^2_Z \right]}{16(s + M^2_Z)} + O(\delta).$$  \hspace{1cm} (3.7c)

Then we compute the normalized azimuthal angular distribution functions $f^j_{\phi_*}$ as follows,

$$f^0_{\phi_*} = \frac{1}{2\pi} - \frac{3 \pi^2 (c^2_L - c^2_R)^2 M_Z \sqrt{s} (s + M^2_Z) \cos \phi_* + 8(c^2_L + c^2_R)^2 M^2_Z s \cos 2 \phi_*}{16\pi (c^2_L + c^2_R)^2 \left[ (s - M^2_Z)^2 + 2(s^2 + M^4_Z) \ln(\sin^2 \frac{\theta_*}{2}) \right]} + O(\delta),$$  \hspace{1cm} (3.8a)

$$f^1_{\phi_*} = \frac{1}{2\pi} - \frac{9 \pi^2 \sqrt{s} (s + M^2_Z) \cos \phi_* - 32 M^2_Z s \cos 2 \phi_*}{128\pi M_Z (s + M^2_Z)} + O(\delta),$$  \hspace{1cm} (3.8b)
Figure 3. Normalized angular distribution in the polar angle $\theta_*$ in the Z decay frame for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ terms, respectively, where the red and blue curves exactly overlap. We use a laboratory polar angle cut $\delta = 0.2$ for illustration.

\[ f_{\phi_*}^{\ast} = \frac{1}{2\pi} + \frac{9\pi(c_L^2 - c_R^2)^2 M_Z \sqrt{s} \cos\phi_*}{128(c_L^2 + c_R^2)^2(s + M_Z^2)} + \mathcal{O}(\delta), \]  

(3.8c)

where the coefficients $(c_L, c_R) = (s_W^2 - \frac{1}{2}, s_W^2)$ are the gauge couplings of the Z boson to the (left, right)-handed leptons. Here we have again chosen a lower cutoff $\delta \ll 1$ on the polar scattering angle $\theta$, which corresponds to the lower cut on the transverse momentum of the final state photon, $P_T^\gamma > q \sin\delta$.

As a side remark, we note that if the Z boson were a stable particle, one could in principle measure its polarization directly to extract the new physics signal of the dimension-8 operator. However, since the Z decays rapidly into fermion pairs, the contributions of the out-going longitudinal $Z_L$ and transverse $Z_T$ intermediate states interfere in the angular distributions of the fermions produced in $Z_L$ and $Z_T$ decays. Such interference effects appear as the $\phi_*$ angular dependence in Eq.(3.8).

Using the above results, we present numerical results for the normalized angular distribution functions of $\theta, \theta_*$, and $\phi_*$, in Fig. 2, Fig. 3, and Fig. 4, respectively. In each fig-
Figure 4. Normalized angular distribution in the azimuthal angle $\phi_*$ for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ terms, respectively, where the blue and black curves nearly overlap. We use a laboratory polar cut $\delta = 0.2$ for illustration.

It is of interest to examine the behaviours of the angular distribution functions $f^j_\xi$ in the high-energy limit $s \gg M_Z^2$. For all the functions $f^j_\theta$ and $f^j_{\theta_*}$, the coefficients of all trigonometric functions approach constants. This is why Figs. 2 and 3 show that the distributions in $\theta$ and $\theta_*$ are not sensitive to the collision energy $\sqrt{s}$, as we vary the collision energy $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$ in the four plots. For the angular functions $f^0_{\phi_*}$ and $f^2_{\phi_*}$, the coefficients of $\cos \phi_*$ are suppressed by $M_Z/\sqrt{s}$, so they approach the constant term $\frac{1}{2\pi}$ for $s \gg M_Z^2$. This is why in Fig. 4 the angular functions $f^0_{\phi_*}$ and $f^2_{\phi_*}$ (shown as the black and blue curves) appear fairly flat and largely
overlap each other. In contrast, for the angular function \( f^1_{φ^*} \), the coefficient of \( \cos φ^* \) is enhanced by an energy factor \( \sqrt{s}/M_Z \), and can be much larger than the constant term. For \( s \gg M_Z^2 \), we can approximate Eq.(3.8b) in the following form:

\[
 f^1_{φ^*} = \frac{1}{2\pi} \left( \frac{1}{2} + \cos^2 φ^* \right) - \frac{9\pi \sqrt{s}}{128 M_Z} \cos φ^* + O\left(\frac{M_Z^2}{s}, \delta \right)
\]

\(|S_a - S_b|\) is much larger than 1 for the angular function \( f^1_{φ_e} \), while \(|S_a - S_b|\) is subject to a strong cancellation in the SM contribution \( f^0_{φ^*} \). We can make use of this feature to suppress the SM background and enhance significantly the \( O(Λ^{-4}) \) signal at the same time. To this end, we define new functions

\[
 O_j \equiv |σ_j| \left( \int_{\frac{3\pi}{2}}^{\pi} - \int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) f^j_{φ^*} dφ^*, \quad (3.10)
\]

for \( j = 0, 1, 2 \). Furthermore, Fig. 4 shows that the \( O(Λ^{-8}) \) distribution \( f^2_{φ^*} \) is fairly flat and largely overlaps with \( f^0_{φ_e} \) of the SM. Thus, the \( O(Λ^{-8}) \) contributions to \(|S_a - S_b|\) also cancel strongly and become negligible. Hence, for the signal analysis, here we only need to consider the \( O(Λ^{-4}) \) contributions.

We define the signal and background event numbers as follows:

\[
 S = O_1 \times L \times ϵ, \quad (3.11a)
\]
\[ B = \mathcal{O}_0 \times \mathcal{L} \times \epsilon, \quad (3.11b) \]

where \( \mathcal{L} \) denotes the luminosity and \( \epsilon \) is the detection efficiency. With these definitions, we find that the SM background \( B \) is quite small due to the large cancellation between regions (a) and (b), whereas its statistical error \( \Delta_B \) is not so small:

\[ B = N_a - N_b, \quad (3.12a) \]

\[ \Delta_B = \sqrt{\Delta_a^2 + \Delta_b^2} = \sqrt{N_a^0 + N_b^0} = \sqrt{\sigma_0 \times \mathcal{L} \times \epsilon}, \quad (3.12b) \]

where \( N_a^0 \) and \( N_b^0 \) are the SM event numbers in regions (a) and (b), respectively. We estimate the signal significance by

\[ Z_4 = \frac{S}{\Delta_B} = \frac{\mathcal{O}_1(Z\gamma)}{\sqrt{\sigma_0(Z\gamma)}} \times \sqrt{\text{Br}(Z \rightarrow \ell\ell) \times \mathcal{L} \times \epsilon}. \quad (3.13) \]

We note that \( \mathcal{O}_1 \) and \( \sigma_0 \) are functions of the angular cut \( \delta \) (corresponding to the photon transverse momentum cut \( P_T^\gamma > q \sin \delta \) ). Fig. 4 shows that the magnitude \( |f_{\phi_c}^1| \) is very small around \( \phi_c = \frac{\pi}{2}, \frac{3\pi}{2} \) since it is dominated by the \( \cos \phi_c \) term as in Eq.(3.9). So we may cut off the nearby area to reduce the SM backgrounds. For this, we introduce a cut parameter \( 0 < \phi_c < \frac{\pi}{2} \), using which region (a) reduces to \([0, \phi_c] \cup [2\pi - \phi_c, 2\pi]\) and region (b) becomes \([\pi - \phi_c, \pi + \phi_c]\). We then compute the corresponding signal observable \( \mathcal{O}_1^c \) and the background fluctuation \( \sqrt{\sigma_0^c} \). In Fig. 4, the angular function \( f_{\phi_c}^0 \) appears rather flat, so we obtain a simple expression for \( \sigma_0^c \), as follows:

\[ \mathcal{O}_1^c = |\sigma_1| \left( \int_{\pi - \phi_c}^{\pi + \phi_c} - \int_0^{\phi_c} - \int_{\pi - \phi_c}^{2\pi} \right) f_{\phi_c}^1 d\phi_c \]

\[ \simeq \frac{3\alpha(1 - 4s_W^2)M_Z(s - M_Z^2)[3(\pi - 2\delta)(s - 3M_Z^2) \sin 2\delta] \sin \phi_c}{256 s_Wc_W \Lambda^4 s_W^2}, \quad (3.14a) \]

\[ \sigma_0^c \simeq \frac{2\phi_c^0}{\pi} \sigma_0 \]

\[ = \frac{\alpha^2(1 - 4s_W^2 + 8s_W^4)[- \cos \delta(s - M_Z^2)^2 + 2(s^2 + M_Z^2)\ln(\cot \frac{\delta}{2})]}{c_W^2 s_W^2(s - M_Z^2)s^2} \phi_c \quad (3.14b) \]

where \( \alpha = e^2/4\pi \) is the fine structure constant.

For our analysis, we choose the values of \( \phi_c = \phi_c^m \) and \( \delta = \delta_m \) such that the signal significance \( Z_4 = (\mathcal{O}_1^1/\sqrt{\sigma_0^c}) \sqrt{\text{Br}(Z \rightarrow \ell\ell) \times \mathcal{L} \times \epsilon} \) is maximized. Thus, \( \phi_c = \phi_c^m \) corresponds to the maximum of the function \( \sin \phi_c/\sqrt{\phi_c} \), and we derive \( \phi_c^m \approx 1.17 \), which is independent of the collision energy \( \sqrt{s} \). The value of \( \delta_m \) required to obtain the maximal significance of \( Z_4 \propto (\mathcal{O}_1^1/\sqrt{\sigma_0^c}) \) depends on the collision energy \( \sqrt{s} \) : at high energies \( s \gg M_Z^2 \), we find that \( \delta_m \approx 0.329 \).

We present in Fig. 5 the signal significance obtained in this way for the collision energy \( \sqrt{s} = 3 \text{ TeV} \). We input the total leptonic branching fraction \( \text{Br}(Z \rightarrow \ell^- \ell^+) \approx
Figure 5. Analysis of the significance $Z_4 = S/\Delta B$. Plot (a) depicts $Z_4$ as a function of $\delta$. Plot (b) presents $Z_4$ for $\delta = \delta_m$ as a function of the new physics scale $\Lambda$. For illustration, we choose the collision energy $\sqrt{s} = 3$ TeV and the integrated luminosity $L = 2$ ab$^{-1}$.

0.10, and assume an integrated luminosity $L = 2$ ab$^{-1}$ and an ideal detection efficiency $\epsilon = 1$ for simplicity. In Fig. 5(a), we depict the significance $Z_4$ as a function of the angular cut $\delta$, which exhibits the maximum at $\delta_m \simeq 0.33$ for $\Lambda = 1$ TeV, as expected. Thus, under the angular cuts $(\phi_c, \delta) = (\phi^m_c, \delta_m)$, we derive

$$(\sigma^c_0, O^1) \simeq \left(23.1, 11.1 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{fb}. \quad (3.15)$$

For illustration, we may then use Eq.(3.13) to estimate the signal significance as follows:

$$Z_4 \simeq 32.7 \left(\frac{\text{TeV}}{\Lambda}\right)^4 \simeq 5.0 \left(\frac{1.60 \text{ TeV}}{\Lambda}\right)^4, \quad (3.16)$$

which is plotted in Fig. 5(b). From this, we find that the probe of the new physics scale can reach $\Lambda = (2.0, 1.8, 1.6)$ TeV at $(2\sigma, 3\sigma, 5\sigma)$ level, respectively.

We note that the practical detection efficiency would be smaller than 100%, so the actual sensitivity may be somewhat weaker. But, as we show later in Eqs.(3.25) and (3.29) of Sec. 3.3, the sensitivity reach for $\Lambda$ has rather weak dependences on the integrated luminosity and detection efficiency, namely, $\Lambda \propto (L \epsilon)^{1/8}$ at $\mathcal{O}(\Lambda^{-4})$ and $\Lambda \propto (L \epsilon)^{1/16}$ at $\mathcal{O}(\Lambda^{-8})$. Hence, increasing $L$ or $\epsilon$ only has minor effect on the sensitivity reach of the new physics scale $\Lambda$. In contrast, raising the collision energy $\sqrt{s}$ can do more to improve the sensitivity reach of $\Lambda$ because $\Lambda \propto (\sqrt{s})^{1/2}$ at $\mathcal{O}(\Lambda^{-4})$ and $\Lambda \propto (\sqrt{s})^{5/8}$ at $\mathcal{O}(\Lambda^{-8})$.

3.2.2. Analysis Including the $\mathcal{O}(\Lambda^{-8})$ Contribution

In this subsection, we include in the analysis the contribution of $\mathcal{O}(\Lambda^{-8})$. Since the $\mathcal{O}(\Lambda^{-8})$ term has a higher power of energy dependence, it may have better sensitivity
when the collision energy is higher, e.g., $\sqrt{s} = 3$ TeV, even though it is suppressed by $\Lambda^{-8}$.

We see from Fig. 4 that both the distributions $f_{\phi^*}^0$ and $f_{\phi^*}^2$ are rather flat, and thus insensitive to the $O(\Lambda^{-8})$ contribution. Hence, in order to enhance the signal sensitivity to the $O(\Lambda^{-8})$ contribution, we study instead the distributions in $\theta$ and $\theta_*$. For this, we choose the region $\theta \in [\delta, \pi - \delta]$ and $\theta_* \in [\delta_*, \pi - \delta_*]$. With the angular cuts $(\delta, \delta_*)$, we compute the SM contribution $\sigma_c^0(Z\gamma)$, the $O(\Lambda^{-4})$ contribution $\sigma_c^1(Z\gamma)$, and the $O(\Lambda^{-8})$ contribution $\sigma_c^2(Z\gamma)$ as follows:

\[
\sigma_c^0(Z\gamma) = \frac{e^4(8s_W^4 - 4s_W^2 + 1)}{32\pi s_W^2 c_W^2 (s - M_Z^2) s^2} \times \frac{1}{16} [4\cos \delta (9 \cos \delta_* - \cos 3\delta_*) M_Z^2 s - (15 \cos \delta_* + \cos 3\delta_*) (s^2 + M_Z^2) (\cos \delta + 2 \ln \tan \delta)] , \tag{3.17a}
\]

\[
\sigma_c^1(Z\gamma) = \pm \frac{e^2(1 - 4s_W^2) M_Z^2 (s - M_Z^2)}{32\pi s_W c_W \Lambda s^2} \left[ 2(5 - \cos 2\delta_*) s + (\cos 2\delta_* + 7) M_Z^2 \right] \cos \delta \cos \delta_* , \tag{3.17b}
\]

\[
\sigma_c^2(Z\gamma) = \frac{(8s_W^4 - 4s_W^2 + 1) M_Z^2 (s - M_Z^2)^3}{192\pi \Lambda s^2} \times \frac{\cos \delta}{64} \left[ (7 + \cos 2\delta)(9 \cos \delta_* - \cos 3\delta_*) s + (5 - \cos 2\delta)(15 \cos \delta_* + \cos 3\delta_*) M_Z^2 \right] , \tag{3.17c}
\]

where the $\pm$ signs of the $O(\Lambda^{-4})$ term correspond to the two possible signs of a given dimension-8 operator, $\text{sign}(c_j) = \pm$, as shown in Eq.(2.1). If we take the limit $(\delta, \delta_*) \rightarrow 0$ in the above formulas, we find that they reduce consistently to Eq.(3.3), as expected.

We can then estimate the corresponding signal significance to be

\[
Z_8 = \frac{S}{\Delta_B} = \frac{|\sigma_c^1(Z\gamma) + \sigma_c^2(Z\gamma)|}{\sqrt{\sigma_c^0(Z\gamma)}} \times \sqrt{\text{Br}(Z \rightarrow \ell\ell) \times \mathcal{L} \times \epsilon} . \tag{3.18}
\]

For the collision energy $\sqrt{s} = 3$ TeV, we find that the $O(\Lambda^{-8})$ term dominates. To obtain the maximal signal significance, we derive the corresponding values of the angular cuts $(\delta, \delta_*) = (\delta_m, \delta_{sm})$, which are $(\delta_m, \delta_{sm}) \simeq (0.623, 0.820)$. Inputting $\text{Br}(Z \rightarrow \ell\ell) \simeq 0.10$ and choosing $\sqrt{s} = 3$ TeV, $\mathcal{L} = 2 ab^{-1}$ and $\epsilon = 1$, we compute the cross section for $Z\gamma$ production:

\[
\sigma(Z\gamma) = \left[ 10.1 \pm 0.0251 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.554 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb} . \tag{3.19}
\]

Thus, from Eq.(3.18) we estimate the signal significance

\[
Z_8 \simeq \left| \pm 1.79 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 631 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right| \simeq \left| \pm 0.112 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 2.46 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right| . \tag{3.20}
\]

\footnote{We note that the interference of higher-dimensional operators with the SM may appear at the same order for high energy scales. However, such interference terms could in general be distinguished by different angular dependences.}
Finally, we may combine $Z_4$ and $Z_8$ to achieve a better sensitivity reach to the new physics scale $\Lambda$:

$$Z = \sqrt{Z_4^2 + Z_8^2},$$

which is depicted by the red curve in Fig. 6. In this way, we find that the new physics scale can be probed up to $\Lambda \simeq (2.2, 2.0, 1.9)$ TeV at the $(2\sigma, 3\sigma, 5\sigma)$ levels, respectively. These numbers apply for both $\pm$ signs in Eq. (3.20), since we find that the case of minus sign in Eq. (3.20) only causes a tiny difference in the $\Lambda$ bound by less than 1%. Hence, the $O(\Lambda^{-4})$ term in Eq. (3.20) has negligible effect for the collider energy $\sqrt{s} = 3$ TeV. As we will show in Sec. 3.3, this feature applies to all cases with $\sqrt{s} \gtrsim 1$ TeV.

### 3.3. Analysis of Different Collision Energies

In this subsection, we further extend our analysis of $\sqrt{s} = 3$ TeV case to different collision energies $\sqrt{s} = (250, 500, 1000, 5000)$ GeV, in each case with a sample integrated luminosity $L = 2$ ab$^{-1}$.

Using the same method as we presented in Sec. 3.2.1-3.2.2, we analyze the SM backgrounds and signal contributions for different collider energies. For each given collider energy $\sqrt{s}$, we derive the optimal angular cuts for realizing the maximal signal significance $Z_4$ and $Z_8$. Namely, for the analysis of $Z_4$, we use the angular cuts $(\delta_m, \phi^m_c)$ for angles $(\theta, \phi^c_c)$; while for the analysis of $Z_8$, we use the angular cuts $(\delta_m, \delta^*_m)$ for angles $(\theta, \theta^*_c)$. We summarize the optimal angular cuts for different collider energies in Table 1. As we noted below Eq. (3.14), the dominant contribution to the signal significance $Z_4$ depends on the cut $\phi^c_c$ only through a simple function $\sin \phi^c_c/\sqrt{\phi^c_c}$,
Table 1. Summary of the optimal angular cuts for realizing the maximal signal significance. For the signal significance $Z_4$, we impose the cuts $(\delta_m, \phi^m_c)$ on the angular distributions of $(\theta, \phi^c_\star)$ whereas, for the signal significance $Z_8$, we set the cuts $(\delta_m, \delta^*_m)$ on the angular distributions of $(\theta, \theta^*_c)$.

which does not depend on energy $\sqrt{s}$ and reaches its maximum at $\phi^m_c \simeq 1.17$. This is why the optimal cut $\phi^m_c$ is nearly independent of the collider energy $\sqrt{s}$, as shown in Table 1.

With the optimal kinematical cuts in Table 1, we compute the SM contributions and the $O(\Lambda^{-4})$ contributions for different collider energies,

$$\sqrt{s} = 250 \text{ GeV}, \quad (\sigma^c_0, \Theta^c_\ell) = \left(3936, 0.913 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.22a)$$

$$\sqrt{s} = 500 \text{ GeV}, \quad (\sigma^c_0, \Theta^c_\ell) = \left(860, 1.85 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.22b)$$

$$\sqrt{s} = 1 \text{ TeV}, \quad (\sigma^c_0, \Theta^c_\ell) = \left(209, 3.71 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.22c)$$

$$\sqrt{s} = 3 \text{ TeV}, \quad (\sigma^c_0, \Theta^c_\ell) = \left(23.1, 11.1 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.22d)$$

$$\sqrt{s} = 5 \text{ TeV}, \quad (\sigma^c_0, \Theta^c_\ell) = \left(8.30, 18.5 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.22e)$$

where we include the case of $\sqrt{s} = 3 \text{ TeV}$ from Eq.(3.15) for comparison.

With these, we derive the following signal significances at each collision energy, for the leptonic branching fraction $\text{Br}(Z \to \ell\ell) \simeq 0.10$ and an integrated luminosity $L = 2 \text{ ab}^{-1}$,

$$\sqrt{s} = 250 \text{ GeV}, \quad Z_4 \simeq 3.29 \left(\frac{0.5\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.23a)$$

$$\sqrt{s} = 500 \text{ GeV}, \quad Z_4 \simeq 2.18 \left(\frac{0.8\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.23b)$$

$$\sqrt{s} = 1 \text{ TeV}, \quad Z_4 \simeq 3.62 \left(\frac{\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.23c)$$

$$\sqrt{s} = 3 \text{ TeV}, \quad Z_4 \simeq 2.05 \left(\frac{2\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.23d)$$

$$\sqrt{s} = 5 \text{ TeV}, \quad Z_4 \simeq 2.33 \left(\frac{2.5\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}. \quad (3.23e)$$
We note that the signal significance is nearly proportional to the squared centre-of-mass collision energy \((\sqrt{s})^2\). At high energies \(s \gg M_Z^2\), we have
\[
Z_4 \propto \frac{M_Z s}{\Lambda^4} \sqrt{\mathcal{L}} \times \epsilon .
\] (3.24)

Thus, for a given significance \(Z_4\), the corresponding reach of the new physics scale \(\Lambda\) is
\[
\Lambda \propto \left(\frac{M_Z \sqrt{\mathcal{L}} \times \epsilon}{Z_4}\right)^{\frac{1}{4}} \times (\sqrt{s})^{\frac{1}{2}} .
\] (3.25)

We see that the collision energy \(\sqrt{s}\) has the most sensitive effect on the reach of the new physics scale \(\Lambda\). For instance, raising the collision energy from \(\sqrt{s} = 250\) GeV to \(\sqrt{s} = 3\) TeV, the reach of the new physics scale is improved by a significant factor \(\Lambda(3\) TeV)/\(\Lambda(250\) GeV) \(\simeq 3.46\). On the other hand, \(\Lambda\) has a rather weak dependence on the significance, \(\Lambda \propto Z_4^{-\frac{1}{4}}\), so the 5\(\sigma\) reach is only slightly weaker than the 2\(\sigma\) reach: \(\Lambda(5\sigma)/\Lambda(2\sigma) \simeq 1/1.26\). Furthermore, we note that \(\Lambda\) depends much more weakly on the integrated luminosity and the detection efficiency, \(\Lambda \propto (\mathcal{L} \times \epsilon)^{\frac{1}{8}}\). For instance, increasing the integrated luminosity from \(\mathcal{L} = 2\) ab\(^{-1}\) to \(\mathcal{L} = 6\) ab\(^{-1}\), would enhance the reach of the new physics scale by only a factor of \(\Lambda(6\) ab\(^{-1}\))/\(\Lambda(2\) ab\(^{-1}\)) \(\simeq 1.15\). Also, if the detection efficiency is increased from \(\epsilon = 40\%\) to \(\epsilon = 90\%\), the reach of the new physics scale would be slightly extended by a factor of \(\Lambda(90\%)/\Lambda(40\%) \simeq 1.11\).

Next, extending Section 3.2.2 to different collision energies, we include contributions up to \(\mathcal{O}(\Lambda^{-8})\) in a similar manner. We apply the optimal kinematical cuts as in Table 1 and compute the cross sections of \(e^+e^-\rightarrow Z\gamma\) as follows:

\[
\sqrt{s} = 250\text{ GeV}, \quad \sigma(Z\gamma) = \left[2427 \pm 6.62\left(\frac{0.5\text{ TeV}}{\Lambda}\right)^4 + 1.39\left(\frac{0.5\text{ TeV}}{\Lambda}\right)^8\right]\text{ fb}, \quad (3.26a)
\]

\[
\sqrt{s} = 500\text{ GeV}, \quad \sigma(Z\gamma) = \left[417 \pm 0.996\left(\frac{0.8\text{ TeV}}{\Lambda}\right)^4 + 0.624\left(\frac{0.8\text{ TeV}}{\Lambda}\right)^8\right]\text{ fb}, \quad (3.26b)
\]

\[
\sqrt{s} = 1\text{ TeV}, \quad \sigma(Z\gamma) = \left[94.0 \pm 0.404\left(\frac{\text{ TeV}}{\Lambda}\right)^4 + 1.73\left(\frac{\text{ TeV}}{\Lambda}\right)^8\right]\text{ fb}, \quad (3.26c)
\]

\[
\sqrt{s} = 3\text{ TeV}, \quad \sigma(Z\gamma) = \left[10.1 \pm 0.0252\left(\frac{2\text{ TeV}}{\Lambda}\right)^4 + 0.554\left(\frac{2\text{ TeV}}{\Lambda}\right)^8\right]\text{ fb}, \quad (3.26d)
\]

\[
\sqrt{s} = 5\text{ TeV}, \quad \sigma(Z\gamma) = \left[3.63 \pm 0.0103\left(\frac{2.5\text{ TeV}}{\Lambda}\right)^4 + 0.718\left(\frac{2.5\text{ TeV}}{\Lambda}\right)^8\right]\text{ fb}, \quad (3.26e)
\]

where for comparison we have also included the result from Eq.(3.19) for the case of \(\sqrt{s} = 3\) TeV. The above can be compared to the cross sections (3.3) with only a preliminary angular cut \(\delta > 0.2\) (corresponding to a lower cut on the photon transverse momentum \(P_T^\gamma = q \sin \delta\)). We see that under the final angular cuts on \((\theta, \theta_*, \phi_*)\) the
SM contribution is substantially reduced in each case, whereas the signal contributions at $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ are little changed.

Using the above, we analyze the signal significance up to $\mathcal{O}(\Lambda^{-8})$, for different collider energies. With inputs of the leptonic branching fraction $\text{Br}(Z\to\ell\ell)\approx 0.10$ and an integrated luminosity $\mathcal{L} = 2\text{ ab}^{-1}$, we arrive at

\[
\sqrt{s} = 250\text{ GeV}, \quad Z_8 = \left| \pm 1.90 \left( \frac{0.5\text{ TeV}}{\Lambda} \right)^4 + 0.400 \left( \frac{0.5\text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon},
\]

(3.27a)

\[
\sqrt{s} = 500\text{ GeV}, \quad Z_8 = \left| \pm 0.689 \left( \frac{0.8\text{ TeV}}{\Lambda} \right)^4 + 0.432 \left( \frac{0.8\text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon},
\]

(3.27b)

\[
\sqrt{s} = 1\text{ TeV}, \quad Z_8 = \left| \pm 0.589 \left( \frac{\text{ TeV}}{\Lambda} \right)^4 + 2.53 \left( \frac{\text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon},
\]

(3.27c)

\[
\sqrt{s} = 3\text{ TeV}, \quad Z_8 = \left| \pm 0.112 \left( \frac{2\text{ TeV}}{\Lambda} \right)^4 + 2.46 \left( \frac{2\text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon},
\]

(3.27d)

\[
\sqrt{s} = 5\text{ TeV}, \quad Z_8 = \left| \pm 0.0764 \left( \frac{2.5\text{ TeV}}{\Lambda} \right)^4 + 5.32 \left( \frac{2.5\text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}.
\]

(3.27e)

From the above, we note that for the relevant reach of $\Lambda$, the $\mathcal{O}(\Lambda^{-4})$ terms give the dominant contributions for collision energies $\sqrt{s} < 1\text{ TeV}$, while the $\mathcal{O}(\Lambda^{-8})$ terms become dominant for $\sqrt{s} \gtrsim 1\text{ TeV}$. When $\mathcal{O}(\Lambda^{-8})$ becomes dominant at high energies, we have $\Lambda \propto Z_8^{-\frac{1}{8}}$. In such cases, the reach in $\Lambda$ becomes rather insensitive to the significance $Z_8$. For instance, at high energies $\sqrt{s} \gtrsim 1\text{ TeV}$, we have $\Lambda(5\sigma)/\Lambda(2\sigma) \simeq 1/1.12$ for $Z_8$, whereas we previously found $\Lambda(5\sigma)/\Lambda(2\sigma) \simeq 1/1.26$ for $Z_4$.

At high energies $s \gtrsim (1\text{ TeV})^2 \gg M_Z^2$, the $\mathcal{O}(\Lambda^{-8})$ terms become dominant, so we have the approximate relation

\[
Z_8 \propto \frac{M_Z^2(\sqrt{s})^5}{\Lambda^8} \sqrt{\mathcal{L} \times \epsilon},
\]

(3.28)

and hence

\[
\Lambda \propto \left( \frac{M_Z^2 \sqrt{\mathcal{L} \epsilon}}{Z_8} \right)^{\frac{1}{2}} (\sqrt{s})^{\frac{5}{8}}.
\]

(3.29)

We see from Eqs. (3.25) and (3.29) that the sensitivity to $\Lambda$ increases with the collision energy with the power $(\sqrt{s})^{\frac{5}{8}}$ or $(\sqrt{s})^{\frac{5}{2}}$, a relatively slow rate of increase. We note also that the sensitivity to the new physics scale $\Lambda$ is rather insensitive to the integrated luminosity $\mathcal{L}$ and the detection efficiency $\epsilon$, owing to their small power-law dependence $(\mathcal{L} \epsilon)^{\frac{1}{16}}$.

Finally, we compute from Eqs. (3.23) and (3.27), the combined significance, $Z = \sqrt{Z_4^2 + Z_8^2}$, for each collider energy. With these, in Table 2 we present the corresponding combined sensitivity reaches to the new physics scale $\Lambda$ at different $e^+e^-$ collider energies. In the last row of this Table, the two numbers in the parentheses correspond
to the case of the dimension-8 operator whose coefficient has a minus sign, whereas in all other entries the effects due to the coefficient having a minus sign are negligible. In Fig. 7, we further present the reaches of the new physics scale Λ as functions of the $e^+e^-$ collision energy $\sqrt{s}$. In Fig. 7(a) we show the Λ reaches for the signal significances $(Z_4, Z_8)$ at 2σ and 5σ levels, respectively. Then, in Fig. 7(b), we depict the combined sensitivity $Z = \sqrt{Z_4 + Z_8} = (2, 3, 5)\sigma$, by the red, green and blue curves. For reference, we also show two lines $\Lambda = \sqrt{s}$ and $\Lambda = \sqrt{s}/2$ in each plot.

Before concluding this Section, we mention that we have performed a numerical Monte Carlo simulation based on the analytical formula (3.14). We used for this purpose CUDAlink in Mathematica, so as to exploit the CUDA parallel computing architecture on Graphical Processing Units (GPUs), which can generate millions of events in seconds. We have computed the probability density function of $\theta$, $\theta^*$ and $\phi^*$ for the case of $\sqrt{s} = 3$ TeV and $\Lambda = 2$ TeV. Eq.(3.3d) shows that the SM contribution dominates the total cross section. According to Eq.(3.15), we have $O_c^c/\sigma^c_0 \simeq 0.03$. For comparison, our Monte Carlo simulation yielded the following event counts: $|N_a - N_b| = 3054$.

---

3In the effective theory approach, the exact relation between the cutoff scale Λ and the mass $M$ of the lowest underlying new state $X$ is unknown, and one expects $M/\Lambda = O(1)$. If the new state $X$ could only be produced in pairs in the $e^+e^-$ collisions (e.g., the production of dark matter particles), requiring $M > \sqrt{s}/2$ will suffice to allow applications of the current effective theory with new physics cutoff $\Lambda = M/O(1)$.
Table 2. Combined sensitivity reaches to the new physics scale \( \Lambda \) at the 2\( \sigma \) and 5\( \sigma \) levels, for different collider energies. Here the two numbers in the parentheses correspond to the case of the dimension-8 operator whose coefficient has a minus sign, while in all other entries the effects due to the coefficient having a minus sign are negligible. For illustration, we have input a fixed representative integrated luminosity \( \mathcal{L} = 2 \text{ab}^{-1} \) and an ideal detection efficiency \( \epsilon = 100\% \).

<table>
<thead>
<tr>
<th>( \sqrt{s} ) (GeV)</th>
<th>250</th>
<th>500</th>
<th>1000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda_{2\sigma} ) (TeV)</td>
<td>0.59</td>
<td>0.84</td>
<td>1.2</td>
<td>2.2</td>
<td>2.9</td>
</tr>
<tr>
<td>( \Lambda_{5\sigma} ) (TeV)</td>
<td>0.48(0.46)</td>
<td>0.68(0.65)</td>
<td>1.0</td>
<td>1.9</td>
<td>2.6</td>
</tr>
</tbody>
</table>

and \( N_a + N_b = 104752 \), corresponding to \( |N_a - N_b|/(N_a + N_b) \approx 0.029 \). This agrees well with the ratio \( \mathcal{O}_1/\sigma_0 \approx 0.03 \) inferred from our analytical formula (3.14), serving as a consistency check on our Eq.(3.15). Our Monte Carlo simulation package may be used to generate other distributions and quantities that may be of interest for experiments.\(^4\)

4. Conclusions

As we have discussed in this paper, the process \( e^+e^\rightarrow Z\gamma \) provides a rare opportunity to probe an effective dimension-8 operator in the SMEFT. The \( ZV\gamma \) vertices (\( V = Z, \gamma \)) have no tree-level SM contributions, and nor do they receive any contributions from dimension-6 operators, opening up the possibility of probing the new physics scale associated with one particular dimension-8 operator. We have used a general analysis of the angular distributions for \( Z\gamma \) production in the laboratory frame and for \( Z \rightarrow \ell^+\ell^- \) in the \( Z \) rest frame to identify particular angular distributions and cuts that maximize the statistical sensitivity to the possible new physics scale \( \Lambda \), either including only the \( \mathcal{O}(\Lambda^{-4}) \) contributions that interfere with the SM contributions, or including together the \( \mathcal{O}(\Lambda^{-8}) \) contributions of the dimension-8 operator.

As seen in Fig. 7(b) and Table 2, the prospective sensitivities to \( \Lambda \) extend into the multi-TeV range. As one would expect from the energy dependences of the dimension-8 contributions to the cross section for \( e^+e^-\rightarrow Z\gamma \), the prospective sensitivities increase with the collision energies. However, since we assume a constant integrated luminosity, the sensitivities increase more slowly than \( \sqrt{s} \). The sensitivities at the 2\( \sigma \), 3\( \sigma \) and 5\( \sigma \) levels of significances are not greatly different, as discussed in the text and seen by comparing the red, green and blue curves in Fig. 7(b).

We have also drawn in Fig. 7 the two reference lines \( \Lambda = \sqrt{s} \) and \( \Lambda = \sqrt{s}/2 \). In general, one would expect the SMEFT approach to be suitable only for energy scales that are small compared to \( \Lambda \). However, the way that we have defined \( \Lambda \) in Eq.(2.1) of this paper corresponds to the true new physics scale \( \tilde{\Lambda} \) only if the unknown coefficient

\(^4\)Further details may be obtained from RQX.
has a magnitude of unity. If, on the other hand, the true magnitude of \( c_j \gg 1 \), the true new physics scale \( \tilde{\Lambda} \) could be much larger than the value of \( \Lambda \) extracted from our analysis, and the SMEFT approach would have broader applicability.

It is interesting to compare the sensitivity to the dimension-8 coefficient found here with that found previously in studies of the dimension-8 operator contributions to light-by-light scattering and the process \( gg \rightarrow \gamma \gamma \) at the LHC. The former is sensitive to a dimension-8 scale that is \( \mathcal{O}(100) \text{ GeV} \) [9], whereas the latter is sensitive to a dimension-8 scale that is \( \mathcal{O}(1) \text{ TeV} \) [11]. The dimension-8 operators studied in those analyses are fully different from what we studied here, and hence probe different aspects of dimension-8 physics. But, in this work we found it encouraging that future \( e^+e^- \) colliders (such as the ILC, CEPC, FCC-ee, and CLIC) may be able to provide very competitive sensitivities to probing the scale of new physics. We therefore encourage further detailed studies of the \( e^+e^- \rightarrow Z\gamma \) process by our experimental colleagues.

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Appendix

A. Helicity Amplitudes for \( Z\gamma \) Production with \( Z \) Decays

In this Appendix we present the helicity amplitudes for the production process \( e^-e^+ \rightarrow Z(q_1, \epsilon_\lambda)\gamma(q_2, \epsilon_\lambda') \) and then we include leptonic \( Z \) decays. These results are used in the analyses of Sections 2 and 3 in the main text.

A.1. Helicity Amplitudes for \( Z\gamma \) Production

The helicity amplitudes for \( e^- (p_1) e^+ (p_2) \rightarrow Z(q_1, \epsilon_\lambda)\gamma(q_2, \epsilon_\lambda') \) can be written as

\[
T_{\lambda\lambda'}^{ss'} = \bar{s}'(p_2) \left[ \frac{e^2}{s_W c_W} \left( f_{\lambda'}^*(q_2)(q_1 - p_1) f_{\lambda}^*(q_1) + f_{\lambda}^*(q_1)(q_2 - p_1) f_{\lambda'}^*(q_2) \right) \right]
\]

\[
- \frac{i2 M_Z^2}{\Lambda^4} \epsilon^{\mu\nu\alpha\beta} \gamma_\mu \epsilon^*_{\lambda\nu}(q_1) c^*_{\lambda',\alpha}(q_2) q_{2\beta} \left( c_L P_L + c_R P_R \right) u^s(p_1),
\]

(A.1)
where we have used the standard spinor notations $u^s(p_1)$ and $\bar{u}^{s'}(p_2)$ [22] for the initial state of $e^-$ and $e^+$, and $(\epsilon_L, \epsilon_R)$ denote the polarization vectors of the final state gauge bosons ($Z, \gamma$). In the above, $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are the chirality projection operators, and the coefficients $(c_L, c_R) = (s_W^2 - \frac{1}{2}, s_W^2)$ arise from the (left-, right)-handed gauge couplings of electrons to the $Z$ boson. In the above, we have used the Mandelstam variables $t = (p_1 - q_1)^2 = -\frac{1}{2}(s-M_Z^2)(1-\cos\theta)$ and $u = (p_1 - q_2)^2 = -\frac{1}{2}(s-M_Z^2)(1+\cos\theta)$.

We then express the three polarization vectors $\epsilon_\lambda(q_1, \theta)$ of the $Z$ boson as follows:

\[
\epsilon^Z_\pm (\theta) = \frac{1}{\sqrt{2}} (0, \pm \cos \theta, -i, \pm i \sin \theta), \tag{A.2a}
\]

\[
\epsilon^Z_0 (q_1, \theta) = \frac{1}{M_Z} (q_1, E_Z \sin \theta, 0, E_Z \cos \theta), \tag{A.2b}
\]

where $E_Z = \sqrt{q_1^2 + M_Z^2}$. The final-state photon has two transverse polarization vectors that are similar to those of $Z$ boson,

\[
\epsilon^\pm_\lambda (\theta) = \epsilon^Z_\pm (\theta + \pi) = \epsilon^Z_\pm (\theta). \tag{A.3}
\]

The first two terms in Eq.(A.1) arise from the SM contributions via the $t$- and $u$-channel exchanges, respectively, while the third term is contributed by the dimension-8 operator. For the final-state $Z(\lambda)\gamma(\lambda')$ helicity combinations $\lambda\lambda' = (---, --+, +++, +++)$ and $\lambda\lambda' = (0-, 0+)$, we compute the SM contributions to the scattering amplitudes as follows:

\[
\mathcal{T}^{ss',T}_{\text{sm}}(---+) = \frac{2e^2}{s_W c_W (s-M_Z^2)} \begin{pmatrix} (e_L \cot \frac{\theta}{2} - e_R \tan \frac{\theta}{2}) M_Z^2 & -e_L \cot \frac{\theta}{2} + e_R \tan \frac{\theta}{2} s \\ (e_L \tan \frac{\theta}{2} - e_R \cot \frac{\theta}{2}) s & -e_L \tan \frac{\theta}{2} + e_R \cot \frac{\theta}{2} M_Z^2 \end{pmatrix}, \tag{A.4a}
\]

\[
\mathcal{T}^{ss',L}_{\text{sm}}(0-, 0+) = \frac{2\sqrt{2}(e_L + e_R)e^2 M_Z}{s_W c_W (s-M_Z^2)} \sqrt{\frac{\Lambda^2}{1-1}}, \tag{A.4b}
\]

where $(e_L, e_R) = (c_L \delta_{s,-\frac{1}{2}}, c_R \delta_{s,\frac{1}{2}})$ with the subscript index $s = \pm \frac{1}{2}$ denoting the initial-state electron helicities. We note that in Eq.(A.4a), the identical-helicity amplitudes $\mathcal{T}^{ss',T}_{\text{sm}}(\pm \pm)$ in the diagonal entries are proportional to the mass-factor $M_Z^2$ (unlike the off-diagonal entries $\propto s$). This is expected because in the massless limit $M_Z \to 0$ the identical-helicity amplitudes should vanish exactly after ignoring the tiny electron mass (as in the pair-annihilation process $e^- e^+ \to \gamma \gamma$ in QED [22]). Hence the nonzero amplitudes $\mathcal{T}^{ss',T}_{\text{sm}}(\pm \pm) \propto M_Z^2$.

Next, we compute the corresponding helicity amplitudes from the new physics contributions of the dimension-8 operator as follows:

\[
\mathcal{T}^{ss',T}_{(8)}(---+) = \frac{(e_L + e_R) \sin \theta M_Z^2 (s-M_Z^2)}{\Lambda^4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.5a}
\]

\[
\mathcal{T}^{ss',L}_{(8)}(0-, 0+) = \frac{\sqrt{2} M_Z (s-M_Z^2)/\sqrt{s}}{\Lambda^4} \begin{pmatrix} e_L \sin^2 \frac{\theta}{2} - e_R \cos^2 \frac{\theta}{2}, & e_R \sin^2 \frac{\theta}{2} - e_L \cos^2 \frac{\theta}{2} \end{pmatrix}. \tag{A.5b}
\]

We note that in Eq.(A.5a) the off-diagonal amplitudes $\mathcal{T}^{ss',T}_{(8)}(---)$ and $\mathcal{T}^{ss',T}_{(8)}(+++) \propto M_Z^2$ vanish exactly. There is a nontrivial reason behind this, which can be understood by
noting that the $Z\gamma Z^*$ vertex [cf. Eqs.(2.3) and (A.1)] contains the rank-4 antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$, which contracts with the $Z$ and $\gamma$ polarization vectors $\epsilon^Z_{\lambda,\nu}(\theta)$ and $\epsilon^{\gamma}_{\lambda,\alpha}(\theta)$ as in Eq.(A.1) below. For the off-diagonal $Z\gamma$ helicity combinations $\lambda\lambda' = +-,--$, we deduce

$$e^{\mu\nu\alpha\beta}\gamma_{\mu}\epsilon^Z_{\alpha\nu}(\theta)\epsilon^{\gamma}_{\beta\alpha}(\theta) = e^{\mu\nu\alpha\beta}\gamma_{\mu}Z^*_\alpha(\theta) = 0,$$

according to Eq.(A.3).

### A.2. Helicity Amplitudes Including Z Decays

In this Appendix, we further incorporate the leptonic decays of $Z$ boson. We first consider the $Z$ decays in its rest frame, $Z \to \ell^-(k_1)\ell^+(k_2)$, where the final-state leptons have momenta:

$$k_1 = k(1, \sin\theta_* \cos\phi_*, \sin\theta_* \sin\phi_*, \cos\theta_*),$$

$$k_2 = k(1, -\sin\theta_* \cos\phi_*, -\sin\theta_* \sin\phi_*, -\cos\theta_*),$$

where the leptons are nearly massless and $k = |\vec{k}| \simeq \frac{1}{2} M_Z$. In the $Z$ rest frame, the massless lepton spinors are defined as follows,

$$u_+(k_1) = \sqrt{2k}\left(0, 0, e^{-\frac{i\phi_*}{2}} \cos\frac{\theta_*}{2}, e^\frac{i\phi_*}{2} \sin\frac{\theta_*}{2}\right),$$

$$u_-(k_1) = \sqrt{2k}\left(0, 0, e^{-\frac{i\phi_*}{2}} \sin\frac{\theta_*}{2}, e^\frac{i\phi_*}{2} \cos\frac{\theta_*}{2}, 0, 0\right),$$

$$v_+(k_2) = \sqrt{2k}\left(e^{-\frac{i\phi_*}{2}} \cos\frac{\theta_*}{2}, e^\frac{i\phi_*}{2} \sin\frac{\theta_*}{2}, 0, 0\right),$$

$$v_-(k_2) = \sqrt{2k}\left(0, 0, e^{-\frac{i\phi_*}{2}} \sin\frac{\theta_*}{2}, -e^\frac{i\phi_*}{2} \cos\frac{\theta_*}{2}\right),$$

where $u_+$ ($u_-$) correspond to spin-up (-down) and $v_+$ ($v_-$) correspond to spin-down (-up) along their moving directions in Eq.(A.7).

Then, we write down the left-handed and right-handed spinor currents in the $Z$ boson rest frame,

$$\tilde{C}_L^\mu = \bar{v}_L\gamma^\mu u_L = M_Z(0, -\cos\theta_* \cos\phi_- i \sin\phi_*, -\cos\theta_* \sin\phi_+ \cos\phi_*, \sin\theta_*),$$

$$\tilde{C}_R^\mu = \bar{v}_R\gamma^\mu u_R = M_Z(0, -\cos\theta_* \cos\phi_+ i \sin\phi_*, -\cos\theta_* \sin\phi_- \cos\phi_*, \sin\theta_*),$$

where $(u_L, u_R) = (u_-, u_+)$ and $(v_L, v_R) = (v_+, v_-)$. After making a Lorentz boost $\hat{L}$ back to the lab frame (c.m. frame of $Z\gamma$) and rotating the axis $z'$ back to the axis $z$ by the rotation $\tilde{R}$, we have new currents $C^{\mu}_{L,R} = \tilde{R}\tilde{C}^{\mu}_{L,R}$ in the lab frame. The Lorentz boost $\hat{L}$ acts on the $(0,3)$ components, with $\hat{L}_{00} = \hat{L}_{33} = \gamma$ and $\hat{L}_{03} = \hat{L}_{30} = \gamma\beta$, where $(\beta, \gamma) = (p_Z/E_Z, E_Z/M_Z)$. The rotation matrix $\tilde{R}$ acts on the $(1,3)$ components, with
elements $\hat{R}_{11} = \hat{R}_{33} = \cos \theta$ and $\hat{R}_{13} = -\hat{R}_{31} = \sin \theta$. Thus, we can derive the currents $C_{\mu}^{L,R}$ and express them in terms of $Z$ boson polarization vectors,

\begin{align}
C_{L}^{\mu} &= M_Z \left( \sin \theta e_0^Z - \sqrt{2} \sin \frac{\theta}{2} e^{-i\phi} e_0^Z + \sqrt{2} \cos \frac{\theta}{2} e^{-i\phi} e_0^Z \right), \\
C_{R}^{\mu} &= M_Z \left( \sin \theta e_0^Z + \sqrt{2} \sin \frac{\theta}{2} e^{-i\phi} e_0^Z + \sqrt{2} \cos \frac{\theta}{2} e^{-i\phi} e_0^Z \right).
\end{align}

Next, we can obtain the amplitude of $e^-e^+\rightarrow \ell^-\ell^+\gamma$ by replacing the $Z$ polarization vector $e_0^Z(q_1)$ in Eq.(1.1) with $C_{\mu}^{L,R}(q_1)D_Z$, where $D_Z = 1/(q_1^2 - M_Z^2 + iM_Z \Gamma)$ is from the $Z$ propagator. Since $q_{1\mu}C_{\mu}^{L,R}(q_1) = 0$, we can drop the $q_1^\mu$ term in the $Z$-propagator. Then, we derive the $\ell^-\ell^+\gamma$ amplitude as follows,

\begin{align}
&\mathcal{T}_{\sigma\sigma\lambda}(\ell\bar{\ell}\gamma) = \frac{e f_{L,R}^Z D_Z}{s_W^2 c_W} v^s(p_2) \left[ \frac{e^2}{s_W c_W} \left( f_\lambda^s(q_2)(q_2 - P_1)C_{L,R}(q_1) + C_{L,R}(q_1)(q_2 - P_1)f_\lambda^s(q_2) \right) \right] \\
&\quad - \frac{i2M_Z^2}{A^4} e^{\mu\nu\beta\gamma} \epsilon_{\mu\nu\beta\gamma} C_{L,R}(q_1) e_0^s(q_2) q_{2\beta} \left( c_L P_L + c_R P_R \right) u^s(p_1),
\end{align}

where $(\sigma, \sigma', \lambda)$ denote the helicities of the final-state particles ($\ell^-, \ell^+, \gamma$), and we have defined the coefficients $(f_{L}^Z, f_{R}^Z) = \left( c_L \delta_{\sigma, -\frac{1}{2}}, c_R \delta_{\sigma, \frac{1}{2}} \right)$.

Substituting Eq.(A.10) into Eq.(A.11), we can express the amplitude (A.1) in terms of the helicity amplitudes (A.4)-(A.5) of Appendix A.1,

\begin{align}
&\mathcal{T}_{\sigma\sigma\lambda}(\ell\bar{\ell}\gamma) = \frac{e M_Z D_Z}{s_W^2 c_W} \left[ \sqrt{2} e^{i\phi} \left( f_{R}^s \cos^2 \frac{\theta}{2} - f_{L}^s \sin^2 \frac{\theta}{2} \right) T_{ss'}^T(\pm \lambda) \right. \\
&\quad + \sqrt{2} e^{-i\phi} \left( f_{L}^s \sin^2 \frac{\theta}{2} - f_{R}^s \cos^2 \frac{\theta}{2} \right) T_{ss'}^L(\mp \lambda) \left. + (f_{R}^s + f_{L}^s) \sin \theta \sigma T_{ss'}^L(0\lambda) \right],
\end{align}

where $T_{ss'}^T(\pm \lambda)$ and $T_{ss'}^L(0\lambda)$ are the on-shell helicity amplitudes of $e^-e^+\rightarrow Z\gamma$,

\begin{align}
&\mathcal{T}_{ss'}^T(\pm \lambda) = \mathcal{T}_{ss',T}(\pm \lambda) + \mathcal{T}_{s's',T}(\mp \lambda), \\
&\mathcal{T}_{ss'}^L(0\lambda) = \mathcal{T}_{ss',L}(0\lambda) + \mathcal{T}_{s's',L}(0\lambda),
\end{align}

which sum up the contributions from both the SM and dimension-8 operator as derived in Eqs.(A.4)-(A.5) of Appendix A.1. From Eq.(A.12), we see that the full cross section of $e^-e^+\rightarrow \ell^-\ell^+\gamma$ depends on the angle $\phi$ due to the interference between the terms with different $Z$ boson helicities $\lambda = +, -, 0$. Eq.(A.12) also explicitly shows the $\theta$ dependence associated with each $Z$ boson helicity. We have used Eqs.(A.12)-(A.13) in the analysis of angular observables in Section 3.

References


