LECTURES ON BEAM OPTICS

by

B. de Raad, A. Minten and E. Keil

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In the above series the following lectures were given:-

Introduction, by B. de Raad
Unseparated beams, by J. Geibel *)
Low-energy separated beams, by A. Minter
Medium-energy separated beams, by J.M. Perreau *)
R.F. Separated beams, by E. Keil
Beam envelope formalism and analog computer method,
   by K.G. Steffen (DESY) **)

*) The subject was mainly treated in the light of practical examples, therefore the author did not wish to submit a manuscript for publication.

**) The subject has been treated by the author in his recent book on "High-energy beam optics" (Interscience monographs and texts in physics and astronomy, New York, 1965).
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I. INTRODUCTION TO BEAM OPTICS

B. de Raad

Accelerator Research Division
1. **INTRODUCTION**

In most experiments with accelerators it is necessary to use well-collimated beams of secondary particles with a small momentum spread. Focusing is obtained with quadrupoles and momentum analysis with bending magnets. We shall give here some equations and some general considerations that were found useful in designing beams for the CERN Proton Synchrotron (CPS). In all these beams the angles between the particle trajectories and the axis of the beam transport system are of the order of 10^{-2} radian, so that the tangents and sines can be taken as equal to the angles themselves with high accuracy. This paraxial approximation simplifies the mathematics.

The basic theory of alternating gradient focusing is due to Courant and Snyder [1]. Survey articles about beam transport techniques have been given by Chamberlain [2], Luckey [3] and King [4]. Some of the subjects that we shall discuss are treated in an elementary way in the book by Livingood [5]. Several articles about quadrupoles have been published by Septier [6]. For an extensive treatment of the theory of high-energy beam optics, the reader is referred to the recent book by Steffen [7]. Several computer programmes are in use at CERN to calculate trajectories in the stray field of the CPS, and to calculate lens strengths, image sizes, etc., for secondary beams. In this report we shall discuss the basic concepts of beam optics, the knowledge of which should enable the reader to make an efficient use of the computer programmes.

2. **EQUATIONS OF MOTION IN A QUADRUPOLE FIELD**

A schematic cross-section of a quadrupole magnet is shown in Fig. 1. The profile of each pole is a rectangular hyperbola with the equation

\[ xy = \frac{R^2}{2} \]

where R is the radius of the inscribed circle. The axis of the quadrupole is the z axis.
The magnetic field distribution can be derived from the magnetic scalar potential

\[ V = -G \, x y \]  

\[ \text{(2)} \]

For convenience we shall first assume that \( G > 0 \). This corresponds to a horizontally focusing quadrupole. The case \( G < 0 \) will be discussed at the end of Section 3. The components of the field are then

\[ B_x = -\frac{\partial V}{\partial x} = G \, y \]

\[ B_y = -\frac{\partial V}{\partial y} = G \, x \]  

\[ \text{(3)} \]

We also note that

\[ \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = G \]  

\[ \text{(4)} \]

From these equations we see that a quadrupole is completely described by giving its gradient \( G \). The maximum value of \( G \) that can be realized in a practical quadrupole is limited by saturation of the steel and lies in the range from \( 1/R \) to \( 2/R \) \( T/m \), depending on the amount of tapering of its poles.\(^*\)

The equations of motion of a particle with charge \( e \) and relativistic mass \( m \), travelling in the \( z \)-direction with constant velocity \( v_z = v \) are

\[ \frac{d}{dt} \left( m \frac{dx}{dt} \right) = F_x = -ev \, B_y = -eG \, vx \]

\[ \frac{d}{dt} \left( m \frac{dy}{dt} \right) = F_y = +ev \, B_x = +eG \, vy \]  

\[ \text{(5)} \]

\( * \) We shall use MKS units, expressing distances in m, magnetic fields in T (1 Tesla = \( 10^4 \) gauss), and gradients in T/m.

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The direction of the forces is also indicated in Fig. 1, which shows that $F_x$ is everywhere directed towards the plane $x = 0$, while $F_y$ is always directed away from the plane $y = 0$. Therefore the quadrupole shown in Fig. 1 is horizontally focusing and vertically defocusing. The roles of the two planes can be exchanged by reversing the polarity of the quadrupole.

The magnetic forces are always perpendicular to the direction of motion so that the momentum $p = mv$ is constant. Eliminating $t$ with the relation $z = vt$ we can rewrite Eq. (5) as

$$\frac{d^2 x}{dz^2} + \frac{eE}{p} x = 0$$

$$\frac{d^2 y}{dz^2} - \frac{eE}{p} y = 0.$$  \hspace{1cm} (6)

The quantity $p/e$ is called the magnetic rigidity and is usually indicated by the symbol $B_p$. A particle with magnetic rigidity of 1 Tm has a momentum of 0.300 GeV/c.

From Eq. (6) we see that the differential equation for the $x$ motion is independent of the $y$ co-ordinate of the particle, and vice versa. The fact that the motion in the $x$ and $y$ plane is independent leads, of course, to great simplifications. If the quadrupole were rotated over an angle around the $z$ axis this would no longer be true.

Since a quadrupole is focusing in one plane and defocusing in the other plane, it is not possible with a single quadrupole to make an image in the same way as with glass lenses. Therefore the extremely useful properties of quadrupoles for beam optics were realized only after the principle of alternating gradient focusing had been discovered more or less by accident about 15 years ago. To make an image in both planes one can use, for example, two quadrupoles of opposite polarity some distance apart. The reader can easily convince himself that such
an arrangement can indeed be made focusing in both planes. Using familiar formulae from glass optics one can, for instance, show that the combination of a converging and diverging lens with equal strengths, some distance apart, is always focusing independent of the order of the lenses.

3. THICK LENS EQUATIONS FOR A QUADRUPOLE

If the length \( L \) of the quadrupole is so small that the change of \( x \) and \( y \) inside the quadrupole can be neglected, Eq. (6) can be integrated directly and becomes

\[
\Delta \left( \frac{dx}{dz} \right) = - \frac{GL}{E_p} x \\
\Delta \left( \frac{dy}{dz} \right) = + \frac{GL}{E_p} y .
\]

(7)

In this approximation the quadrupole behaves as a thin lens with focal length

\[
f_0 = \frac{E_p}{GL}
\]

(8)

that is focusing in one plane and defocusing in the other plane. Although this formula is very useful for a first estimate, it is not sufficient for accurate work. We shall therefore derive the exact solution of (6) and prove that the quadrupole can be represented by a thick lens with different focal distances and principal planes for the motion in the focusing and defocusing planes, respectively.

Let us introduce a constant

\[
K^2 = \frac{E}{E_p} .
\]

(9)

Taking \( z = 0 \) at the entrance of the quadrupole we can write the solution of Eq. (6) as
\[ x = x_0 \cos K z + \frac{x'_0}{K} \sin K z \]
\[ y = y_0 \cosh K z + \frac{y'_0}{K} \sinh K z \]

(10)

where \( x_0, x'_0, y_0, y'_0 \) are the initial values of \( x, \frac{dx}{dz}, y, \) and \( \frac{dy}{dz} \).

By differentiation of Eq. (10) we find

\[ x' = -K x_0 \sin K z + x'_0 \cos K z \]
\[ y' = K y_0 \sinh K z + y'_0 \cosh K z \]

(11)

Assume now that a particle is emitted from a point \( P \) located off the \( z \) axis in the plane \( z = z_p < 0 \) as shown in Fig. 2. In the field-free region the particle travels in a straight line and will therefore enter the quadrupole with the initial conditions

\[ x_0 = x_p - x'_p z_p, \ x'_0 = x'_p \]
\[ y_0 = y_p - y'_p z_p, \ y'_0 = y'_p \]

(12)

By substituting Eq. (12) into Eqs. (10) and (11), where we also put \( z = L \), we find the position and angles of the trajectory at the exit of the quadrupole. From there on, the particle travels again in a straight line given by

\[ x = \left[ \cos KL - K(z - L) \sin KL \right] x_p + \left[ \frac{1}{K} \sin KL + (z - L - z_p) \cos KL + Kz_p (z - L) \sin KL \right] x'_p \]
\[ y = \left[ \cosh KL + K(z - L) \sinh KL \right] y_p + \left[ \frac{1}{K} \sinh KL + (z - L - z_p) \cosh KL - Kz_p (z - L) \sinh KL \right] y'_p \]

(13)
It is always possible to find a point \( P' \) so that for \( z = z(P') \) the coefficient of \( x' \) vanishes. In the plane \( z = z(P') \), all particles emitted from point \( P \) pass through the same point so that this is the image plane of the plane \( z = z_P \) for the horizontal motion, but always \( z(x') \neq z(P') \), except when \( K = 0 \), which is a trivial case.

It is shown in geometrical optics that a system with the imaging properties as given by Eq. (13) can be described completely by giving its principal planes and focal length. These can be found most easily by considering an incident ray with \( x' = y' = 0 \). In the image principal plane for the horizontal motion \( x = x_P \), so that the coefficient of \( x_P \) in Eq. (13) must be unity. This gives for the position of the image principal plane:

\[
z(x') = L + \frac{\cos kL - 1}{K \sin kL}. \quad (14)
\]

This ray crosses the \( z \) axis at the image focal point

\[
z(x) = L + \frac{\cos kL}{K \sin kL} \quad (15)
\]

and the focal length is

\[
f_x = \frac{1}{K \sin kL}. \quad (16)
\]

The corresponding equations for the vertical plane are

\[
z(y') = L - \frac{\cosh kL - 1}{K \sinh kL} \quad (17)
\]

and

\[
z(y') = L - \frac{\cosh kL}{K \sinh kL} \quad (18)
\]

\[
f_y = \frac{1}{K \sinh kL}. \quad (19)
\]
In practical cases \( kL \ll 1 \) so that the trigonometric and hyperbolic functions can be developed in a power series, retaining only the first two terms. Using also Eqs. (8) and (9) we then find for the focusing plane

\[
z(x') = \frac{L}{2} - \frac{L^2}{24f_0}
\]

\[
f_x = f_0 + \frac{L}{6}
\]

and for the defocusing plane

\[
z(y') = \frac{L}{2} + \frac{L^2}{24f_0}
\]

\[
f_y = f_0 - \frac{L}{6}
\]

Note that the middle of the quadrupole is at \( z = L/2 \).

By considering a trajectory which is parallel to the \( z \) axis at the exit of the quadrupole, we can calculate the position of the object principal planes \( H_x \) and \( H_y \) and the object focal points \( F_x \) and \( F_y \). One then finds that \( H_x \) and \( H_x' \) lie symmetrically with respect to the middle of the quadrupole, while the same is true for \( F_x \) and \( F_x' \), etc. This result is obvious since the quadrupole is symmetrical around its middle.

Comparing the thick- and thin-lens formulae, we see from Eq. (21) that in the focusing plane a quadrupole is weaker than would follow from the thin-lens approximation. Inspection of Fig. 2 shows that the actual trajectory is closer to the axis and therefore in a weaker field than would follow from the thin-lens approximation. The reverse is true in the defocusing plane. The fact that the principal planes are slightly different from the symmetry plane of the quadrupole is of little importance.
As was mentioned already in Section 2, focusing in both the x and y plane can only be obtained with at least two quadrupoles of opposite polarity, and in general a high-energy particle beam consists of a substantial number of quadrupoles. To describe such a system we shall still use Eq. (6), but with the convention that $\zeta > 0$ in a horizontally focusing quadrupole and $\zeta < 0$ in a horizontally defocusing quadrupole. Equation (9) then becomes

$$K^2 = \frac{|\zeta|}{E_p},$$  \hspace{1cm} (24)

and when $\zeta < 0$, the trigonometric and hyperbolic functions in Eqs. (10) to (23) are interchanged.

4. **DOUBLETS AND TRIPLETS**

A doublet consists of two quadrupoles with approximately equal strength, but opposite polarity. The beam envelope in the x and y plane of a doublet is shown in Fig. 3. Particles are emitted from a point P on the axis and must be brought to an image $P'_x$ in the horizontal plane and $P'_y$ in the vertical plane. In practical beams with momentum analysis and separators it is often advantageous to choose $P'_x$ and $P'_y$ differently. The projection of the particle motion on the x plane has a larger excursion from the z axis in the second quadrupole, which is focusing in the x plane, than in the first one, which is defocusing in the x plane. Therefore the focusing effect of the second quadrupole is stronger than the defocusing effect of the first quadrupole and the difference of these two effects gives the net focusing action of the doublet in the x plane. The same argument is valid for the y plane.

It follows from Eqs. (21) and (23) that in the focusing plane a quadrupole is weaker, while in the defocusing plane it is stronger than would follow from the thin-lens approximation. Since the net focusing action of a doublet depends on the difference of the effects of the two
quadrupoles, a doublet is considerably weaker than would follow from
the thin-lens approximation. The same is true for other combinations
of quadrupoles.

The task of the beam designer is now to find the gradients
$G_1$ and $G_2$ in the two quadrupoles (which have equal length $L$) that give
images $P'_x$ and $P'_y$ for a given object $P$. Even in this very simple
case, an explicit solution for $G_1$ and $G_2$ can only be derived in the
thin-lens approximation of Eq. (8) which can give quite large errors.
When the proper equations (20) and (23) are used, this approach becomes
hopeless.

In the literature$^2,8$ one finds graphs for the focusing prop-
ties of doublets and triplets. However, these assume that $P'_x$ and $P'_y$
coincide, that the triplet (see below) is symmetrically excited, and
choose some specific values of the distance in between the quadrupoles.
Therefore they are of limited use.

In practice, all beams are designed with the help of computer
programmes. The designer specifies the position, length, and polarity
of the quadrupoles, and the positions of the image points in the $x$ and $y$
planes. The computer then calculates the required gradients of the quad-
ru poles by a process of successive approximations. The computer programme
can, of course, also perform other calculations, such as adjusting the
position of the quadrupoles to obtain a prescribed magnification, or
tracing the particle orbits through the system.

Some general comments can be made about a doublet. Even if $P'_x$
coincides with $P'_y$, the values of $|G_1|$ and $|G_2|$ are generally different.
If $t < t'$, then $|G_1| > |G_2|$, and vice versa. Only when $t = t'$ we have
$|G_1| = |G_2|$, as can readily been seen from the symmetry of the system in
that case.

Inspection of Fig. 3 shows that the angular magnification in the
plane $DP$ is larger than in the plane $FD$. It is well known from geometri-
cal optics that the transverse magnification is the inverse of the angular
magnification. This also follows from Liouville's theorem which will be discussed later. Therefore the transverse magnification in the plane PD is usually at least twice as large as in the plane DF. There is no objection to this, and in many cases one can make good use of it. The difference in magnification can be enhanced by increasing s. Note that by doing so one also decreases the maximum angle in the DF plane for which particles emitted from P will pass through the doublet.

The alternative arrangement is a triplet, consisting of a central quadrupole of length L and a quadrupole with length \( \frac{1}{2} \) L on each side. It is customary to have the same gradient in the two outer quadrupoles, which will in general be different from the gradient in the central quadrupole, but this is not at all necessary. Figure 4 shows the beam envelopes in a triplet. In this case the magnifications in the x and y plane are rather equal, so that the focusing properties of a triplet are more similar to those of a glass lens than in the case of a doublet. It is clear from Fig. 4 that for equal object and image distances the sum of the absolute values of the angular deflections in a triplet is larger than in a doublet. Therefore a triplet requires considerably stronger quadrupoles than a doublet for the same imaging function and is therefore uneconomical at high energies.

5. MATRIX FORMULATION

Let us consider any optical system in which the displacement \( x_1 \) and slope \( x'_1 \) of a particle trajectory at the exit are related to the corresponding values \( x_0 \) and \( x'_0 \) at the entrance by the linear equations

\[
\begin{align*}
x_1 &= a_{11} x_0 + a_{12} x'_0 \\
\gamma'_1 &= a_{21} x_0 + a_{22} x'_0
\end{align*}
\]

(25)

Using matrix notation we can write Eq. (25) as
\[
\begin{pmatrix}
x_1 \\ x'_1
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\ a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x_0 \\ x'_0
\end{pmatrix}.
\] (26a)

Assume now that this system is followed by a second optical system that transforms the vector \((x_1, x'_1)\) into the vector

\[
\begin{pmatrix}
x_2 \\ x'_2
\end{pmatrix} =
\begin{pmatrix}
b_{11} & b_{12} \\ b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x'_1
\end{pmatrix}.
\] (26b)

The relation between \((x_2, x'_2)\) and \((x_0, x'_0)\) is then

\[
\begin{pmatrix}
x_2 \\ x'_2
\end{pmatrix} =
\begin{pmatrix}
b_{11} & b_{12} \\ b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} \\ a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x_0 \\ x'_0
\end{pmatrix}.
\] (27)

where the matrices \(A\) and \(B\) are multiplied by the well-known rules for matrix multiplication. In this way we can trace a ray through any number of optical elements by simply multiplying the corresponding matrices. Note that the order of the matrices in this product is the inverse of the order in which the particles traverse the corresponding optical elements. This order must be carefully observed, since in general for matrices \(A\) and \(B\)

\[
A B \neq B A.
\] (28)

If \(A\) is the transfer matrix for particles passing from left to right through some optical system, the matrix for particles passing from right to left is \(A^{-1}\), defined by the relation

\[
A A^{-1} = 1.
\] (29)

We shall show in Section 7 that the determinant of all matrices representing combinations of quadrupoles and drift spaces is unity, that is

\[
a_{11}a_{22} - a_{12}a_{21} = 1.
\] (30)
It is readily verified therefore that

\[ A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \]  

Equation (30) provides a convenient method for detecting errors in calculations involving matrix multiplications.

To find the transfer matrix of a long beam transport system, the most convenient method is to trace (in general with the computer) one ray with \( x_0 = 1, x'_0 = 0 \) and another ray with \( x_0 = 0, x'_0 = 1 \). From the former one can find \( a_{11} \) and \( a_{21} \), and from the latter one finds \( a_{22} \) and \( a_{12} \) of the transfer matrix up to any point along the beam.

The matrix formulation is especially useful for the calculation of beam transport systems whose focusing properties are a periodic function of \( z \). Its main application is therefore in the theory of particle accelerators, but also for the calculation of beams the use of matrices can be very convenient. Suppose, for example, that one wants to find the effect of varying a particular quadrupole somewhere half-way in a long beam. Having traced the two rays mentioned above, we can write the matrix for the total beam as the product of three matrices, corresponding to the part of the beam before the variable quadrupole, the variable quadrupole itself, and the part after the variable quadrupole. In this way the change in the total transfer matrix due to the variable quadrupole is readily calculated.

The transfer matrix for a thin focusing lens with focal length \( f_0 \) is

\[ \begin{pmatrix} 1 & 0 \\ -1/f_0 & 1 \end{pmatrix}. \]
and for a drift space of length $s$ it is

$$\begin{pmatrix}
1 & s \\
0 & 1
\end{pmatrix}.$$  \hspace{1cm} (33)

From Eqs. (10) and (11) we see that the matrices for the focusing and defocusing plane of a quadrupole are

$$\begin{pmatrix}
\cos KL & \frac{\sin KL}{K} \\
-K \sin KL & \cos KL
\end{pmatrix}$$

and

$$\begin{pmatrix}
\cosh KL & \frac{\sinh KL}{K} \\
K \sinh KL & \cosh KL
\end{pmatrix}.$$ \hspace{1cm} (34)

6. **CHROMATIC ABERRATIONS**

The focusing properties of a quadrupole depend on the momentum of the particles. This effect, called chromatic aberration, limits the momentum band that can be accepted if a given unsharpness of the image is not to be exceeded. In this section we shall derive a general expression for the chromatic aberration of any optical system. We consider the $x$ plane and assume that the equation of motion is

$$\frac{d^2 x}{dz^2} + \frac{eG(z)}{p} x = 0$$ \hspace{1cm} (35)

where $G$ depends on $z$ and can be either positive or negative, as explained at the end of Section 3.
Let us consider a beam transport system, shown in Fig. 5, that images P in P′ for particles with momentum p₀. The transfer matrices from P to some intermediate point R and from R to P′ are B and A, respectively. The transfer matrix M from P to P′ is then
\[
M = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
= \begin{pmatrix}
a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\
a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}
\end{pmatrix}.
\]  
(36)

Since P′ is the image of P
\[
a_{11}b_{12} + a_{12}b_{22} = 0
\]  
(37)

and the magnification is
\[
\frac{x_{1}'}{x_0} = \frac{x_{2}'}{x_1} = a_{11}b_{11} + a_{12}b_{21}.
\]  
(38)

The trajectory of a particle with \(p = p_0 + \Delta p\) is the same as that of a particle with \(p = p_0\) with all gradients changed by an amount
\[
\Delta G = -G \frac{\Delta p}{p}.
\]  
(39)

We now assume first that only in a short piece of quadrupole, located at R, is the gradient changed by this amount \(\Delta G\). The modified transfer matrix \(M'\) is then
\[
M' = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{\Delta Gd_2}{Bp} & 1
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}.
\]  
(40)
Using Eq. (37) we find

\[ n_{12}' = -\frac{a_{12} b_1 a_0 \Delta G d\sigma}{Bp} \]  \hspace{1cm} (41)

Therefore a trajectory starting from P with \( x_0 = 0, x_0' = a_0 \) passes point P' at a distance

\[ x_1 = -\frac{a_0 a_{12} b_1 a_0 \Delta G d\sigma}{Bp} \]  \hspace{1cm} (42)

and appears to come from an object at P located at a distance

\[ \delta = \frac{a_0 a_{12} b_1 a_0 \Delta G d\sigma}{(a_{11} b_{11} + a_{12} b_{21})Bp} \]  \hspace{1cm} (43)

from the axis.

It is clear that

\[ x_R = b_{12} a_0 \]  \hspace{1cm} (44)

Moreover, using Eqs. (31) and (38) we find that

\[ x_R = -a_{12} a_1 = -\frac{a_{12} a_0}{a_{11} b_{11} + a_{12} b_{21}} \]  \hspace{1cm} (45)

Combining Eqs. (39), (43), (44) and (45) we get

\[ \delta = -\frac{1}{a_0} \frac{x^2}{R} \frac{6}{Bp} \frac{\Delta p}{p} \int d\sigma \]  \hspace{1cm} (46)

The total apparent displacement of the object at P is found by integrating Eq. (46) from P to P'. If particles are emitted from P in the angular interval \(-\alpha_0 < \theta < +\alpha_0\), the total apparent increase of the object size for a particle with momentum \( p_0 + \Delta p \) is

\[ d = \frac{2}{a_0} \frac{\Delta p}{p} \int_{P}^{P'} x^2 \frac{6}{Bp} d\sigma \]  \hspace{1cm} (47)
In this equation \( G \) is, of course, a function of \( z \). Note that a particle with \( p = p_0 - \Delta p \) has the same \( \lambda \).

Using Eq. (6) we can rewrite Eq. (47) as

\[
d = - \frac{2}{a_0} \frac{\Delta p}{p} \int_{P}^{P'} x x'' \, dz
\]

and by partial integration we obtain

\[
d = \frac{2}{a_0} \frac{\Delta p}{p} \int_{P}^{P'} (x')^2 \, dz \cdot
\]

The length of the projection on the \( x \) plane of a particle trajectory between \( P \) and \( P' \) is

\[
L_t = \int_{P}^{P'} ds = \int_{P}^{P'} \sqrt{1 + (x')^2} \, dz = \int_{P}^{P'} dz + \frac{1}{2} \int_{P}^{P'} (x')^2 \, dz
\]

Therefore \( L_t \) exceeds the distance \( z_P - z_{P'} \) measured along the \( z \) axis by an amount

\[
\Delta L = \frac{1}{2} \int_{P}^{P'} (x')^2 \, dz
\]

and we can write Eq. (49) as

\[
d = \frac{4}{a_0} \frac{\Delta p}{p} \Delta L
\]

From the latter equation we see that it is not possible with quadrupoles alone to make a beam without chromatic aberration, since any trajectory is always longer than the direct distance \( z_P - z_{P'} \). The
smallest $\Delta L$ is obtained with trajectories that are as short as possible and everywhere make small angles with the $z$ axis. This argument can be used to judge qualitatively the relative merits of different quadrupole systems. To find the chromatic aberration in practice, it is usually quicker to trace a ray with momentum $p_0 + \Delta p$ through it, rather than to use (49) or (52).

In separated beams it is desirable to have the smallest possible chromatic aberration in the plane of separation, which is usually the vertical plane, whereas the aberrations in the horizontal plane are of much smaller importance. Van der Meer\textsuperscript{9}) has shown, that by placing a sextupole near the momentum-analysing slit where the horizontal images of different momenta are separated due to the dispersion of the bending magnet, one can correct the chromatic aberration in the vertical plane. This correction only works if the dispersion is large compared to the horizontal image size, while the vertical image should be reasonably far away from the sextupole. The scalar potential of the magnetic field in a sextupole is given by

$$V = \frac{-2a_3}{3} (3x^2y - y^3)$$  \hspace{1cm} (53)

so that

$$\frac{\partial B}{\partial y} = 2a_3 x .$$  \hspace{1cm} (54)

The sextupole therefore adds a correction to the vertical focusing of the beam which is proportional to $x$ and therefore to $\Delta y/p$.

Ticho\textsuperscript{10}) has made a separated K beam with $\Delta p/p = \pm 3\%$ by non-linear shimming of bending magnets.

7. PHASE-SPACE REPRESENTATION AND LIOUVILLE'S THEOREM

The motion of a particle in the focusing plane of a quadrupole can be represented by drawing its actual trajectory as shown in Fig. 6a,
but we can also study how $x$ and $x'$ change along the particle orbit. Such a phase-space trajectory is shown in Fig. 6b. The equation of motion in a focusing quadrupole is

$$x'' + K^2 x = 0 .$$  \hspace{1cm} (55)

If we multiply this equation with $x'$ and integrate we obtain

$$\left( \frac{x'}{K} \right)^2 + x^2 = \text{constant} = a^2$$  \hspace{1cm} (56)

where the value of the constant is determined by the initial conditions. Therefore the phase-space trajectory of the particle is an ellipse which can be transformed into a circle by using $x'/K$ as the ordinate. The solution of Eq. (55) can be written as

$$x = a \cos (Kz + \delta)$$

$$\frac{x'}{K} = - a \sin (Kz + \delta)$$  \hspace{1cm} (57)

where $a$ is the amplitude of the motion and $\delta$ is a phase angle. Using $x$ and $x'/K$ as co-ordinates we see that in the phase space the particle moves along a circle with radius $a$, and that in passing through a quadrupole of length $L$ its representative point describes an arc of length

$$\alpha \gamma = aKL$$  \hspace{1cm} (58)

in the clockwise direction. This is illustrated in Fig. 7a.

In the defocusing plane, the phase-space trajectory in normalized co-ordinates is given by the orthogonal hyperbola

$$\left( \frac{y'}{K} \right)^2 - y^2 = \text{constant} = \pm b^2 .$$  \hspace{1cm} (59)
The solution of the equation of motion can be written as

\[ y = b \cosh (Kz + \delta) \]

\[ \frac{y'}{K} = b \sinh (Kz + \delta) \]  

(60)

where \( b \) is the minimum value of \( y'/K \) or \( y \) depending on whether the particle crosses the axis or not, and \( \delta \) is a phase angle. This is illustrated in Fig. 7b. Let us assume that the phase-space point moves from A to B in a defocusing quadrupole of length \( L \). We denote the distances from A and B to the line \( y + y'/K = 0 \) by \( AA' \) and \( BB' \), respectively. It can readily be shown \(^1\) that then

\[ \frac{BB'}{AA'} = e^{-\frac{2L}{K}} \]  

(61)

In a field-free region the phase-space trajectory is a straight line parallel to the x axis.

With the help of a sheet of paper on which a family of concentric circles and orthogonal hyperbolas has been drawn, one can graphically follow the motion of a particle through a focusing system. Although its accuracy is limited, this method is much faster than hand computation for thick lenses.

Let us now consider the ensemble of representative points of all particles emitted by a source and passing through a beam transport system. When the particle beam is limited by several diaphragms its boundary curve in a phase space at any given point along the beam is a complicated polygon, but in many cases it can be approximated by an ellipse. Such an ellipse is transformed into a new ellipse by any linear transformation such as given by the matrix (26). It can be shown that the ratio of the areas of the new and old ellipse is equal to the determinant of the transfer matrix. It is clear that the determinants of the
matrices (33), (32), and (34) which represent drift spaces, thin quadrupoles, and thick quadrupoles, respectively, are equal to one, and therefore the area of the phase-space ellipses is constant along a beam. Successive phase-space ellipses in a focusing quadrupole are shown in Fig. 8.

What we have proved is a special case of Liouville's theorem, which is one of the bases of statistical mechanics\(^1\) and which for our purpose can be formulated as follows. Let each particle of a beam be represented by its transverse displacement \(x\) and transverse momentum \(p_x = mv_x\), and let us assume that the ensemble of the phase-space points of all particles in a beam lies within a closed curve of arbitrary shape. If this beam passes through a focusing system containing linear and non-linear lenses and is also accelerated, the area inside the final curve which encloses all phase-space points is equal to that of the original curve. The effect of a non-linear lens is not to increase the phase-space area of a beam, but to deform it. For practical purposes, this often amounts to the same as an increase, since the minimum image size that can be made with such a deformed curve by means of quadrupoles is larger than with an ellipse that has the same area.

If the particle is not accelerated, as in the case of secondary beams, one can use \(dx/dz\) instead of \(p_x\), but that is not allowed for an accelerator such as the CPS. If we consider, for example, a mid-F or mid-D point in the CPS, the phase-space ellipse representing the beam must be upright for reasons of symmetry, and Liouville's theorem says that

\[
x(\text{max}) p_x(\text{max}) = x(\text{max}) mv_z \left( \frac{dx}{dz} \right)_{\text{max}} = p_z x(\text{max}) \left( \frac{dx}{dz} \right)_{\text{max}} = \text{const}.
\]

\(^{\star}\) These are so-called canonically conjugate variables.
On the other hand, the focusing properties of the GPS remain the same during the acceleration cycle and therefore

$$\frac{x_{\text{max}}}{(dx/dz)_{\text{max}}} = \text{constant} = \beta$$  \hspace{1cm} (63)

where $\beta$ is a characteristic parameter of the focusing structure. Multiplying Eq. (62) and (63) we find

$$p_z x^2(\text{max}) = \text{const}$$  \hspace{1cm} (64)

and therefore the amplitude of the betatron oscillations is inversely proportional to $p_z^{1/2}$ or $B^{1/2}$, where $B$ is the magnetic field. A similar argument\(^{13}\) can be used to describe the damping of phase oscillations in a synchrotron.

8. MATCHING AND REPRESENTATION OF PHASE-SPACE ELLIPSES BY COMPLEX NUMBERS

Let us consider a long focusing system, for example a linear accelerator with quadrupoles inside the drift tubes. The beam emerging from the linac can be described by two phase-space ellipses, the so-called emittance ellipses, one for the horizontal $(x,x')$ and one for the vertical $(y,y')$ plane. In general, the ratio of the axes and the orientation of the horizontal and vertical emittance ellipses will be different. Their areas are often equal but this is not at all necessary.

We now consider another long focusing system, for example the GPS, into which one wants to inject a beam of particles which will then be accelerated. If particles are injected with a large angle with respect to the GPS equilibrium orbit they will evidently hit, sooner or later, the vacuum chamber wall. One can therefore draw phase-space ellipses for the two planes which contain the representative points $(x,x')$ and $(y,y')$ of all particles whose betatron amplitudes are sufficiently

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p/he
small so that they will not hit the wall of the chamber (taking into account closed orbit distortions due to magnet errors, etc.). These ellipses are called the horizontal and vertical admittance ellipses.

To avoid beam loss at beam transfer, the admittance\(^*\) of the CPS must be larger than the emittance of the linac. The aperture required to accommodate the injected beam will be a minimum when the shape and orientation of its emittance ellipses at the point where it enters the synchrotron, after having passed through the stray field, are the same as those of the admittance ellipses of the synchrotron at that position. In other words, the emittance ellipses of the beam should be matched to the admittance ellipses of the synchrotron.

The problem of an accelerator designer is now to design a system of quadrupoles which can produce simultaneous matching in the horizontal and vertical planes. For matching purposes one is not interested in the correspondence of individual points on the initial (before) and final (after the matching system) ellipses, nor in the area of these ellipses, since we know from Liouville's theorem that this area is conserved anyhow. It is therefore sufficient to describe each ellipse by two parameters, which might, for example, be the ratio of its axes and the angle between one axis and the x axis. For matching in both planes we therefore need four degrees of freedom.

An elegant method of calculation, using complex numbers, has been developed by Hereward\(^{14}\). With \(a\), \(b\) and \(c\), as shown in Fig. 9, we define

\[
R = \frac{a}{b}
\] (65)

and

\[
X = \frac{c}{b}.
\] (66)

The ellipse is then completely described by the single complex number

\[
Z = R + jX
\] (67)

\(^*\) The area of the admittance and emittance ellipses will be called admittance and emittance.
and its area is

\[ E = \pi ab. \]  \hspace{1cm} (68)

The half-width of the beam is

\[ \hat{x} = \left( \frac{E}{\pi R} \right)^{1/2} \left( x^2 + \hat{x}^2 \right)^{1/2} = \left( \frac{E}{\pi R} \right)^{1/2} |z|. \]  \hspace{1cm} (69)

If the transfer matrix of the matching system is given by Eq. (26a) the transformation rule for the complex number \( Z \) is

\[ z_1 = a_{11} \frac{Z_0 + j a_{12}}{a_{22} - j a_{21} \frac{Z_0}{Z_0}} \]  \hspace{1cm} (70)

For a proof of the latter equation we refer to Hereward. If one has a computer that can handle complex numbers, the programme writing is considerably shortened by the use of the complex notation. It is obvious that this notation is not restricted to matching systems, but that with the help of Eqs. (69) and (70) one can calculate the beam envelope along any beam transport system.

The general matching problem requires solving four equations of the fourth order to find the strengths of four quadrupoles. It is not possible to give a general recipe for this problem, but an important simplification can be obtained by using triplets. In a triplet one can adjust the focal length for the \( x \) and \( y \) plane more or less independently by a suitable excitation of the middle lens and the two outer lenses. Therefore with two triplets, some distance apart, one can simulate a system of two lenses that have independent focal lengths in the \( x \) and \( y \) plane. One can therefore calculate independently the focal lengths for matching in the two planes, so that the problem is reduced to solving twice two quadratic equations with two unknowns. Subsequently, one designs the two triplets which give the required focal lengths for the \( x \) and \( y \) plane.

As mentioned in Section 4, triplets are uneconomical at high energies. In a general case one would therefore choose reasonable
spacings of the four quadrupoles and reasonable values for the initial
gradients. By successive approximations, the computer then tries to
find the lens strengths for matching. If one has been unlucky in the
choice of lens spacings and initial strengths, there may not be any
solution or the computer may not be able to find it. However, the
computation is very fast and therefore it is possible for the computer
to survey in a short time practically all the combinations of lens
strengths and spacings that are of interest.

9. BENDING MAGNETS

Practically all beams contain bending magnets for momentum
analysis. From a beam optics point of view, the main effect of a
bending magnet is its dispersion, that is, the fact that off-momentum
particles are deflected away from the beam centre line. In addition
to this, there occurs a small focusing effect. The latter is in-
creased if the entrance and exit faces make large angles with respect
to a plane perpendicular to the beam axis, but even so this edge focus-
ing is always very weak. For high-energy beams it is therefore customary
to use bending magnets with a homogeneous field and with parallel
end faces, and to do all the focusing with quadrupoles.

The dispersion of a bending magnet can most conveniently be
taken into account by adding a third column to the $2 \times 2$ transfer matrix
discussed in Section 5. If a beam passes symmetrically through a
bending magnet with length $L$ and deflection angle $\varphi$, as shown in Fig. 8,
its effect can be written as

$$
\begin{pmatrix}
  x' \\
  y' \\
  \Delta p \\
  p'
\end{pmatrix} =
\begin{pmatrix}
  1 & L \cos \frac{1}{2} \varphi & -L \sin \frac{1}{2} \varphi & 0 \\
  0 & 1 & -2 \tan \frac{1}{2} \varphi & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x_0 \\
  y_0 \\
  \Delta p_0 \\
  p_0
\end{pmatrix}
$$

(71)
and

\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6
\end{pmatrix} = \begin{pmatrix}
1 - \phi \tan \frac{1}{2} \phi & \frac{L\phi}{2 \sin \frac{1}{2} \phi} \\
2 \sin \frac{1}{2} \phi & - \left(2 \tan \frac{1}{2} \phi + \phi \tan^{2} \frac{1}{2} \phi\right) & 1 - \phi \tan \frac{1}{2} \phi \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}.
\]

(72)

The sign convention for \(x\) and \(\phi\) is shown in Fig. 10. For a deflection in the direction of the negative \(x\) axis we have \(\phi < 0\). Matrices for arbitrary entrance and exit angles have been given by Penner\(^{15}\) to whom we also refer for a proof of Eqs. (71) and (72).

To get a feeling for the physical significance of these matrices it is convenient to assume that \(\phi\) is small so that \(\cos \phi = 1\) and \(\sin \phi = \tan \phi = \phi\). In that case, we see that in the horizontal plane the magnet behaves as a drift space of length \(L\) for all particles, but that the trajectories of off-momentum particles make an angle \(\Delta x' = - \phi \left(\Delta p/p\right)\) with the beam axis and have an extra displacement \(\Delta x = - \frac{1}{2} L \phi \left(\Delta p/p\right)\) at the magnet exit (note that \(\phi < 0\) in our convention). Vertically the magnet behaves in this approximation as a drift space of length \(L\) with, at each end, a thin focusing lens of focal length

\[
f_e = \frac{2L}{\phi^2}.
\]

(73)

This is the familiar formula for edge focusing.

Assume now that we have a bending magnet producing a given deflection and we ask ourselves how it could be used together with the doublet shown in Fig. 3, so that we obtain the best possible momentum analysis. The total deflection angle for particles with momentum \(p_0\) is \(\phi\), while for particles with a lower momentum \(p_0 - \Delta p\) it is \(\phi + \Delta \phi\). To
compare the relative merits of the positions A and C, shown in Fig. 3, we draw the phase-space ellipses in these points. One can readily show that the area of each ellipse shown in Fig. 11 is

\[ \iota = \pi \, \text{ed} \]  

(74)

while according to Liouville's theorem

\[ \pi(\text{ed})_A = \pi(\text{ed})_C \]  

(75)

The solid ellipses in Fig. 9 are the phase-space ellipses for \( p = p_0 \), and the ones drawn in interrupted lines are for \( p = p_0 - \Delta p \) after the bending magnet. At point C we have \( |\Delta \phi| > 2d \) so that the two ellipses are completely separated. Therefore the images at \( P'_x \) for \( p_0 \) and \( p_0 - \Delta p \) are also completely separated. At point A we have \( |\Delta \phi| < 2d \) so that the ellipses and therefore also the images at \( P'_x \) partially overlap. From this discussion we see that the momentum resolution obtained with a given \( \phi \) is approximately proportional to the width of the beam in the bending magnet, which therefore should be made as large as possible. Note that the beam cross-section in point C fits better in the bending magnet gap than at point A, since normally the width of the poles is several times larger than the height of the gap.

The dispersion obtained by placing the bending magnet in between the two quadrupoles is larger than at A, but in general somewhat smaller than at C. This arrangement can be interesting if a very different magnification in the x and y plane, and therefore a large \( s \), is used.

Looking at Fig. 4 we see that a large \( \delta \) in the bending magnet can be obtained by splitting the triplet at its plane of symmetry and inserting the bending magnet there. One has then two doublets, the first of which makes a parallel beam, while the second focuses the beam down to an image. This situation is shown in Fig. 12. Such a system, with \( P'_x \) and \( P'_y \) coinciding and with \( l = l' \), so that it has unit magnification has,
for example, been discussed by Courant\textsuperscript{16}). The disadvantage is that now four quadrupoles are required instead of two. However, if one wants to make high intensity beams in the 25 GeV range, the quadrupoles become so large that even if the single doublet of Fig. 3 is used one would, in general, compose each quadrupole of two separate quadrupoles of length $\frac{1}{4}L$.

The arguments given above are also valid for the separation obtained with electrostatic separators. Since especially in this case one wants to obtain the maximum possible separation, the arrangement of Fig. 12, with a parallel or slightly converging beam inside the separator, is frequently used\textsuperscript{17}). It is also possible, of course, to use a triplet to make the beam parallel inside the separator and then a second triplet to bring it to an image again, instead of the two doublets shown in Fig. 12.

To obtain a good momentum analysis, the width of the beam at the momentum analysing slit should be as large as possible. On the other hand, it is desirable that the final image is dispersion free, so that the final image width is as small as possible and independent of the momentum band which is selected by the momentum-analysing slit. This requirement can be met in a beam with two successive deflections. In general the second deflection at the same time serves other purposes, for example to remove background produced in the edges of the momentum-analysing slit or to obtain a sufficient separation between adjacent beams. Figure 13 shows a beam with two successive deflections in which the final image size, but not the angular divergence, is independent of the momentum bite.

Particles emitted from $P$ are brought to a horizontal and vertical image at $P'$ by the doublet $Q_1Q_2$. The accepted momentum bite can be chosen by varying the width of the momentum-analysing slit $S$. The latter can be as wide as the aperture of the quadrupoles. To avoid loss of particles, the slit is immediately followed by a single, horizontally
focusing, quadrupole $Q_3$ which acts as a field lens and images $Q_2$ on $Q_4$. The vertically defocusing effect of $Q_3$ has no influence on the beam since it has a vertical image $P'$ close to or inside $Q_3$. A final horizontal and vertical image $P''$ is formed by the doublet $Q_4Q_5$. After this doublet follows a second deflecting magnet $BM_2$.

Let the deflections produced by $BM_1$ and $BM_2$ at $P'$ and $P''$ be $d_1$ and $d_2$, respectively. We take the trajectory of a particle emitted from $P$ with $x_0 = 0$, $x'_0 = 0$, and $p = p_0$ as the $z$ axis. A particle emitted from $P$ with $x_0 = 0$ and with $p = p_0 + \Delta p$ will be deflected less and passes $P'$ at distance

$$x_{P'} = \frac{\Delta p}{p} d_1 .$$

(76)

For simplicity we assume that the image $P'$ coincides with the centre of $Q_3$ so that the latter acts exclusively as a field lens and has no influence on the position of the images of the beam. Under the influence of the doublet $Q_4Q_5$, that images $P'$ in $P''$ with magnification $m_2 < 0$, the particle passes $P''$ at a distance from the axis

$$x_{P''} = (m_2 d_1 + d_2) \frac{\Delta p}{p} .$$

(77)

To obtain a dispersion-free image at $P''$, the images for different momenta should coincide. The condition for this is

$$\frac{d_2}{d_1} = -m_2 ,$$

(78)

that is, the two bending magnets should deflect in the same direction, and the ratio of their deflections must be equal to the magnification of the second doublet. For systems with more than two deflections, similar relations can be derived.

By making a beam which is completely symmetrical with respect to its half-way point, one can achieve that both the final image size and
angular divergence are independent of the momentum bite. An example of this is given by the beams in the SLAC beam switchyard\textsuperscript{18}, and is shown in Fig. 14. The field lens $O_3$ makes an image of the centre of BM$_1$ on the centre of BM$_2$. Orbits of particles with different momenta but the same initial conditions at P will again coincide after BM$_2$ so that the final beam emittance is not at all affected by the momentum analysing system.
REFERENCES


4) N.M. King, Progr. in Nuclear Physics 2, 71 (1964).


   A. Septier, "Strong focusing lenses" in Advances in Electronics and  


9) S. van der Meer, CERN 60-22.


11) P. Lapostolle, CERN/PD-PL2, Feb. 1955, and CERN Symposium 1956,  
    Vol. 1, p. 178.

12) C. Kittel, Elementary Statistical Mechanics (J. Wiley and Sons,  
    1958).


16) E.D. Courant and R. Cool, Proc.Int.Conf. on High-Energy Accelerators  

17) H.K. Ticho, Proc.Int.Conf. on High-Energy Accelerators and  

FIG. 1 Components of field and force in a quadrupole.
FIG. 2 PRINCIPAL PLANES AND FOCAL POINTS OF QUADRUPOLE
FIG. 3 BEAM ENVELOPE IN DOUBLET

FIG. 4 BEAM ENVELOPE IN TRIPLET
Fig. 5: Calculation of chromatic aberration

Fig. 6: Particle motion and corresponding phase space plot in the focusing plane
Fig. 7: Phase space trajectories in the focusing and defocusing plane of a quadrupole.

Fig. 8: Successive phase space ellipses in focusing plane.
Fig. 9: Notation to represent a phase space ellipse by a complex number

\[ R = \frac{a}{b} \]
\[ X = \frac{c}{b} \]
\[ Z = R + jX \]

Fig. 10: Beam passing symmetrically through a bending magnet
FIG. 11 SEPARATION OF PHASE SPACE ELLIPSES BY BENDING MAGNET

FIG. 12 MOMENTUM ANALYSIS WITH BENDING MAGNET IN BETWEEN TWO DOUBLETS
Fig. 13: Dispersionfree beam with two successive deflections
Fig. 14  SLAC switchyard beam with final emittance independent of momentum bite
II. LOW-ENERGY SEPARATED BEAMS

A. Hinten

Track Chamber Division
1. **SEPARATED BEAMS**

Physics experiments, and especially bubble chamber experiments, require beams of particles separated in both momentum and mass. Separated beams are designed to separate

\[
\begin{align*}
\pi^+ & \text{ from } p^+ \text{ or } p^+ \text{ from } \pi^+ \\
K^+ & \text{ from } p^+ \text{ and } \pi^+ \\
K^- & \text{ from } \pi^- \\
\bar{p} & \text{ from } \pi^-.
\end{align*}
\]

Since pions and protons are about equally abundant, the separation \(\pi^+ p\) is normally easy to obtain, and \(\pi^-\) beams can even be obtained without separation. \(K^+\), \(K^-\) and \(\bar{p}\) are rare particles and pose the real problem.

The deviation \(a_m\) of particles of momentum \(p\) in a magnetic field is

\[
a_m = \frac{Bl}{mv} = \frac{Bl}{p}
\]

where \(B\) is the (homogenous) magnetic field and \(l\) its length. To separate masses after a momentum selection, one has to make use of the different velocity \(v\) for different masses \(m\), as is done by time-of-flight measurement or Čerenkov counters in electronic experiments. For spatial separation one uses the deviation in an electric field \(E (c=1)\)

\[
a_e = \frac{E l}{mv^2} = \frac{El}{p}\beta
\]

In practice the electrostatic separator is a plate condensor of 3 to 10 metres length, a plate distance of 5 to 10 cm, and a vertical electric field \(E\) of 50 - 100 kV/cm. In order to keep the beam horizontal, the deviation is compensated by a cross magnetic field (100 kV/cm ~ 300 gauss) or by outside short magnets with higher field. In this latter case the trajectories have a sagitta.
\[
\frac{\xi}{\beta} = \frac{E_0}{2p_0} \\
\] 

The angular separation is then \( \Delta \alpha = \frac{E_0}{p} \Delta \beta \)

\[
= \frac{E_0}{p^2} \Delta \epsilon
\]

with \( \epsilon = \sqrt{m_0^2 + p^2} \), \( \beta = \frac{p}{\epsilon} \), and for high momenta \( p \sim \epsilon \)

\[
\Delta \epsilon = \frac{\Delta m_0^2}{2\epsilon} \sim \frac{\Delta m_0^2}{2p}
\]

\[
\Delta \alpha = \frac{E_0}{2p^2 \epsilon} \Delta m_0^2 \sim \frac{E_0}{2p^2} \Delta m_0^2.
\]

We see immediately that \( p\nu \) separation is much easier than \( K\pi \) separation.

\( p\nu : \Delta m_0^2 = 0.88 - 0.02 = 0.86 \text{ GeV}^2 \)

\( K\pi : \Delta m_0^2 = 0.25 - 0.02 = 0.23 \text{ GeV}^2 \).

The geometrical arrangement of a separation stage is given in Fig. 1a. The beam is made parallel by a quadrupole doublet, passes through the separator, and is focused by a final doublet on the mass slit. In the separator different masses are separated by an angle

\[
\Delta \alpha = \frac{E_0}{p^2} \Delta \epsilon
\]

which is transformed to a spatial separation in the final image which is

\[
\delta = f_2 \cdot \Delta \alpha.
\]

The quality of the separation is given by the separation factor

\[
S = \frac{\delta}{1}
\]
where $I$ is the image size that is approximately given by $I = mT$. The magnification is $m = f_2/f_1 = \varphi_2/\varphi_1$ and $T$ is the target size

$$S = \frac{f_2 \Delta \alpha}{mT}$$

$$= \frac{D}{T} \frac{\Delta \alpha}{\alpha_1}.$$

To obtain high separation the vertical aperture $D$ should be large, whereas target size $T$ and acceptance angle $\alpha_1$ should be small.

To gain space one can have a convergent beam passing through the separator (Fig. 1b). The angular separation would be the same of course, whereas the spatial separation is

$$\delta = \frac{f_2}{2} \Delta \alpha$$

and the separation factor

$$S = \frac{1}{2} \frac{D}{T} \frac{\Delta \alpha}{\alpha_1}.$$

The separation $\delta$ should not be compared here, since $f_2$ will be different in the two cases, and with it $\delta$ and $I$. The separation factor $S$, however, decreases by a factor of two in the convergent case.

This condition can be demonstrated in the phase space representation (Fig. 2):

a) shows the phase space area at the target,
b) after a drift space,
c) in the separator,
d) at the image.

So the decrease in separation in the convergent case is indicated by the smaller beam size and an increase of angular spread in the separator.
2. **LOW-ENERGY BEAMS**

Low-energy beams are beams below 1 GeV/c.

The particle flux in a beam is given by the formula

\[
F = P(p) \frac{dp}{d\Omega} e^{-L/L_0} \frac{\text{particles}}{10^{11} \text{protons}}.
\]

Here we understand

- \(P[\text{particles/10}^{11} \text{protons}\cdot\text{sr}\cdot\text{GeV/c}]\) production factor, which is a complicated function of primary proton energy, target conditions, production angle and secondary momentum \(p[\text{GeV/c}]\);
- \(dp[\text{GeV/c}]\) momentum bite;
- \(d\Omega[\text{sr}]\) solid angle;
- \(e^{-L/L_0}\) decay factor, where \(L\) is the length of the beam and \(L_0\) the decay length;
- \(L_0 = \beta\gamma r\) decay length which is for kaons \(L_{0K} = 7.5 \text{p[m]}\) and pions \(L_{0\pi} = 55 \text{p[m]}\).

Low-energy beams are characterized by the fact that production and decay factors are small compared to higher energies. On the other hand, momentum bite and solid angle can be large since conditions set by separation are not restrictive. In any case, the length is as small as possible, both to increase the flux and to keep the ratio of kaons to the relatively stable pions and muons as high as possible. As a practical example, we compute the flux at a momentum of 0.8 GeV/c. At a proton momentum of 19.2 GeV/c and a production angle of 15° we have:

\[
\begin{align*}
P(K^-) &= 0.4 \times 10^9 \\
P(K^+) &= 1.2 \times 10^9 \\
P(\bar{p}) &= 2.5 \times 10^6 \\
P(\pi^-) &= 2.5 \times 10^{10}.
\end{align*}
\]
For a momentum bite of 10 MeV/c, a solid angle of $10^{-3}$ sr and decay lengths of 6 m for kaons and 45 m for pions, we get for a length of 30 m

$$F(K^-) = 28$$
$$F(K^+) = 84$$
$$F(p) = 25$$
$$F(\pi) = 10^5$$

From these figures we can compute the necessary rejection $R$, which we defined from the equation for the pion contamination

$$\frac{1}{R} \frac{F(\pi)}{F(K^-)} = 0.05$$

if we request five per cent pions as admissible contamination. From that we get

$$R = 0.7 \times 10^5$$

In practice the rejection is even higher.

A second independent problem of low-energy beam arises from the $\pi\mu$ decay. By this decay, muons are produced which are phase-space independent from the original beam and which cannot be separated. By means of a Monte-Carlo programme, we computed the probability of muons originating from pion decays being transmitted through the mass slit. We found

$$t \approx 0.01$$

This produces a muon flux after a 15 m stage

$$N_\mu \approx 0.01 \cdot F(\pi) \frac{d\Omega}{d\Omega} \frac{L}{L_{0\mu}}$$

$$\approx 10^3 \frac{\text{muons}}{10^{11} \text{protons}}.$$
The method for overcoming this is to build a low-energy beam consisting of two stages, the first one working for pion separation, and the second for muon separation.

3. **EXAMPLE: THE k₄ BEAM**

For a low energy one expects a beam which is short and has two separation stages, with momentum and mass separation in the first stage and a dispersion-free second stage with another mass separation.

Figure 3 shows the layout of the k₄ beam. It starts with a production angle of 15° from target 10 in the PS North Hall. A special C-shaped magnet bends the particles by another 15° to get them passing along the next PS magnet. A vertical and a horizontal collimator, both remote controlled, define the aperture and regulate the flux. The doublet Q₁Q₂ focuses in both planes, the magnet BM₁ bends by 22° to obtain a momentum selection. The first mass separation is done by a "Zadig" separator, with a length of 3 m, 10 cm gap, 50 kV/cm and, as requested, external or crossed magnetic field. Q₃ focuses vertically on the first mass slit, which together with the momentum slit inside or quadrupole Q₄ serves as a field lens.

The second stage starts with a magnet BM₂ which bends the beam by another 22°, first for geometrical reasons but secondly to clean the beam, since a slit never absorbs or stops the unwanted particles but only degrades them, and they have to be separated magnetically. Quadrupole Q₅ focuses again vertically to get the beam passing parallel through the second separator, a 6 m CERN-MPA separator with outside compensation magnets, and from outside adjustable plates to adapt them to the curved trajectories of the particles. Q₆ and Q₇ finally focus again, vertically into the second mass slit, horizontally into the entrance of the bubble chamber which is the narrowest limitation in this plane. A vertical bending magnet cleans after the second mass slit and gives the particles the right angle at which to enter the bubble chamber.
Figure 4 shows the beam envelopes in both planes. $Q$, focuses horizontally to achieve high acceptance and high flux, and defocuses vertically to have low acceptance and high separation. The beam is vertically parallel in the separators, has both images together after the first stage, and a different position after the second stage. The dashed curve shows an axial ray which is off-momentum by 1%. Its first image point has a dispersion of 7 cm, its trajectory is kept in the boundaries by the field lens. At the final image it has a dispersion of 1 cm so the beam is not really dispersion-free, but the final size is small enough to enter the chamber.

Let us compute some relevant quantities:

**Separation:** For $K\pi$ separation at 0.8 GeV/c we get

$$\epsilon_K = 0.94 \text{ GeV}$$
$$\epsilon_\pi = 0.81 \text{ GeV}$$

$$\Delta \alpha = \frac{E}{p} \Delta \epsilon$$
$$= 3 \times 10^{-3} \text{ rad}$$

$$\delta = f \cdot \Delta \alpha$$
$$f = 2.5 \text{ m}$$
$$= 0.75 \text{ cm}$$

I = $m_\gamma T_\gamma$
$$m_\gamma = 0.5, T_\gamma = 0.2 \text{ cm}$$
$$= 0.1 \text{ cm}$$

This image size cannot be used to compute the separation factor $S$ since it does not include any aberration. Contributions to the image size are:

i) halo, i.e. strange particle decay around the target,

ii) non-linearity, i.e. deviations of elements from the exact field distribution,

iii) multiple scattering on windows etc.,
\[ \theta_{\text{proj}} = \frac{15 \text{ MeV}}{p\sigma \text{ MeV}} \sqrt{\frac{\ell}{\ell_{\text{rad}}}} \]

which is with \( \ell = 15 \text{ m} \), and for air with \( \ell_{\text{rad}} = 300 \text{ m} \),

\[ \theta_{\text{proj}} = 4.2 \times 10^{-3} \text{ rad} \]

which is more than the angular separation. Therefore the beam has to pass through vacuum.

**Chromatic aberration:** Off-momentum particles are focused in a distance

\[ \Delta z = f \cdot \frac{\Delta p}{p} \]

from the nominal focus. For \( Q_3 \) and \( \Delta p/p = 0.01 \) this is

\[ \Delta z = 2.5 \text{ cm} \]

Taking into account also the first doublet \( Q_1 \), \( Q_2 \) we get

\[ \Delta z = 7 \text{ cm} \]

This causes in the position of the nominal focus a width

\[ \Delta y = \pm y' \Delta z \]
\[ = \pm 0.14 \text{ cm} \]

which increases the theoretical image by a factor of three.

**Dispersion:** For a bending angle \( \varphi \) and without subsequent focusing, the dispersion over a drift length \( \ell \) is

\[ d = \ell \cdot \varphi \cdot \frac{\Delta p}{p} \]
\[ = 3 \text{ cm} \]

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with $Q_s$ defocusing we get

$$d = 7.5 \text{ cm}.$$ 

We use this dispersion to compensate chromatic aberration. We tilt the mass slit by $45^\circ$ to follow the line of vertical foci as a function of momentum. This procedure is limited by the resolution.

**Resolution:** The resolution is the resolvable momentum separation $\Delta p/p$

$$r = \frac{I}{d}$$

$$= \frac{m_x T_x}{d}$$

$m_x = 8, T_x = 0.3$

$$= 0.004.$$ 

Finally we summarize the properties and parameters of the $k_s$ beam.

**Table 1**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>target (beryllium)</td>
<td>$2 \times 3 \times 38 \text{ mm}^3$</td>
</tr>
<tr>
<td>production angle</td>
<td>$15^\circ$</td>
</tr>
<tr>
<td>length to second mass slit</td>
<td>$28 \text{ m}$</td>
</tr>
<tr>
<td>length to bubble chamber</td>
<td>$31 \text{ m}$</td>
</tr>
<tr>
<td>momentum bite</td>
<td>$\pm 1.08%$</td>
</tr>
<tr>
<td>solid angle</td>
<td>$0.45 \times 10^{-3} \text{ sr}$</td>
</tr>
<tr>
<td>dispersion</td>
<td>$7.4 \text{ cm}/1% \frac{\Delta p}{p}$</td>
</tr>
<tr>
<td>resolution</td>
<td>$0.35%$</td>
</tr>
<tr>
<td>momentum range</td>
<td>$0.6 - 1.2 \text{ GeV/c}$</td>
</tr>
</tbody>
</table>
Figure 5 shows a typical separation curve, taken by moving the image of pions and kaons over the second mass slit by means of the compensating magnets.

4. SPECIAL DESIGNS

In this closing section we want to discuss some methods which have been used or are proposed to increase the kaon flux by decreasing the length and increasing the solid angle.

The $k_3$ beam \(^1\) preceding the $k_4$ beam was built at the same position in the PS North Hall (Fig. 6). It used edge focusing of bending magnets for vertical focusing (Fig. 7). The focal length has been given by de Raad in Section 1 of this report (Introduction to Beam Optics), as

$$f_e = \frac{2L}{\varphi}.$$ 

To obtain a 4 m focal length with a 1 m bending magnet one has to bend by 44°. This means a field of 20 kgauss and saturation effects in the magnet. The effective field length tends to be oval rather than rectangular. Since one has to obtain constant vertical focusing independent from the $x$ co-ordinate, the magnets had to be shimmed. Since the field shape changes with momentum, the shimming limits the beam to one momentum. So the $k_3$ was a fixed-momentum beam for 0.8 GeV/c, with a total length of 24 m.

M. Morpurgo and G. Petrucci \(^4\) recently proposed an extremely short kaon beam for counter use (Fig. 8). They gain length by putting the bending magnets on the drift spaces close to target and second mass slit, by having the beam convergent through the separators and, generally, by using special elements, such as 1 m separators. They finally reach the following parameters:

- length 10 m
- solid angle $5 \times 10^{-3}$ sr
- momentum bite ± 1.5%
The septum separator\(^5\) was recently developed at Berkeley by J. Murray. In principle, the shortest possible solution of a beam would be one separation stage with momentum analysis. This is conventionally impossible because of \(\pi\mu\) decay.

For \(p = 400\) MeV/c, Murray developed a multiple separation system in one stage. This is the septum separator (Fig. 9) consisting of about 30 plates of 1.5 mm each, at a distance of 1.5 mm and a field of 150 kV/cm with alternating polarity. For a proper value of a crossed field \(B\) to compensate \(E\), one gets several separations over the length of one metre in each second gap, so that the optimum relative aperture is 0.25. Some months ago the system was tested and did not work, probably for reasons of multiple scattering in the plates. The mechanical set-up was made with short separated plates. Taking out each second section, suppressing the magnetic field and with proper tuning of the electric field, the kaons oscillate through a periodic field, whereas the pions are out of phase and get lost.

Parameters:
- length: 12 m
- solid angle: 0.5 sr
- momentum bite: ± 2%
- flux at 380 MeV/c: 10 K\(^-\)/10\(^{12}\) p

At the CERN PS a similar beam would have a flux about 10 times higher since one would start with 20 rather than at 6 GeV proton energy.

---

**REFERENCES**

1) S. van der Meer, CERN 63-3.
4) G. Morpurgo and G. Petrucci, CERN NP internal report.
5) R.D. Tripp, private communication.
Figure captions

Fig. 1: The principle of a separation stage.

Fig. 2: Separation of particles in phase space representation.

Fig. 3: Layout of the k_4 beam.

Fig. 4: Optics of the k_4 beam.

Fig. 5: Separation curve.

Fig. 6: Layout of the k_3 beam.

Fig. 7: Principle of edge focusing.

Fig. 8: The Morpurgo-Petrucci proposal.

Fig. 9: Principle of the septum separator.
\[ \mu^+ + \pi^+ \]

- \( p = 800 \text{ MeV}/c \)
- \( E_1 = 40 \text{ KV/cm} \)
- \( E_2 = 42 \text{ KV/cm} \)
- \( MS_1 = 2.3 \text{ mm} \)
- \( MS_2 = 4.0 \text{ mm} \)

**FIG. 5**
Fig. 7
III. RADIO FREQUENCY SEPARATED BEAMS

E. Keil

Accelerator Research Division
1. **PRINCIPLE OF r.f. SEPARATORS**

Assume that a momentum analyzed and nicely collimated pencil beam containing two types of particles comes from the left in Fig. 1.

In a first r.f. cavity r.f.1, a transverse force deflects the particles and modulates the exit angle at the frequency of the r.f. At a certain moment the particles may have the angle indicated in Fig. 1. The beam is then focused into the second r.f. cavity r.f.2 by a lens system which is represented here by a single lens.

Note that the two different types of particles are deflected in exactly the same way when they pass the first cavity at the same time. However, their arrival times at the second cavity are slightly different because of their velocities. The distance between a pion and a kaon at 10 GeV/c, which were together in the first cavity, is about 5 cm in the second cavity.

If we decide to adjust the relative r.f. phase between r.f.1 and r.f.2 such that the "full line" particle passes r.f.2 at the same phase as r.f.1 it will again be deflected by the same amount. In this way the second deflection exactly cancels the first one because the magnification of the optical system was assumed to be -1. If this phase relation between the cavities is maintained all the time this cancellation obviously happens for all entry phases into r.f.1.

The "dashed line" particle arrives at r.f.2 at a different time, and therefore its deflections do not cancel such that it leaves r.f.2 with some residual deflection which, of course, is also modulated by the r.f.

Since the r.f. frequency is about 3000 MHz even single FS bunches with a duration of about 10 nsec occupy several oscillation periods of the r.f. Therefore the incoming particle beam has to be considered as continuous from the r.f. point of view, and, as a consequence, all entry phases are almost equally likely.
Because of this fact a r.f. separator of this kind can only make sure that the "full line" particle is always near to the optical axis whereas the other one is always swept across the whole aperture. Since we want a pure as possible a beam of wanted particles we are forced to call the "full line" particles the unwanted ones and to eliminate them with a central beam stopper (BS). This causes a certain loss of wanted particles as well since there are entry phases where the final deflection is zero or very small, but this cannot be overcome in this type of r.f. separator.

2. SOME FORMULAE

After these qualitative arguments we want to be a little more quantitative and write down some simple formulae.

The phase angle \( \tau \) (in radians) between the two types of particles is given by

\[
\tau \equiv \frac{L}{2c} \frac{\mathcal{W}_1^2 - \mathcal{W}_2^2}{(\zeta \rho)^2} \text{ for } \beta \approx 1 ,
\]

(1)

where \( L \) is the distance between r.f.1 and r.f.2, \( \omega = 2\pi \tau \) is the circular frequency of the r.f., \( \rho \) is the momentum, and \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are the rest energies of the particles. The formula is valid in the highly relativistic approximation \( \beta \approx 1 \).

The deflection \( \psi \) of the wanted particle at the exit of r.f.2 becomes

\[
\psi = -\varphi \sin \omega t + \varphi \sin(\omega t + \tau)
\]

(2)

where \( \varphi \) is the deflection in each cavity separately, and the sign comes from the magnification of the optical system between r.f.1 and r.f.2.

We can transform this expression into

\[
\psi = 2\varphi \cos \left(\omega t + \frac{\tau}{2}\right) \sin \frac{\tau}{2}
\]

(3)
which gives the final deflection as a product of two terms: a rapidly oscillating cosine term showing the r.f. modulation; and a constant sine term containing only the phase angle $\tau$. This formula shows what is already obvious from intuition: the deflection of the wanted particles is maximum if the phase angle between the two particle types is $\pi$ or, more generally, $\pi + 2\pi n$. In this case the final deflection of the wanted particles is the sum of the deflections in the two cavities.

Since it is the final deflection which determines the acceptance of the r.f. separated beam, smaller deflections in each single cavity, and thus also less r.f. power, are required at $\tau = \pi + 2\pi n$ than at neighbouring values of $\tau$.

Therefore we call the momentum where $\tau = \pi$ the design momentum $p_0$ of the r.f. separator and define it by the equation

$$\frac{L}{\lambda(p_0 c)^2} = \frac{1}{\omega^2 - \omega_0^2}.$$  \hspace{1cm} (4)

For the present CSRN separator we have $L = 50$ m, $\lambda = 0.105$ m, $p_0 = 10.32$ GeV/c for K\(\pi\) separation.

It is also obvious [Eq. (3)] that there are momenta where $\tau = 2\pi n$ and where therefore also the deflections of the wanted particles cancel because they are r.f.-wise identical to the unwanted ones. Thus there are momenta where two particular types of particles cannot be separated from each other. This will be useful later.

3. ACCEPTANCE

Electrostatic separator tanks have an acceptance much bigger than the emittance that can be achieved in a high-energy electrostatically separated beam. They can therefore almost be considered as pieces of vacuum pipe during the optical design of the beam. This is quite different for r.f. separators and r.f. separated beams. Here the acceptance is much smaller than that of the vacuum pipe, and the emittance of the
beam is bigger than that of a comparable electrostatic one. In fact it turns out that acceptance and emittance are about equal. Therefore considerations of how to fit the emittance of the beam into the limited acceptance of the r.f. cavities must enter fairly early into the design of a r.f. separated beam.

Let us then look at the vertical phase plane in the middle of the first cavity. If we consider a cavity as a fairly long square pipe its vertical acceptance looks like Fig. 2. The two pairs of parallel lines correspond to the aperture limitations due to the entrance and due to the exit of the r.f. cavity.

Figure 1 can also be drawn in the vertical phase space and then becomes Fig. 3 a-e. The unwanted particle is represented by a full circle, and the wanted particle by an open one. Figure 3a shows the situation before the deflections which are assumed to be kicks in the centre of r.f.1 and r.f.2: the two particles are together at the origin. Figure 3b shows the situation after the deflection in r.f.1: the two particles are still together, but away from the origin. Figure 3c is before the deflection in r.f.2. The picture is Fig. 3b turned by 180° due to the -1 magnification. Figure 3d is after the deflection in r.f.2: the two particles are now separated in angle, but not yet in space. Figure 3e is plotted at some distance behind r.f.2 where the particles have drifted apart. A central beam stopped, represented by the vertical hatched area, may now catch the unwanted ones.

In order to be useful for practical purposes, the initial beam must occupy a finite area in phase space. Let us assume that one of the acceptance limitations (it will be the target in practical cases) is focused into r.f.1. Then we have a situation as given in Fig. 4a. The vertical lines correspond to the target and the horizontal ones to some other limiting aperture, say a collimator. They need not be exactly horizontal but should be approximately so to maximize the hatched area.

On top of this initial area we need space for the r.f. deflection which has to go in the y' direction and may turn Fig. 4a into Fig. 4b.
This shape of the accepted area is now transformed into r.f. 2 by the lens system between r.f. 1 and r.f. 2. It is obvious that the shape should not change much when the two cavities are identical, which they probably are for engineering reasons. Also, it is quite clear that the optimum way of using r.f. power is to have equal deflections which are achieved by having the angular magnification between r.f. 1 and r.f. 2 equal to ± 1.

Altogether, these arguments and some others to be given below, indicate that a matrix of the form

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]  

(5)

would be a very good choice. In this case Fig. 4b still gives the phase-space configuration in r.f. 2 before the deflection. After the deflection we have Fig. 4c with twice the deflection for the wanted particles and no deflection for the unwanted ones.

There is obviously an optimum where the maximum phase-space area of wanted particles is transmitted. If one starts with a very small area at the beginning, one can deflect very much and one loses rather little on the BS. If one almost fills the acceptance of the cavities at the beginning, one can only deflect very little and lose almost everything on the BS. It can be shown that for uniform particle density in phase space the optimum is achieved by dividing the total acceptance in the ratio 1 : 1 : 1. This case is plotted in Fig. 4.

A further reason that makes the matrix (5) very desirable is the following fact:

A practical deflecting structure must have finite length and thus the deflection occurs smoothly over its entire length. When we use travelling waves moving in the direction of the particles it can be shown that, if the deflecting field is uniform along its length, the effect

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of such a finite deflecting structure is exactly equal to that of a
single kick of the same angle in the middle of the cavity if the
particle velocity is equal to the phase velocity \( v_{ph} \) of the r.f. wave.
If this is not the case, a reduction of the total deflection occurs
and also some virtual displacement \( a \). The two cases are shown in
Fig. 5.

Since for the unwanted particles the two deflections are in
the same direction but with a \(-1\) magnification in between the effects
of the deflections including the small virtual displacement do cancel
if the matrix has the form (5).

4. **OPTICAL SYSTEM BETWEEN r.f.1 and r.f.2**

The property (5) can be achieved in the vertical plane with
a symmetrical triplet of quadrupoles. Everything apart from \( a_{21} = 0 \)
is obviously achieved with any symmetrical triplet just because of the
symmetry, and only the condition \( a_{21} = 0 \) imposes particular positions for
the quadrupoles.

In thin lens approximation they can be calculated very easily.
We consider only half the triplet because it contains already all the
parameters.

In Fig. 6 we want the two beams starting on the optical axis
to be parallel to it in the middle, and on top of this we want the verti-
cal parallel off-axis ray to cross the optical axis in the middle of the
system. It is obvious that the first lens must be vertically focusing
to achieve this.

So we want to have the following matrix elements zero:

\[
M_V = \begin{pmatrix}
0 & a_{12} \\
 a_{21} & 0
\end{pmatrix}
\]

(6)

\[
M_H = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & 0
\end{pmatrix},
\]
where the lower zeros render the beam parallel in the middle, and the upper zero in $M_Y$ makes the parallel beam cross the optical axis.

Taking the matrices of the elements, multiplying them together, and solving the equations for the matrix elements yields

$$Z_2 = \left( \frac{3}{2} - \sqrt{\frac{5}{4}} \right) (Z_1 + Z_2)$$

$$F_1 = \pm Z_2$$

$$F_2 = \pm \frac{1}{2} Z_1 \ .$$

The second equation is obvious from an inspection of the trajectories in Fig. 6.

The thick-lens solutions obtained with a computer programme are in close agreement with Eq. (7). In the actual $a_2$ beam the triplet was replaced by a quadruplet because we wanted the central space for an electrostatic separator. Also in this case the positions of the outer lenses do not change much; the inner ones are just shifted towards them.

In the horizontal plane the matrix element $b_{21} \neq 0$, it cannot be made equal to zero at the same time as $a_{21}$. Thus in the horizontal plane some change of the shape of the accepted phase area between r.f.1 and r.f.2 and a corresponding loss in acceptance cannot be avoided.

5. **PRE- AND POST-CAVITY OPTICAL SYSTEMS**

In order to obtain the horizontal limiting lines in the vertical acceptance diagram in Fig. 4a we need a collimator, the angle-defining slit, which is imaged to infinity in r.f.1. This can be done with a quadrupole pair in front of r.f.1 which at the same time images the target into r.f.1. A ray diagram of this system is shown in Fig. 7.

The reversed system may be used after r.f.2 to image the horizontal limiting lines in Fig. 4c onto the BS and to produce another vertical image of the target somewhat later, as shown in Fig. 8.
In both cases the polarities have to be chosen in the way indicated in order to get real positions for the angle-defining slit and the BS.

The optical systems in front, between, and after the r.f. cavities form the separation stage of a r.f. separated beam. As usual in all separated beams it is preceded by a momentum-analysing and beam-defining part and followed by another momentum-analysing and beam-shaping section.

The momentum analyser in the $\text{O}_2$ beam is very simple; it just contains two quadrupoles and two bending magnets. In order to make the beam dispersion free in position and in angle over most of its length, two bending magnets were added to the stage just in front of r.f.1. Two horizontally focusing quadrupoles in the middle between the two groups of bending magnets then yield this dispersion-free property.

The momentum analyser at the end is shorter than that at the beginning because of lack of drift space. As a consequence, its resolution is fairly poor.

A general problem of r.f. separated beams is matching the emittance of the target to the acceptance of the r.f. cavities. Since targets usually have a few mm$^2$ cross-section, whereas the beam size that fits the cavity dimensions best is about 2 cm $\times$ 2 cm, a very big magnification is wanted between the target and r.f.1, especially in the vertical plane where r.f. power limits the practicable deflections to about 1 mrad.

Usually, simple quadrupole doublets tend to be rather symmetrical in layout, and therefore have magnifications $m_H$ and $m_V$ such that $m_H \cdot m_V \approx 1$. Thus one loses in one plane what one gains in the other one.

Brookhaven attempt to overcome this difficulty by suppressing the intermediate vertical focus in front of the angle-defining slit.
6. **BACKGROUND CONSIDERATIONS AND COLLIMATOR ARRANGEMENT**

In the $\alpha_2$ beam the background, especially due to muons from decay in flight of pions, was expected to be rather high, mainly because of the following two reasons:

i) The phase area in both planes handled by the beam was rather high, much higher than in the electrostatic version of the $\alpha_2$ beam. This is entirely due to the targeting technique used where the circulating proton beam is kicked onto the internal target with a fast kicker.

ii) The emission of muons is more and more peaked in the forward direction when the momentum of the pions is increased. Correspondingly, their density in phase space is higher than at lower momenta.

In order to obtain a reasonably pure beam despite these additional difficulties, we tried to define the beam well in front of and behind the separation stage with a large number of adjustable collimators.

Five collimators define the beam before the separation stage: two of them limit the horizontal and vertical divergence, one is a momentum slit, one redefines the vertical target image size, and one is the angle-defining slit.

At the end, there are four collimators and the beam stopper: two of them sit on images of the target, and the other two and the BS sit on images of collimators which do not contain target images. In this way the phase area defined by the entrance collimators is rigorously redefined at the end. One should expect that only background which is really inside the phase area of the wanted particle beam can get into the bubble chamber.

7. **OPERATION OF A R.F. SEPARATED BEAM**

Up to now only the separation of two particle types from each other was considered. However, there are normally three types of
particles in a beam: p, π, and K. If we want to eliminate two of them
at a time we have to choose the momentum and drift-space between the
cavities such that the phase difference between the two unwanted particles
is a multiple of 2π. This is the case for p and π at 10.12 GeV/c
which fortunately is also very near to 10.32 GeV/c, the design momentum
for Kπ separation. Because one can simultaneously eliminate π and p and
get the K's with high efficiency at 10.12 GeV/c, this momentum was chosen
for the initial running of the Ω beam.

The fluxes of K's achieved were just sufficient for bubble
chamber use, something like seven K⁻ per picture in the first run, and
five K⁻ per picture in the second one. This low flux is mainly due to
the low efficiency of the target, and could be substantially improved by
an external target.

The background was about the same as the K⁻ flux, and consisted
mainly of μ⁻ when the beam was well adjusted. Although this figure
agrees with the expected one within a factor of two it is still very high.
The elimination of this background puts severe conditions on the design
and performance of any future r.f. separated beam. However, it appears
possible to improve the background substantially by starting with a smal-
lier initial phase area, as one does with an external target, and by
building a final momentum analyser with the same resolution as the initial
one.

8. COMPARISON WITH ELECTROSTATIC SEPARATORS

One may ask the question why r.f. separators are really so much
superior to electrostatic ones. The basic reason is, of course, that
r.f. separators are time-of-flight devices with a time resolution of, say,
half a r.f. cycle or 0.16 nsec. This means that one does not rely on the
small difference of two large deflections but uses the time variation of
the deflecting force to make the deflections for different particles dif-
ferent.
This argument can be written down precisely. The difference in angle $\Delta x'$ is for electrostatic separators in the relativistic approximation:

$$\Delta x' = \frac{E_s}{p_c} \Delta \left( \frac{1}{\beta} \right) \approx \frac{E_s}{2(p_c)^3} (\mathcal{W}_1^2 - \mathcal{W}_2^2)$$

(8)

and for r.f. separators

$$\Delta x' = \frac{E_s}{p_c} \sin \left[ \frac{\pi}{2} \left( \frac{p_0}{p} \right)^2 \right],$$

(9)

where $p_0$ is the design momentum, $E$ the electric field strength, and $s$ the total length of the separator(s). In the neighbourhood of $p_0$ the sine is close to unity and $\Delta x'$ varies only as $p^{-1}$. Comparing Eq. (8) and Eq. (9) shows that the small term $\Delta(1/\beta)$ is absent in Eq. (9).

At momenta well above $p_0$ one can replace the sine by its argument. We can then use Eq. (4) with Eq. (9) to give:

$$\Delta x' \approx \frac{E_s}{2(p_c)^3} (\mathcal{W}_1^2 - \mathcal{W}_2^2) \left[ \frac{4\pi L}{\lambda} \right].$$

(10)

This expression differs from that for electrostatic separators [Eq. (8)] by the additional factor $-4\pi L/\lambda$ which can easily increase the deflection angle by three orders of magnitude.
Figure captions

Fig. 1: Principle of r.f. separation.

Fig. 2: Vertical acceptance of a r.f. cavity.

Fig. 3: Phase-space representation of r.f. separation.

Fig. 4: Separation of a beam with finite emittance.

Fig. 5: Influence of the r.f. phase velocity.

Fig. 6: Ray diagram of one-half of the optical system between r.f. 1 and r.f. 2.

Fig. 7: Ray diagram of the pre-cavity optical system.

Fig. 8: Ray diagram of the post-cavity optical system.
Fig. 3

- unwanted particle
- wanted particle
Fig. 4

unwanted particles

wanted particles
Fig. 5

Fig. 6