CONTINUOUS GEOMETRY AND SUPERSELECTION RULES

by

J.M. Jauch

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Introduction.

This is an expanded version of a seminar talk at CERN given on March 8, 1961. In keeping with the informal character of these seminars, this report is on a similar informal level. It is an attempt to present several of the important results on the theory of operator rings in a Hilbert space on an intermediate level of sophistication and to show several applications of this theory to problems in quantum mechanics. Some of these applications have already appeared in the literature. Others are in preparation for future publication.

The mathematics draws heavily from the monumental work of Murray and von Neumann on rings of operators. Indeed the first part of these notes is merely a short exposition of some aspects of this theory in a context of physical interpretations.

The relevance of this mathematics for modern theoretical physics derives from the fact that there are physical systems with superselection rules. The observables of such systems generate therefore operator rings which are not trivial and it becomes possible to distinguish structural properties of physical systems which have nothing to do with the dynamical properties of the system. We shall refer to them as kinematical structures. The possibilities for such structural varieties turn out to be much richer than one might have expected either from the classical analogy or from the quantum mechanics in finite dimensional Hilbert spaces. Thus it becomes a problem of some importance to gain an insight in the physical meaning of such structural varieties.
SECTION I

Observables and rings of bounded operators

(1) State vectors and observables

We take as starting point conventional quantum mechanics: pure states are represented as normalised elements of a separable Hilbert space $\mathcal{H}$ with positive definite norm and observables are represented by self adjoint operators on $\mathcal{H}$.

The association of self adjoint operators with measurements has always remained a bit of a mystery. The simplest assumption is that every such operator can actually be measured and hence represents an observable. In certain simple cases, this association can be explicitly given. In most cases this is not so. In the classical literature on the subject this question is usually by-passed by a bold assumption. For instance Dirac \(^1\) answers the question: "Can every self-adjoint operator be measured?" in the following words: "The answer theoretically is yes. In practice it may be very awkward, or perhaps beyond the ingenuity of the experimenter, to devise one apparatus which could measure some particular observable, but the theory always allows one to imagine that the measurement can be made"

This assumption is considerably weakened by the discovery that there are important physical systems for which not every self-adjoint operator is an observable \(^2\).

This remark accentuates the empirical aspect of the question. The system of observables express the relations of the physical system to the possible measurements which can be performed on the system and nothing can be said a priori about this relation. We shall therefore take the empiricist's point of view according to which the structure of this relation should be ultimately determined by the observations on the system.
For us the problem becomes thus to give an explicit and precise formulation of the possible structure of operators which may represent observables and to investigate the different possible physical consequences which may be associated with the different structures.

The simplest self-adjoint operators are the projections. If they are observable they correspond to the type of experiment which has only two possible answers. We call these "yes-no" experiments. The measurement of any quantity can always be replaced by a certain set of yes-no experiments and this corresponds mathematically to the fact that every self-adjoint operator can be completely characterized by a family of projection operators. This theorem of the spectral resolution of self-adjoint operators is fundamental in quantum mechanics and we shall use it implicitly throughout these notes.

(2) The operator ring generated by observables

The last allusion of the preceding section can be put in a more general context, which we shall motivate by a physical consideration. Suppose we have measured an observable \( A \) and we have found an eigenvalue \( a \) of \( A \) leaving the system in a state \( \psi \) which satisfies an eigenvalue equation

\[
A \psi = a \psi
\]

Such a measurement can also be considered a measurement of the observable \( A^2 \) for which we obtain the value \( a^2 \). Similarly if \( u(A) \) is some function of \( A \) we may consider it measured too with a value \( u(a) \).

A measurement of a quantity \( A \) is thus simultaneously a measurement of a host of other quantities, namely all those which can be represented as a function of \( A \).

In order to make these heuristic considerations mathematically precise we must give a meaning to a function of an operator. Here we shall use a short cut which allows a great simplification and circumventor of technical
details which are of no interest to physics. Instead of defining the class of functions $u(\lambda)$ for which an operator function $u(A)$ has a meaning, we shall directly define the class of operators which can be thus represented. I shall precede this with a few definitions.

**Definition:** Let $A$ be a self-adjoint operator. A bounded operator $T$ is said to commute with $A$ if it commutes with all the spectral projections of $A$.

Why this roundabout way of defining a commuting operator? The reason is that $A$ may be unbounded, in which case it can only be defined on a dense linear manifold of $\mathcal{H}$. The operator $T$ may mix up this manifold in such a way that the commutator of $T$ with $A$ may not be definable on a dense set. For bounded operators $A$ the above definition is equivalent with the usual one. For unbounded operators it is the correct generalisation of this definition.

**Definition:** Let $\mathcal{J}$ be a set of self-adjoint linear operators. The **commutant** $\{\mathcal{J}'\}'$ of $\mathcal{J}$ is the set of all bounded operators $T$ which commute with all the operators in $\mathcal{J}$.

The commutant is an operator algebra. This means with $T_1$ and $T_2$ it contains $\lambda T_1 + \mu T_2$ and $T_1 T_2$ for arbitrary complex $\lambda$ and $\mu$. It also is a $*$-algebra since with $T$ it also contains $T^*$. It has a topological closure property which says that the algebra is closed in the weak topology.

Such an algebra shall be called a von Neumann algebra. Every von Neumann algebra $\mathcal{H}$ satisfies the double commutant relation $\mathcal{H}'' = \mathcal{H}$.

**Definition:** The von Neumann algebra $\mathcal{H}$ generated by a set $\mathcal{J}$ of self-adjoint operators is the double commutant of $\mathcal{J}$

$$\mathcal{J}'' = \mathcal{H}$$

It is the smallest von Neumann algebra which contains the set $\mathcal{J}$. 

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The idea is to replace the study of sets of observables by the study of von Neumann algebra which they generate. The algebra is mathematically a simpler object than sets of observables because it is unique while representative sets of observables can be chosen in many different ways. Note that a von Neumann algebra consists only of bounded operators. This is from the mathematical point of view an enormous simplification. It does not mean that we exclude the unbounded operators from our considerations. If we did we would have lost too much, since some of the most important observables are unbounded.

For unbounded self-adjoint operators we can introduce the concept of affiliation.

**Definition:** An unbounded self-adjoint operator $A$ is affiliated with a von Neumann algebra $\mathcal{H}$ if all the spectral projections of $A$ are contained in $\mathcal{H}$.

All the operators in $\mathcal{H}$ are affiliated with $\mathcal{H}$.

We are now equipped for defining the operators which can be represented as a bounded function of a self-adjoint operator $A$. The set of such operators is identical with the von Neumann algebra generated by $A$. We denote it by

$$\mathcal{A} = \{ A \}''$$

The algebra $\mathcal{A}$ is abelian. This means it is contained in its commutant

$$\mathcal{A} \subseteq \mathcal{A}''$$

If the commutant $\mathcal{A}''$ is also abelian, we call $\mathcal{A}$ maximal abelian

$$\mathcal{A} = \mathcal{A}''$$

What we have just done for one operator can be easily extended to sets of commuting operators: if $\{ A_i \} = \mathcal{H}$, $i \in I$ is a set of commuting
observables then the bounded functions of all these observables is the abelian von Neumann algebra generated by them

\[ \mathcal{A} = \mathcal{R}'' \]
\[ \mathcal{A} \subseteq \mathcal{A}' \]

**Definition**: A commuting set of observables \( A_i = \mathcal{R} \ i \in \mathbb{I} \) is said to be a complete set of commuting observables if the algebra generated by them is maximal abelian

\[ \mathcal{R}'' = \mathcal{A} = \mathcal{A}' \]

This definition is identical with the conventional (but less general) definition which requires for such a set to have only non-degenerate common eigenvalues when the latter is applicable. It is more general since it also includes the cases of operators with continuous spectrum.

However, the application of this method of generating operator rings is not restricted to commuting observables. It can be applied equally well to the set \( \mathcal{R} \) of all observables. We define

\[ \mathcal{H} = \mathcal{R}'' \]

This algebra is not abelian (if it were, the system would be classical). Its commutant \( \mathcal{H}' \) is therefore in no particular relation to \( \mathcal{H} \).

(3) **Superselection rules and irreducible systems**

What is a superselection rule? The method of operator rings is particularly suitable for answering this question precisely and succinctly:
Definition: A system has a superselection rule if the set $\mathcal{H}$ of all observables generate an operator ring $\mathcal{H} = \mathcal{H}'''$ with non trivial commutant $\mathcal{H}'$.

Non trivial means that $\mathcal{H}'$ contains an operator other than a multiple of the identity.

If, on the other hand, the operator ring $\mathcal{H}'$ is trivial we shall speak of an irreducible system. The ring $\mathcal{H}$ is then identical with the ring $L$ of all bounded operators.

It is obvious that the irreducible systems have no particular structure. Unless one makes a further specification of the physical properties of the system it is not possible to distinguish one irreducible system from any others since the ring of all bounded operators is the only means of identifying this structure.

The situation is quite different when there are superselection rules. Because now the types of systems are distinguished by the different types of operator algebras $\mathcal{H}$ with non trivial commutant $\mathcal{H}'$. If we knew the answer to the question: What are the possible structures of operator algebras $\mathcal{H}$, then we would have gained some insight into the possible structure of physical systems with superselection rules. It is precisely this question which was answered in complete generality by Murray and von Neumann. We shall give a synopsis of their result without, of course, entering into any details of the proofs. The answer can be given in two parts. The first part contains the reduction of the general operator rings to a special subclass of such rings, called factors. The second analyses the structure of factors. This is quite analogous to the theory of group representation except that the irreducible representations of groups correspond to the more general concept of the factors for rings.
(4) Reduction to factors

Definition: The centre $\mathcal{Z}$ of an operator algebra $\mathcal{K}$ is the intersection of $\mathcal{K}$ with its commutant $\mathcal{K}'$.

$$\mathcal{Z} = \mathcal{K} \cap \mathcal{K}'$$

It is easy to verify that the centre is always an abelian von Neumann algebra. The size of the centre is, roughly speaking, something like a measure for the complexity of the algebra $\mathcal{K}$ or $\mathcal{K}'$. For this reason the algebras with trivial centre are good candidates for the building blocks of a reduction theory.

Definition: A factor $\mathcal{K}$ is a von Neumann algebra with trivial centre

$$\mathcal{K} \cap \mathcal{K}' = \{ \lambda I \}$$

In order to implement the reduction theory, it is necessary to say a few things about the direct integral of Hilbert spaces. This is a generalisation of the direct sum which is easy and familiar. So we start with it.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces of dimensions $n_1$ and $n_2$. In each of these spaces we select an arbitrary element $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$. The pairs of elements

$$\Psi = \{ \psi_1, \psi_2 \}$$

form a linear vector space if one defines the linear operations

$$\lambda \Psi + \mu \Psi = \{ \lambda \psi_1 + \mu \psi_2, \lambda \psi_1 \rightarrow \mu \psi_2 \}$$

if $\Psi = \{ \psi_1, \psi_2 \}$.
and the scalar product

\[(\psi, \varphi) = (\psi_1, \varphi_1) + (\psi_2, \varphi_2)\]

This linear vector space is again a Hilbert space and it is the direct sum of the two spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\)

\[\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\]

There is nothing to prevent us from doing this for any finite number of spaces and define more general direct sums. But let us jump over these trivial generalisations and come to the general direct integral.

Let \(\mathcal{P}(\lambda)\) be a positive, non-decreasing function of the real variable \(\lambda\), normalised such that

\[\int^{\infty}_{-\infty} \mathcal{P}(\lambda) = 0\]
\[\int^{+\infty}_{-\infty} \mathcal{P}(\lambda) = 1\]
\[\mathcal{P}(\lambda + 0) = \mathcal{P}(\lambda)\]

Then we can define a Lebesgue-Stieltjes integral with respect to \(\mathcal{P}(\lambda)\) in a standard fashion. Let \(n(\lambda)\) be a measurable function of \(\lambda\) with values \(0, 1, 2, \ldots, \infty\) and consider the family of Hilbert spaces \(\mathcal{H}_\lambda\) with dimension \(n(\lambda)\). We consider collections of vectors \(\psi_\lambda \in \mathcal{H}_\lambda\) such that \((\psi_\lambda, \varphi_\lambda)\) is measurable and integrable and we define a linear vector space for the quantities \(\psi = \{\psi_\lambda\}\) by setting

\[\alpha \psi + \beta \varphi = \{\alpha \psi_\lambda + \beta \varphi_\lambda\}\]

if \(\varphi = \{\varphi_\lambda\}\)

and a scalar product

\[(\psi, \varphi) = \int (\psi_\lambda, \varphi_\lambda) \, d\mathcal{P}(\lambda)\]
These \( \mathcal{H} \) form again a separable Hilbert space and this space is called the direct integral

\[
\mathcal{H} = \bigoplus \mathcal{H}_\lambda \sqrt{d\rho(\lambda)}
\]

**Remarks:**

The direct integral is characterised by two functions:

1. the dimension function \( \gamma(\lambda) \)
2. the weight function \( \int \rho(\lambda) \)

There are some trivial modifications which can be made on these functions which do not change the direct integral. For instance the numbers \( n(\lambda) \) can be permuted in some way, or they can be changed on a set of measure zero with respect to \( \rho \). Similarly \( \rho(\lambda) \) can be replaced by another function \( \sigma(\lambda) \) which has the same sets of measure zero as the function \( \rho(\lambda) \). All these changes are considered trivial and two direct integrals which differ only by much trivial changes are not considered different from one another.

We come now to the most important theorem of the reduction theory.

**Theorem:**

Every von Neumann algebra \( \mathcal{H} \) determines a unique decomposition of the Hilbert space into a direct integral in such a manner that every operator \( T \in \mathcal{H} \) can be represented as a collection of operators \( T_\lambda \) operating only on the component spaces \( \mathcal{H}_\lambda \). As \( T \) runs through \( \mathcal{H} \) the operators \( T_\lambda \) run through a factor \( \int \).

This theorem reduces the study of general operator algebras to that of factors. The whole complexity of the general algebras is contained in the functions \( \int \rho(\lambda) \) and \( n(\lambda) \) and in the structure of factors.
(5) The structure of factors

The preceding sketch of the reduction theory has no surprising aspects. Indeed these results are closely related to the corresponding results in the representation theory of groups.

However, the structure theory of factors is quite a different story. One might get at the study of factors in the obvious way by studying them first in finite dimensional spaces and then generalising the results to infinite dimensional ones. The finite dimensional factors are very easy. Some quite elementary considerations give the following result: to every factor \( f^\prime \) in a finite dimensional Hilbert space \( \mathcal{H} \) corresponds a direct product

\[
\mathcal{H} = \mathcal{K}_1 \times \mathcal{K}_2
\]

of two other spaces such that \( f^\prime \) acts irreducibly in \( \mathcal{K}_1 \) and \( f^\prime \) acts irreducibly in \( \mathcal{K}_2 \). In the special case that \( f^\prime \) is already irreducible \( \mathcal{K}_2 \) is of dimension 1. In any case both \( f^\prime \) and \( f^\prime \) are isomorphic to an irreducible algebra of bounded operators in some finite-dimensional space.

If this result were also true for infinite-dimensional spaces then the entire structure theory of operator rings would be finished with a rather simple result. However, the central discovery of Murray and von Neumann is that this is not so. The factors in infinite-dimensional spaces have a much richer structure, something that could not have been guessed from the analogy to the finite-dimensional case.

The crucial point in the discussion is the occurrence of minimal projections. In the finite-dimensional case every projection contains in its range at least the subspace \( \mathcal{H}_2 \). There is no projection in \( f^\prime \) with a smaller range. If such minimal projections occur in the infinite dimensional case then indeed the structure of the factor is essentially the same as the structure of finite dimensional factors.
But do there always exist such minimal projections in a ring? There is no difficulty proving this in finite dimensions but the proof cannot be extended to infinite dimensions because the assertion is not true.

In order to clarify the structure theory of factors Murray and von Neumann generalized the problem. Instead of establishing the existence of a factor of a particular size they developed a theory of comparability of projections.

There are two basic notions which are needed for such a theory, one of equivalence of projections and the other one an ordering of projections.

**Definition**: Two projections \( E \) and \( F \) in an algebra \( \mathcal{H} \) are said to be equivalent with respect to \( \mathcal{H} \) (\( E \sim F \)) if there exists a partial isometry \( W \in \mathcal{H} \) with initial projection \( E \) and final projection \( F \)

\[
W^*W = E \\
W^*W = F
\]

**Definition**: A projection \( E \in \mathcal{H} \) is inferior to a projection \( F \in \mathcal{H} \) (\( E \preceq F \)) if there exists a projection \( G \leq F \) such that \( E \wedge G \).

One verifies easily that the relation \( \preceq \) provides a partial ordering of projections in the algebra. All this is possible in any algebra, not just factors.

The remarkable point is that for factors the ordering relation \( \preceq \) is complete. In other words any projections in a factor are comparable. This result is crucial for the structure theory of factors.

In order to complete the structure theory of factors another important tool is necessary. This is the dimension function.
The dimension function on factors

It is trivial to verify that two equivalent projections have the same orthogonal dimensions. Therefore it is natural to try the definition of a general dimension function on projections of the factors which would be expected to satisfy the generalisation of the relations for the dimension function on finite projections.

These relations are summarized in the following:

**Definition:** A dimension function on a factor is a mapping \( \mathcal{C}(E) \) from the projections \( E \) in the factor to the non-negative real numbers (including \( \infty \)) which satisfies

1. \( \mathcal{C}(E) > 0 \) for \( E > 0 \)
2. \( \mathcal{C}(0) = 0 \)
3. \( \mathcal{C}(E) = \mathcal{C}(F) \) if and only if \( E \sim F \)
4. \( \mathcal{C}(E \cup F) + \mathcal{C}(E \cap F) = \mathcal{C}(E) + \mathcal{C}(F) \)

Such a function has, from the mathematical point of view, the characteristic properties of a measure. From the physical point of view it has another interpretation. In finite dimensional spaces the dimension function represents the **a priori probability** of a state. For instance, suppose the projection \( E \) represents a yes-no experiment on a system about which nothing is known, then the usual interpretation of quantum mechanics requires that \( \mathcal{C}(E) \) is the (relative) probability that an observation of \( E \) will give the result yes. The need for such an a priori probability in quantum statistics makes the existence of the dimension function a practical necessity.

The mathematical aspect of this question is very satisfactory because Murray and von Neumann could show that the dimension function with the above properties always exists and is (apart from normalisation) unique. The range of the dimension function belongs to one of three classes, giving rise to three different types of factors.
Type I 
\[ \mathcal{P}(E) = 0, 1, \ldots, n \]
subclasses

\[ I_n \quad n < \infty \]
\[ I_\infty \quad n = \infty \]

Type II
subclasses

\[ II_1 \quad 0 \leq \mathcal{P}(E) \leq 1 \]
\[ II_\infty \quad 0 \leq \mathcal{P}(E) \leq \infty \]

Type III
\[ \mathcal{P}(E) = \infty \quad \text{for all} \quad E > 0 \]

The factors of type I are just those factors which one obtains by a generalisation of the factors in finite dimensions. Factors of type II and III have no analogue in finite dimensions. They are new phenomena.

From the physicist's point of view, factors of type III are not interesting since they allow no measurable comparison of a priori probabilities for different projections. Type II factors, on the other hand, are candidates for systems of observables with interesting and non-trivial properties.

By constructing explicit examples factors of each type can be shown to exist.

We mention a few additional results which complete the structure theory of factors and which indicate also some of the complexity of these structures.

The factors \( \mathcal{F} \) and \( \mathcal{F}' \) are always of the same type.

The factor of type I can always be represented on a direct product space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) such that \( \mathcal{F} \) acts on \( \mathcal{H}_1 \) and \( \mathcal{F}' \) acts on \( \mathcal{H}_2 \).

Factors of type II and III cannot be so represented.
A factor is of type I if and only if it contains a minimal projection \( \neq 0 \).

Two factors which are ring isomorphic are of the same type but the converse is not true. There exist factors of type II and III which are not isomorphic. This last result shows that the dimension function is not a complete characterisation of the structure of factors. What the additional invariant characteristics are is not yet known.

A general von Neumann algebra (not necessarily a factor) can always be decomposed uniquely in a direct sum of three algebras each of which is a direct sum of factors of the same type.
SECTION II

Systems with complete sets of commuting observables (c s c.o.)

1. Maximal abelian algebra

We have already defined a c s c o as a set \( \mathcal{F} \) of observables which generate a maximal abelian algebra

\[
\mathcal{A} = \mathcal{F}^{\prime}\prime
\]

\[
\mathcal{A} = \mathcal{A}^\prime
\]

The existence of a c s c.o. is a physical requirement which expresses the fact that there should exist measurements which will determine the state free from redundancy in the description.

This (physically speaking) relatively mild requirement has rather drastic consequences:

From

\[
\mathcal{A} \subset \mathcal{F}^\prime
\]

and

\[
\mathcal{A} = \mathcal{A}^\prime
\]

follows

\[
\mathcal{F}^\prime \subset \mathcal{A}^\prime = \mathcal{A} \subset \mathcal{F}^\prime
\]

This relation says first of all that the commutant \( \mathcal{F}^\prime \) of \( \mathcal{F} \) is abelian and secondly that it is contained in \( \mathcal{A} \). A further consequence is that the centre is identical with the commutant

\[
\mathcal{Z} = \mathcal{F}^\prime
\]
This condition is thus necessary for the existence of a c.s.c.0. We claim now the converse.

**Theorem 3**): The necessary and sufficient condition for the existence of a maximal abelian subalgebra of a von Neumann algebra $\mathcal{H}$ is that the centre $\mathcal{Z}$ of $\mathcal{H}$ is identical with the commutant $\mathcal{H}'$ of $\mathcal{H}$.

**Proof**: Sufficiency is already proved by the preceding remarks. Thus we proceed with establishing necessity.

**Lemma 1**: Let $\mathcal{A} \subset \mathcal{H}$ be an abelian subalgebra of a von Neumann algebra $\mathcal{H}$ with abelian commutator algebra $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{A} = \mathcal{A}'$. Hence $\mathcal{A} = \mathcal{A}'$. In other words if $\mathcal{A}$ is maximal abelian in $\mathcal{H}$ it has no abelian extension outside $\mathcal{H}$.

**Proof of Lemma 1**: Assume $\mathcal{H}' \subset \mathcal{H}$, $\mathcal{A} \subset \mathcal{H}$, and $\mathcal{A} = \mathcal{A}' \cap \mathcal{H}$. We show first that $\mathcal{Z} \subset \mathcal{A}$. If this were not so there would exist an operator $z \in \mathcal{Z}$ and $z \notin \mathcal{A}$. But this operator is both in $\mathcal{H}$ (since it is in $\mathcal{Z} \subset \mathcal{H}$) and in $\mathcal{A}$ (since $\mathcal{Z} = \mathcal{H}' \subset \mathcal{A}'$). Hence $\mathcal{H}' \subset \mathcal{A}'$ and since $\mathcal{Z} = \mathcal{H}'$ also $\mathcal{H}' \subset \mathcal{A}$. Consequently $\mathcal{A}' \subset \mathcal{H}' = \mathcal{H}$. Thus $\mathcal{A} = \mathcal{A}' \cap \mathcal{H} = \mathcal{A}'$ and this proves Lemma 1.

**Lemma 2**: If $\mathcal{H}' \subset \mathcal{H}$ then every abelian algebra in $\mathcal{H}$ can be extended to a maximal abelian algebra contained in $\mathcal{H}'$. 

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Proof of Lemma 2:

Let $\mathcal{A}$ be the class of all abelian algebras in $\mathcal{H}$. It has non-trivial members, for instance, $\{T\}^*$ for any $T \in \mathcal{H}$, and it is partially ordered by inclusion. If $\mathcal{A}_o$ is a linearly ordered subclass of $\mathcal{A}$, then the union of all elements in $\mathcal{A}_o$ is in $\mathcal{A}$ and is an upper bound for $\mathcal{A}_o$. By Zorn's lemma, $\mathcal{A}$ contains a maximal element $\mathcal{A}_o$ for which

$$\mathcal{A}_o' = \mathcal{A}_o \cap \mathcal{H}$$

By lemma 1 it follows

$$\mathcal{A}_o' = \mathcal{A}_o$$

and this proves lemma 2.

The proof of the theorem is now as follows: $\mathcal{A} = \mathcal{H}'$ implies $\mathcal{H} \subseteq \mathcal{H}$. By lemma 2 there exist maximal abelian algebras in $\mathcal{H}$. This establishes the theorem.

(2) The type of algebras with maximal abelian subalgebras

What can we say about the structure of the algebra generated by observables? We shall answer this question by looking at the factors to which the algebra gives rise in the direct integral decomposition under the assumption of the existence of a c.s.c.c. The result is very simple: all the factors of such an algebra are of type I. This theorem is published. It is an immediate consequence of a somewhat more general theorem which is proved for instance in a paper by Neumark and Fomin. This says the following:

Let $\mathcal{H}$ be a von Neumann algebra, $\mathcal{H} = \mathcal{H} \cap \mathcal{H}'$ its centre and $\mathcal{H}_\lambda$ the Hilbert space of the unique direct integral representation which decomposes $\mathcal{H}$ into factors. Let $T \in \mathcal{H}$ and $T_\lambda$ the operator to which $T$ gives rise in the space $\mathcal{H}_\lambda$, so that
\[ T \{ \psi_\lambda \} = \{ T_\lambda \psi_\lambda \} \quad \psi_\lambda \in \mathcal{H}_\lambda \]

As \( T \) runs through \( \mathcal{M} \), \( T_\lambda \) runs through a factor \( \mathcal{J}_\lambda \). This factor is irreducible for almost all \( \lambda \) if and only if \( \mathcal{J} \) is maximal abelian in \( \mathcal{M}' \).

The application of this theorem to our situation is immediate. If \( \mathcal{J} \) is certainly maximal abelian in \( \mathcal{M}' \). Hence the theorem applies. The factors are almost all irreducible. But an irreducible factor is always of type I, since every projection is contained in it, in particular the minimum projections.

Conversely of course one can conclude: factors of type II and III do not contain maximal abelian subalgebras.

(3) Continuous geometries and hidden variables

Von Neumann has taken a great interest in the algebras of type II for several reasons. One of them is the fact that these algebras give rise to a generalisation of projective geometry which is called continuous geometry (or pointless (!) geometry). Another one is that algebras of type two when interpreted as operator rings generated by observables would describe physical systems with a continuous a priori probability for the yes-no experiments (projections). It is clear that such a system would have to be one with superselection rules. Furthermore, the operators which commute with the observables would have to generate a non-abelian algebra.

This latter condition means, as we have just shown, that for such systems there cannot exist a complete set of commuting observables.
Now such systems have features which resemble what one might like to call hidden variables. Indeed no set of measurements will ever succeed in determining the values of all degrees of freedom which occur in such systems. It could be expected that this situation would lead to indeterminacy or acausality of a much more radical nature than the ones which we are already familiar with in ordinary quantum mechanics.

We have already mentioned that the structure of the algebras generated by observables is ultimately an empirical question and its answer must be found in the fundamental laws of the physical world as they express themselves in the behaviour of physical measurements. We see now that the existence of a complete set of commuting observables is not something that should be considered as axiomatic either since it, too, is closely linked to the structure of such operator rings.

(4) Supersymmetries and essential observables

Every unitary or antiunitary transformation of the Hilbert space which commutes with the Hamiltonian gives rise to a symmetry transformation. The antiunitary among them are those which are associated with the time reversal transformation. A symmetry transformation may leave other observables invariant as well. We shall now study in particular those which leave all the observables invariant. This leads to the concept of supersymmetry.

**Definition**: A unitary operator $U$ different from the identity is a **supersymmetry** if it commutes with all observables.

It is clear that a supersymmetry means a superselection rule. Conversely a superselection rule always means there exists a supersymmetry. The proof for this is an application of a rather useful but simple theorem:

"Every von Neumann algebra is generated by its unitary operators"
Suppose then there is a superselection rule. Then $\mathcal{H}'$ is non-trivial, therefore it contains unitary operators other than the identity. (Since $\mathcal{H}'$ is generated by them). This means there is a supersymmetry. All this is quite general.

Now let us further assume that there exists a c.s.c.o. then there exist maximal abelian algebra in $\mathcal{H}$. Let $A_i$ (i $\in$ I an index set) be the set of these algebras, then their intersection

$$L^c = \bigcap_i A_i$$

is again an abelian subalgebra of $\mathcal{H}$.

**Definition**: The algebra $L^c$ is called the **core** of the algebra $\mathcal{H}$.

**Definition**: An observable in the core is called an **essential observable**.

**Theorem**: If $\mathcal{H}' = \mathcal{J}$ then the core is identical with the centre $\mathcal{J}$.

**Proof**:

Since $\mathcal{J} = \mathcal{H}' \subset A_i$.

it follows that $\mathcal{J} \subset \bigcap_i A_i = L^c$.

In order to prove conversely that $L^c \subset \mathcal{J}$ we need the

**Lemma**: Every operator $T \in \mathcal{H}$ is member of at least one maximal abelian algebra in $\mathcal{H}$.
Proof of lemma:

T generates the algebra \( \{ T \}'' \) and according to lemma 2 in section II.1 this algebra can be extended to a maximal abelian algebra in \( \mathcal{L} \). This proves the lemma.

Let now \( S \in \mathcal{L} \). If \( S \) were not in \( \mathcal{J} \), then there exists an operator \( T \in \mathcal{H} \) which does not commute with \( S \). But \( S \in \mathcal{A}_i \) for all \( i \in I \). Thus \( T \) cannot be in any \( \mathcal{A}_i \). This contradicts the lemma. Thus \( S \in \mathcal{J} \).

\[ \mathcal{L} \subseteq \mathcal{J} \]

and therefore \( \mathcal{J} = \mathcal{L} \). This proves the theorem.

This theorem says that the system has essential observables if and only if it has superselection rules. The essential observables are such that they must be represented in any c.s.c.c. For this reason I called them essential.

(5) An application to systems of identical particles

Let us consider a system of \( n \) identical particles. The space of state vectors \( \mathcal{G} \) of such a system is related to the space \( \mathcal{H} \) for one single particle by the formation of the tensor product

\[ \mathcal{G} = \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H} \]

There exists a natural symmetry group for such a system generated by the permutation operators for the \( n \) particles. Indeed, if the particles are truly indistinguishable then the observables must be invariant under these transformation. Thus the permutation group must give rise to a supersymmetry.
Let $P$ be a permutation and $U(P)$ the unitary representation generated by this permutation in the Hilbert space $\mathcal{H}$. Let $\Sigma$ be the algebra generated by $U(P)$

$$\Sigma = \{ U(P) \}^*$$

This algebra is abelian if and only if the permutation group is abelian. That is

$$\Sigma \not\subset \Sigma'$$

for $n \geq 3$

This is due to the fact that the representation $P \rightarrow U(P)$ is faithful.

Let us, as before, denote by $\mathcal{H}$ the algebra generated by the observables and by $\mathcal{H}'$ the commutant of $\mathcal{H}$. Then, since the $U(P)$ commute with all observables, we must have

$$U(P) \subset \mathcal{H}'$$

and consequently

$$\Sigma \subset \mathcal{H}'$$

If $\mathcal{H}'$ is abelian then $\Sigma$ must also be abelian. Combining this observation with the previous result we conclude:

If there exists a c.s.c.c. and if the number of identical particles is $n \geq 3$ then the space of state vectors $\mathcal{H}$ is a proper subspace of the direct product space $\mathcal{G}$.

How can we determine this proper subspace? Mathematically the problem is to find a projection operator $E'$ in the centre of $\mathcal{H}$ which reduces $\Sigma$ in such a way that the reduction of $\Sigma$ to the range of this projection is abelian.
The solution of this problem was given by Tixairo with the following simple result \(^5\):

Denote by \( S \) and \( A \) the symmetrised and antisymmetrised subspaces of \( G \). Then there exist three possibilities. The range \( \mathcal{K}' \) of \( E' \) is either

\[ S \oplus A, \quad A \quad \text{or} \quad S \oplus A \]

In the first case the particles satisfy Bose statistics, in the second case Fermi statistics. These correspond to well-known physical possibilities. The third case is a more general possibility. Here the particles are a statistical mixture of the Bose and Fermi statistics because the two subspaces are separated by a superselection rule. The supersymmetry transformation corresponds to the permutation of the two spaces and the essential observable is the projection into one of the subspaces. It corresponds to a yes-no experiment which determines the statistics of the particles.

It seems then from this theorem that the mathematical possibility which is obtained from the analysis of the superselection rules due to permutations are slightly more general than the possibilities known to occur in nature.
REFERENCES

1) P.A.M. Dirac, Principles of Quantum Mechanics, 3rd. Ed. p. 37

   It should not be mentioned, in order to avoid confusion, that Dirac’s use of
   the word observable is synonymous with our self-adjoint operator


3) This theorem was communicated to me by G. Tixaire, who has proved it
   with the theory of the direct integral of Hilbert spaces. The proof given
   here, which is due to J.M. Jauch and B. Misra, uses only global methods and
   avoids the measure theoretic finesses of the direct integral


   No reference is made in this paper to Murray and von Neumann’s classification
   but the implication mentioned here is obvious.

5) M.A. Naimark and S.V. Fomin, Uspokhi Mat. Nauk 10, No. 2 (64), 111 (1955)
   (translated into English in Am. Math. Soc. Trans. Series 2, Vol. 5 (1957)).

6) G. Tixaire, private communication.