ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

PHASE SPACE ANALOGUE COMPUTER FOR BEAM MATCHING PROBLEMS

by

B.W. Montague
PHASE SPACE ANALOGUE COMPUTER FOR BEAM MATCHING PROBLEMS

by

B.W. Montague
Phase Space Analogue Computer for Beam Matching Problems

by

B W. Montague

1. Summary.

2. Introduction

3. Principles of the Analogue
   3.1. Matrix formalism of beam matching
   3.2. Matrix representation of system elements
   3.3. Simulation of a matrix operation
   3.4. Representation of a vector
   3.5. Other equivalent systems

4. Design Details of Analogue
   4.1. Feedback amplifier unit
   4.2. Quadrature sine wave generator
   4.3. The complete system

5. Utilisation of the Analogue.
   5.1. Choice of scaling factors
   5.2. Oscilloscope display
   5.3. Setting of boundary conditions
   5.4. Search for solutions

6. Additions and Improvements.
   6.1. Precision
   6.2. Number of stages
   6.3. Phase strobe markers
   6.4. Simultaneous operation in both phase planes.
   6.5. Simulation of non-linear phenomena.

7. Acknowledgements.

Appendix: Theory of Quadrature Sine Wave Generator.
1. Summary.

This report describes an electronic analogue computer built for the solution of problems connected with the injection of the 50 MeV beam into the CERN Proton Synchrotron.

A brief introduction to the nature of these problems is followed by an explanation of the analogical principles involved. The apparatus consists of a sequence of voltage adding networks and buffer amplifiers which simulates the continued multiplication of 2 x 2 matrices with unity determinant. The analogue could be used to simulate any problem with the same mathematical properties.

The design of the electronics is dealt with in some detail and is followed by a section devoted to operation and the application to a typical problem. The report concludes with a discussion of possible improvements and ways of extending the field of application by the addition of certain facilities.

2. Introduction.

Towards the end of 1957 detailed studies of the beam matching between the CERN 50 MeV linac injector and the Proton Synchrotron were in progress. The problem involved the determination of a suitable quadrupole lens system to match the linac output beam into the synchrotron acceptance in beam radius and divergence (or convergence), simultaneously in vertical and horizontal planes. The situation was complicated by the fact that the optimum operating conditions for the linac could not be predicted with any confidence; furthermore there was a strong suspicion that, even if the focusing conditions in the linac could be specified precisely, the emergent beam would not necessarily correspond with the theoretical one.

It was therefore necessary to assume that the linac beam characteristics would lie anywhere within rather wide limits, and to provide a matching system which could be easily adjusted to match any plausible beam. It soon became apparent that the solution of such a problem in 4 variables and an excess of free parameters (necessary to give the system the desired flexibility) was extremely laborious by analytical methods.

The electronic analogue to be described was developed primarily to provide a rapid means of obtaining approximate solutions to the problem, specifically for determining the most suitable positions of the matching lenses. It has subsequently turned out to be much more useful than was anticipated; the suspicions regarding the
unpredictability of the linac beam in the early running-in stages have been fully justified and it has frequently been necessary to determine a set of solutions at short notice for a beam whose characteristics had just been measured.

The present analogue, constructed early in 1958, is in fact the prototype, and has never been tidied up and improved. It gives results accurate to better than 5 c/o rather easily, and with care in calibration can be relied upon to about 3 c/o. With only minor modifications it should be possible to make a similar instrument with 1 c/o precision, and in the last section of this report proposals are made for improving the accuracy and flexibility.

At the time the analogue was being designed it came to the author's notice that an apparatus based on identical principles had been built at the Brookhaven National Laboratory for solving the same type of matching problem on the A.G.S.

Other forms of analogue are possible in principle; two of these are mentioned briefly in section 3.5.


3.1. Matrix Formalism of Beam Matching.

A very convenient method of analysing the action of a lens system on a beam of particles uses a 2 x 2 matrix representation. If \( x \) be the co-ordinate along the beam axis and \( y \) the horizontal transverse co-ordinate then \( y' = \frac{dy}{dx} \) for a particle is the angle in the horizontal plane between the trajectory and the \( x \)-axis. The transverse motion of a particle can therefore be represented in the horizontal plane by a column vector \( \begin{pmatrix} y \\ y' \end{pmatrix} \), and similarly in the vertical plane by \( \begin{pmatrix} z \\ z' \end{pmatrix} \). We are here assuming that there is no acceleration along the \( x \)-axis and that the angles \( y' \) and \( z' \) are small. In an accelerated system it would be necessary to use canonically conjugate variables; here it is convenient to choose angles and displacements. A further assumption that the \( y \) and \( z \) motions are independent enables us to treat them separately. This assumption is quite legitimate in the problem under consideration.

An arbitrary beam optical system can be represented in one co-ordinate by a 2 x 2 matrix, and the equation of transformation between points 0 and 1 at the extremities of the system is:

\[
\begin{pmatrix}
  y_1' \\
  y_1
\end{pmatrix} = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
  y_0 \\
  y_0'
\end{pmatrix}
\]

or

\[
y_1 = A y_0
\]

(1)
A property of the matrix, valid under our assumptions, is that the determinant is unity, this property corresponding to area conservation in phase space (Liouville’s Theorem). Thus, there are only 3 independent parameters to determine a given transformation. It will be shown below that one of these parameters is redundant, and that only 2 independent conditions are necessary in each plane to satisfy the requirements of beam matching.

Let us associate the point 0 with the output end of the linac and the point 1 with the injection point in the P.S. Furthermore, suppose that the vector $Y_1$ has to trace out in the P.S. phase plane an ellipse corresponding to the P.S. acceptance at that position. Then, since (1) is a linear transformation, the vector $Y_0$ must also trace out an ellipse of the same area (since det $A = 1$) but, in general, of a different shape. If $A$ can be chosen such that the ellipse traced out by $Y_0$ corresponds to the linac beam emittance then matching has been achieved. For every point on the $Y_0$ ellipse there corresponds a unique point on the $Y_1$ ellipse, $A$ being given. There exists however an infinity of matrices $A$ which will transform the one ellipse into the other, but the correspondence between the points is different for each matrix, this being equivalent to different phase angles between the vectors. From the point of view of beam matching it is unnecessary to specify this phase angle, which can therefore be arbitrary. Hence one of the 3 parameters mentioned above is redundant.

3.2. Matrix Representation of System Elements.

From the simple thin lens formula of geometrical optics we can write down the equations connecting beam angles and positions immediately before and after a thin lens:

$$\begin{align*}
Y_1 &= Y_0 \\
Y_1' &= Y_0' + f Y_0'
\end{align*}$$

where $f$ is the focal length of the lens. It is often more convenient to use the focal strength $\Delta = 1/f$ and the present formalism requires that $\Delta$ be negative for a focusing lens. We can thus write (2) in matrix form:

$$\begin{pmatrix}
Y_1 \\
Y_1'
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
\Delta & 1
\end{pmatrix}
\begin{pmatrix}
Y_0 \\
Y_0'
\end{pmatrix}$$

or

$$Y_1 =
\begin{pmatrix}
1 & 0 \\
\Delta & 1
\end{pmatrix}
Y_0$$

PS/1500
The corresponding relation for a drift space of length \( L \) can easily be shown to be:

\[
Y_2 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} Y_1
\]

(4)

Representation of a system consisting of a number of thin lenses and drift spaces is obtained by successive multiplication of the individual matrices in the correct order.

We shall not in this report deal with the cases of more complicated system elements such as thick lenses, since the thin lens approximation is quite adequate for the discussion of the analogue. In general, any compound system can be represented by a 2 x 2 matrix, which can in turn represent a system of 3 basic elements, either two thin lenses separated by a drift space or two drift spaces separated by a thin lens. This corresponds to the principle in electrical network theory that a general 4-terminal network always has an equivalent \( \pi \) or \( T \) network, even if the latter are not necessarily realisable physically. In fact, the mathematical formalism is identical for the two problems.

3.3. Simulation of a Matrix Operation.

The operation given by equations (2) and (3) can be represented to a good approximation by a simple adding network if we consider \( y \) and \( y' \) to be represented by voltages \( v \) and \( v' \) respectively. Such a network is shown in Fig. 1.
The approximation is better as the inequalities are better satisfied; furthermore it is assumed that \( v_0 \) and \( v'_0 \) have negligible source impedances. The potentiometer \( R_2 \) controls the lens strength \( \Delta (=\gamma f) \) and in Fig. 1, this can be varied between 0 and +1. A focusing lens requires a negative strength; this can be represented by using an inverting amplifier to supply \(-v_0\) to the top of \( R_2 \).

The equivalent network for representing a drift space is obtained from Fig. 1 by interchanging the corresponding primed and unprimed variables and by substituting \( L \) for \( \Delta \). The units represented by \( L, \Delta, v \) and \( v' \) can be chosen to suit the nature of the problem subject to certain restrictions. The first and obvious requirement is that the equations (3) and (4) be dimensionally homogeneous. The second restriction implied by Fig. 1 and its drift space equivalent is that both \( L \) and \( \Delta \) are variable only over the range 0 to +1. Since \( \Delta = \frac{1}{Y} f \), the shortest focal length representable is unit length, and in many problems this is an undesirable restriction. It could be overcome by allowing that the two arms \( (R_0) \) of the adding network may be different, but this would alter the attenuation factor of the network.

Certain practical limitations of the circuit of Fig. 1 are now evident. The attenuation of one of the variables by a factor of 2 is clearly undesirable, and the increase in this factor if one used input arms of different resistance would be even more inconvenient. Furthermore, the cascading of successive stages would soon make it impossible to satisfy the inequalities to a sufficient degree. It is therefore necessary to include in each stage a buffer amplifier whose functions are to transform impedances in a convenient manner and to maintain unity stage gain.
The basic circuit for a lens stage is shown in Fig. 2, together with the facilities for changing the polarity and the scaling factor. The amplifier has a high voltage gain (∼ 150), and both direct and phase inverted outputs. Negative feedback is applied back to the input circuit so that the stage gain as shown is unity for the \( v_0' \) input. If the loop gain is very much greater than the stage gain, the latter is determined almost entirely by the ratio of feedback to input resistance, and may be different for different input channels. Under these conditions the amplifier input point constitutes a "virtual earth" and the different input channels are almost completely decoupled from each other.

3.4. Representation of a Vector.

So far we have considered the transform properties of a network operating on voltages which represent arbitrary phase space vectors. It would of course be possible to use a series of the networks discussed in conjunction with input voltages \( v_0' \) moved step by step, in order to determine the corresponding voltages \( v_n \), \( v_n' \) at the end of a system of \( n \) elements. We have, however, the time variable at our disposal, and it would be interesting to make use of this to sweep a vector \( Y \) periodically round a contour in the phase plane.

It is perhaps fortunate that strong focusing beam optical systems have acceptances which can be represented by an elliptical boundary in the phase plane. It was this property that suggested the use of sinusoidal voltages in quadrature for \( v_0, v_0' \). These voltages can be displayed as a Lissajous ellipse on an oscilloscope and by a suitable choice of amplitudes can be made to represent the desired elliptical phase plane boundary according to the scaling factors adopted. Since the voltages \( v, v' \) are displayed orthogonally, transformation by a succession of matrix simulating networks with unity determinant conserves the area of the ellipse, and the ellipse shape represents exactly the shape of the transformed phase plane boundary. The analogue therefore enables one to see the phase plane ellipse at any point in the system by displaying the corresponding \( v, v' \) voltages as a Lissajous figure on an oscilloscope. Manipulation of the potentiometers of the network then gives an immediate visual display of the effects of varying the corresponding parameters.

The frequency of the input voltages can conveniently be chosen as 50 Hz, as this is high enough to provide a good display and low enough not to require any special care in the feedback amplifiers.
3.5. Other Equivalent Systems.

The system described above was the first one proposed. It was pointed out by Hereward, however, that there exist other systems possible in principle and rather simpler in detail. Consider for example the four-terminal networks of Fig. 3 (a) and (b).

![Diagram](image)

The network equations are:

(a) \[
\begin{align*}
    v_1 &= v_o + R i_1 \\
    i_1 &= i_o
\end{align*}
\]

(b) \[
\begin{align*}
    v_1 &= v_o \\
    i_1 &= G v_o + i_o
\end{align*}
\]

and it is seen that the formalism is identical to that described in section 3.2. An analogue could therefore be built from a series of such networks, with \( v \) and \( i \) representing the phase space variables. To simulate a focusing lens it must be possible to have either \( R \) or \( G \) negative. This can be done with a rather simple electronic circuit, and is easier if we choose \( G \) as the lens element. The networks are terminated by a reactive element so that current and voltage are in quadrature, thus permitting an elliptical display when the system is fed from a sinusoidal source. It will be noted that the current flows in the opposite direction to the matrix transformation.

A trial analogue based on this principle was constructed by Vosickl and worked according to theory but for one severe limitation. With certain combinations of parameters, which unfortunately could easily arise in practice, the system became
completely unstable. This occurred if at any point in the network the nett resistance was negative. Otherwise stated, if the application of Thévenin's Theorem to any branch of the network resulted in a negative current in that branch then the system was unstable. An analysis showed that, if the terminating reactance were omitted, the beam optical property corresponding to the instability was the presence of a beam "waist" inside the limits of the system. The inclusion of the terminating reactance increases slightly the range of stability but not enough to be of use. Accordingly this analogue had to be abandoned.

Another possible network with the correct transformation properties uses inductive and capacitive elements. It was doubted, however, if variable inductors with sufficiently high $Q$ could conveniently be made and so this system was never tried.

4. **Design Details of Analogue.**

4.1. **Feedback Amplifier Unit.**

The amplifiers are made identical for both lens and drift space functions and are constructed on individual chassies to facilitate servicing. The circuit of an amplifier unit is shown in Fig. 4. It consists of an B80F voltage amplifier followed by an B86CC double triode. One half of the double triode serves as a phase splitter; from its cathode is taken the feedback signal and the negative output sum signal, and from its anode the positive sum signal is fed to the second half of the double triode. The cathode of the latter provides the low-impedance positive sum signal.

The feedback signal is applied, via a low-frequency phase shift correction network, to the lower end of the input resistor chain. The signal input grid of the B80F is tapped on a point of this chain which essentially determines the transfer gain of the stage, since the loop gain is around 150 and the transfer gain between 1 and 4. The signal level at this point is very small and so the point constitutes a "virtual earth". The transfer gain is given by the ratio of the resistance of the feedback and input ends of the chain, and provision is made for one of the two input branches to have its resistance switched in steps, providing scale factors 1, 2 and 4 for the adjustable parameter as described in section 3.3.

From the anode of the B80F a series RC network to earth reduces the voltage gain at high frequencies to prevent instability. Preset potentiometers in the feedback loop provide adjustment of the phase shift correction and the transfer gain. Since the signal frequency is 50 Hz, there are no coupling or phase shift problems in the amplifier.
Resistances in ohms
Capacitances in microfarads, unless otherwise stated
Unspecified values adjusted by trial.

FIG. 4
Quadrature Sine Wave Generator

Resistances in ohms
Capacitances in microfarads

unless otherwise stated

FIG. 5
Adjustment of gain trimming and phase shift correction is achieved very simply by using two resistors exactly equal in value connected in series between the input and negative output points. An oscilloscope is connected to the common junction of these resistors and the trimming controls adjusted alternately for minimum residual signal. If the sine wave input is reasonably free from harmonics a good balance is easily obtained. The anode compensation of the phase splitter is adjusted similarly by balancing the positive and negative outputs. No potentiometers are used here, compensation being arranged by trial of components.

4.2. Quadrature Sine Wave Generator.

One of the facilities required in the analogue for ease and speed of setting up was the possibility of adjusting the area and the axial ratio of the input Lissajous ellipse by means of independent controls. In normal operation one uses a nominal value of emittance (area) for all problems of the same system. On the other hand it is frequently necessary to adjust the axial ratio and keep the area constant; independent control of these two parameters is therefore very desirable.

The circuit of Fig. 5 achieves this requirement quite simply and to very good approximation limited only by the source impedance of the 50 Hz input voltage, the linearity of tracking of the ganged potentiometer and the effectiveness of the integrator. The detailed analysis of the circuit is given in the Appendix.

The 50 Hz signal is obtained from the secondary of a mains transformer at 10 volts and passed through a 3 section RC filter to remove the harmonics. The resultant signal at about 3 volts maximum is tapped off a potentiometer which varies \( v_\text{o} \) and \( v_\text{o}' \) in the same ratio, and hence adjusts the surface area of the ellipse. The signal is fed to two similar potentiometer chains incorporating a ganged potentiometer, the axial ratio control, connected in such a way that the voltage on one slider increases as the other decreases. One of these signals is fed to a cathode follower, the output of which is the signal \( v_\text{o}' \). The other signal is fed directly to the grid of the BSOF integrator valve. The integrator feedback is taken via a cathode follower which also provides the quadrature signal \( v_\text{o} \).

There is no fixed input resistor for the integrator valve; the resistance is provided by the slider-to-earth resistance of the potentiometer chain and varies according to the potentiometer setting. This variation is an essential feature of the circuit to obtain a constant product of the peak values of \( v_\text{o} \) and \( v_\text{o}' \) irrespective of the axial ratio setting. The proof of this property is given in the Appendix.
The sense of rotation of the \((v_o, v'_o)\) vector has been chosen to be positive (anti-clockwise) in the conventional phase plane notation. This is irrelevant in most applications since, as previously mentioned, the one-one phase correspondence is of no interest in our matching problems. However, it is possible to envisage circumstances in which one might be interested in the phase of the vector and in providing a phase strobe marker, so the vector rotation sense has been determined. This possibility will be discussed further in section 6.

4.3. The Complete System

The analogue is made up of a number of amplifier units in cascade, preceded by the sine wave generator, and interconnected in such a way that the stages represent alternately drift spaces and lenses. The parameter potentiometers and adding network interconnections are arranged on the front panel, as they belong rather to the logical system than to the amplifiers. The panel also carries sockets for bringing out the signals from intermediate stages. It is, of course, perfectly possible to have two successive lens or drift space units, but this is not normally necessary. Fig. 6 shows three stages of the normal arrangement.

When the analogue was first constructed, helipots were used for the parameter potentiometers. This is not really necessary and a good single turn wirewound potentiometer with adequate linearity is to be preferred for rapid operation. Separate potentiometers for focusing and defocusing polarities were also fitted but are now considered to be unnecessary.

The present model consists of 3 drift spaces and 2 lenses, making 5 stages in all. This was considered to be the minimum suitable for the P.S. matching problem and since the analogue was needed rather quickly it was decided to make it as simple as possible. Furthermore, it was not realised at that time how convenient and flexible an instrument it could be. With the experience gained in its use one would now be tempted to build an improved version with 12 to 15 stages and certain other facilities. Such possible improvements will be discussed in section 6.

5. Utilisation of the Analogue.

In section 3.1. we mentioned that, in the matching problem for which the analogue was designed, it is possible to consider the vertical and horizontal planes separately. From a theoretical standpoint this amounts to the assumption that there is no coupling between the vertical and the horizontal motions. From a practical
point of view it assumes also that, having found an independent solution for each plane, we can synthesize a lens system which satisfies the individual solutions for the two planes. In principle, since there are two conditions to be met in each plane, either four quadrupole lenses positioned arbitrarily or two quadrupoles in determined positions would meet the requirements. There are, however, certain objections to such arrangements.

In a system using only two quadrupoles it would be necessary to change their positions every time there was a change in the matching conditions. This clearly is an intolerable restriction. With four quadrupoles in fixed positions one would have the essential flexibility, but the lack of symmetry of the system would make adjustment of the four parameters somewhat complicated. The C.P.S. matching system uses two symmetrical triplet lenses, the two outer lenses of each triplet having the same strength. Such a system has the advantage that it can be represented to a rather good approximation by two thin lenses at the centres of the triplets. Furthermore the equivalent thin lenses can have arbitrary strengths in the two planes with the restriction that there must be focusing in at least one plane of each triplet.

With this arrangement in mind one can use the analogue to determine the matching solutions separately for each plane, the assumed lens positions being the same in the two planes. Formulae or graphs are then used to convert the focal strengths of the equivalent thin lenses to focal strengths of the actual triplet lenses. If a higher degree of approximation is required, the individual lens strengths can be used to calculate corrections to the adjacent effective drift space lengths and a further solution determined from the analogue. This process converges rapidly, and in the C.P.S. injection matching we have usually found that the first approximation was sufficiently accurate. The procedure used in solving a problem will now be described.

5.1. Choice of Scaling Factors.

Examination of equations (3) and (4) shows that whatever units are chosen, L and A must be in consistent units so that the dimension of $L \Delta$ is unity. If, for example, we express L in metres then $\Delta$ must be expressed in $m^{-1}$. There is, however, the facility for using a further scaling factor in the adding network (section 3.3) which may be different for a lens from that for a drift space.

The choice of a suitable unit for L usually follows from the nature of the problem. In the case under consideration, the maximum drift space to be represented
is somewhat under 10 m, consequently the unit chosen is 1 dekametre. $\Delta$ must then be in $\text{Dm}^{-1}$, and with the adding network factor of 4, a maximum lens strength of 4 $\text{Dm}^{-1}$ (focal length 2.5 m) can be represented.

The beam radii in the system are of the order of centimetres, so we choose one centimetre as the unit for $y$. The unit for $y' = dy/dx$ now follows, since $x$ must have the same unit as $L$. Hence $y'$ is expressed in $(\text{cm})(\text{Dm})^{-1}$ or milliradians. The unit used for emittance is rather arbitrary, since it does not affect the essential scaling factors of the analogue but only the calibration of the oscilloscope scale. It is usually convenient to choose one unit of $yy'$, i.e. one centimetre milliradian in our case, as this simplifies somewhat the preparation of data. Furthermore it should normally turn out to be of the correct order of magnitude if the other units have been chosen appropriately. Exceptions occur in systems where the beam profile undergoes drastic changes from one end of the system to the other. For example, the 0.5 MeV proton beam injected into the linac emerges from the 500 kV accelerating column as a rather flat beam with circular symmetry and small divergence. It must, however, be injected into the linac with a strong convergence in the vertical plane. The choice of scaling factors for this problem was therefore not quite so obvious and logical as in the case of the 50 MeV beam matching, and different factors were used in the two planes.

5.2. Oscilloscope Display.

Even with the prototype analogue described the precision is near to being limited by the display. The oscilloscope tube should therefore be fairly large with a finely divided graticule and have good amplifier linearity. The vertical and horizontal sensitivities are adjusted to have a known ratio which will usually be unity.

5.3. Setting of Boundary Conditions.

The input phase space ellipse of the system will be represented by the sinusoids in quadrature from the generator, and will therefore be a right ellipse. If the actual system is such that the phase plane ellipses at both ends are skew it will be necessary to add or subtract a fictitious drift length to bring the ellipse into principal axes, corresponding to a beam "waist" or "neck". The first drift space of a system does not affect the axial ratio of the ellipse, so this procedure brings about no further complications.
At the other end of the system we have to specify in general a skew ellipse in a convenient way. The easiest way seems to be by determining the maximum $y$ and $y'$ of the beam at that point for the nominal value of emittance chosen. These parameters are the easiest to measure on an oscilloscope and are readily calculated from any other pair of parameters defining the ellipse.

5.4. Search for Solutions.

Having set in the values of the drift spaces it only remains to adjust the lens strength potentiometers until the required ellipse shape is obtained at the end of the system. With a two-lens system the procedure is quite rapid and only a little practice is necessary to obtain very consistent results in repeated attempts. Furthermore one obtains rapidly an idea of the general behaviour of a matching system, its sensitivity to the adjustment of parameters, the shape of the beam profile along the system, etc.

It is worth noting that in matching problems of this type there are in general two solutions which we designate "weak" and "strong" solutions. The weak solution corresponds to the case where there is no beam "waist" between the lenses and the strong solution to the case of an intermediate waist. Clearly the strong solution requires higher lens strengths and is usually avoided. There are, however, conditions under which it must be used. If for example a weak solution in each plane requires that a lens be defocusing in the two planes the case cannot be realised in practice, this being purely an unfortunate property of magnetic lens systems and nothing to do with the matching problem. In such a case the strong solution must be used in one plane. With certain combinations of system parameters it can happen that both the solutions are imaginary. Both this effect and the weak and strong solutions are easily demonstrated on the analogue.

The results obtained for day-to-day problems in the linac–C.P.S. beam matching are usually used directly. However, we have an Autocode programme on the CERN Mercury Computer which, starting from the analogue results, calculates by means of a searching procedure the correct solutions. This programme takes into account the individual lenses of the triplets and computes one case about every 15 seconds to an accuracy of about 0.2 o/o. Comparison of the results with the starting values show that the analogue, in conjunction with the graphical conversion to the triplet system, is rarely in error by as much as 4 o/o.
6. Additions and Improvements.

The performance of the prototype has always been adequate for the problems treated up to the present time. There are, however, certain improvements that would be desirable if a new version were made and it seems likely that an improved analogue on these lines would have all the accuracy and flexibility necessary to handle most types of beam transport problem. The following suggestions are offered in this connection.

6.1. Precision.

The accuracy could be improved appreciably by studying the loading effect of one stage upon the preceding one. At present the adding networks have integral ratios of resistor values and it seems that the change in loading when changing the scale factor is sufficient to justify individual compensation for each range.

6.2. Number of Stages.

The five stages at present used represents the minimum of real interest. It is suggested that 12 to 15 stages would be sufficient to simulate quite a complicated system. A transistorised version of 15 stages would probably occupy no more space than the present model.

6.3. Phase Strobe Marker.

Although the phase of the ellipse vector is not generally of interest in a beam matching problem there are some related problems in which a knowledge of the phase can be of some use. In the C.P.S. inflector region there is a number of adjustable beam limiting apertures whose function is to define a beam limited in position and in angle in the synchrotron. It is of interest to know how the corresponding limiting lines in the phase plane transform through the system and this information is carried by the phase of the transformed vector.

In the analogue one can determine the phase of the vector by a strobe pulse applied to the grid of the oscilloscope tube. This pulse can be obtained from one of the input sinusoidal voltages via an intermediate phase shifter to enable the marker to be set at any point on the ellipse. One can then display the ellipse corresponding to the position of a limiting aperture and adjust the strobe marker to the maximum value of \( y \) on this ellipse. This is the point at which a limiting aperture would touch the beam phase plane ellipse tangentially. If now the phase
plane is examined at some other point in the system the strobe marker will appear at the contact point of the transformed limiting line.

One could extend this idea by using a strobe pulse variable in duration so that a section of the ellipse is blanked out, corresponding to the interception of part of the beam by an aperture. The same effect could be obtained by a biased diode clipper connected at the appropriate point. Use for multiple strobe markers will also suggest themselves.

6.4. Simultaneous Operation in both Phase Planes.

In simulating a system in which it is impracticable or inconvenient to consider the vertical and horizontal planes as separate problems it would be useful to have a simultaneous display of both planes. One obvious way of doing this would be to use two similar analogues with their outputs displayed on a double beam oscilloscope. A more compact and economical arrangement would use the same amplifiers for both planes and switch in the parametric elements for the two planes alternately. With a 50 Hz signal the switching could be done between two and five times per second which would be easily within the capabilities of relays and still give a satisfactory display. To prevent transients due to contact bounce spoiling the clarity of the ellipses a blanking signal of a few milliseconds duration could be applied to the tube during the commutation period.

6.5. Simulation of Non-linear Phenomena.

The simulation of certain non-linearities in lenses is possible in principle by the use of voltage dependent elements. If in Fig. 2 (section 3.3.) a voltage dependent resistor were put in parallel with the adding network resistor $R_e/k$ the effect would be that of the lens field gradient increasing with radius. The amount of non-linearity introduced could be adjusted by a linear resistor in series with the VDR. A non-linearity of opposite sign could be simulated by connecting the VDR in series with its adjustable resistor between the slider of the potentiometer $R_2$ and earth. Normal silicon carbide VDR's have an operating voltage which is rather high for this application. Pairs of semiconductor diodes in parallel, back-to-back, would probably be suitable.

In high energy beam transport systems the scattering of particles by the edges of a beam defining slit is of some importance. This could be simulated rather easily by a circuit similar to that required for producing a strobe marker. One of the input sine waves would be fed through a phase shifter, a squaring circuit and a
differentiator, producing alternately positive and negative pulses. This signal could then be mixed with the $y'$ signal at the point in the system corresponding to the defining slit and the phase of the pulse signal adjusted so coincide with the extreme points of the $y$ vector. The display in the phase plane at this point would then look something like Fig. 7, which is a reasonable representation

![Fig. 7](image)

cf the scattering effect of the slit. The display at points later in the system would then show the phase plane distribution of the scattered particles. For this facility it would be necessary for the amplifier frequency response to be good up to several kilocycles, but this should not present much of a problem.

7. Acknowledgements.

The author wishes to express his appreciation of the helpful discussions with a number of his CERN colleagues, in particular with Drs. Hereward and Lapostolle and Ing. Vosicki.
APPENDIX.

Theory of Quadrature Sine Wave Generator.

The circuit of Fig. 8 represents the actual conditions under the assumption that the source impedance in series with $V_o$ can be neglected compared with the minimum value of $R_1$ or $R_2$ required.

![Circuit Diagram](image)

Fig. 8.

We put

\[
\begin{align*}
R_1 + R_2 &= R_c \quad \text{(constant)} \\
R_2 &= \rho R_c \\
R_1 &= (1 - \rho) R_c
\end{align*}
\]

Hence

\[
R = \frac{R_1 R_2}{R_1 + R_2} = R_o \rho (1 - \rho)
\]

The source impedance $R$ at either potentiometer slider is $R_2$ and $R_1$ in parallel, i.e.

The source impedance $R$ at either potentiometer slider is $R_2$ and $R_1$ in parallel, i.e.

We can draw an equivalent circuit as in Fig. 9.

![Circuit Diagram](image)

Fig. 9.
in which:

\[ v'_1 = v_o \frac{R_l}{R_o} = v_o (1 - \rho) \]

and if

\[ v_o = \dot{\varphi}_o \sin \omega t \]

then

\[ v'_1 = \dot{\varphi}_o (1 - \rho) \sin \omega t \] (7)

Now

\[
(v'_1 - v_1) = CR \frac{d}{dt} (v_1 - v_a) \\
= CR (1 + \mu) \frac{dv_1}{dt}
\]

Hence, from (7)

\[
\frac{dv_1}{dt} + \frac{v_1}{(1 + \mu) CR} = \frac{\dot{\varphi}_o (1 - \rho)}{(1 + \mu) CR} \sin \omega t
\] (8)

If

\[ v_1 = \dot{\varphi}_1 \sin (\omega t - \varphi) \] (9)

the solution to (8) is

\[
\dot{\varphi}_1 \left[ \omega \cos (\omega t - \varphi) + \frac{\sin (\omega t - \varphi)}{(1 + \mu) CR} \right] = \frac{\dot{\varphi}_o (1 - \rho) \sin \omega t}{(1 + \mu) CR}
\] (10)

with \( \tan \varphi = \omega CR (1 + \mu) \)

If we make \( \omega CR (1 + \mu) \gg 1 \) then \( \varphi \approx \pi/2 \) and we have

\[ \dot{\varphi}_1 \approx \frac{\dot{\varphi}_o (1 - \rho)}{\omega CR (1 + \mu)} \]

and hence from (9)

\[ v_1 = -\frac{\dot{\varphi}_o (1 - \rho)}{\omega CR (1 + \mu)} \cos \omega t \]
Also, since \( v_a = - \mu v_1 \)
\[
v_a = \frac{\dot{v}_o \mu (1 - \rho)}{\omega CR (1 + \mu)} \cos \omega t \tag{11}
\]

Now from Fig. 8 \( v_2 = \frac{R_2}{R_o} v_o = \rho v_o \)
\[
\therefore v_2 = \rho \dot{v}_o \sin \omega t \tag{12}
\]

From (11) and (12) we see that \( v_a \) and \( v_2 \) are in quadrature and the product of their peak values is
\[
\dot{v}_2 \dot{v}_a = \frac{v_o^2 \mu \rho (1 - \rho)}{\omega CR (1 + \mu)}
\]

But from (6) \( R = R_o \rho (1 - \rho) \)
\[
\therefore \dot{v}_2 \dot{v}_a = \frac{v_o^2}{\omega CR_o} \cdot \frac{\mu}{(1 + \mu)} \tag{13}
\]

The area of the ellipse is therefore independent of the position \( \rho \) of the axial ratio potentiometer.

We also have from (6), (11) and (12)
\[
\frac{\dot{v}_a}{\dot{v}_2} = \frac{1}{\omega CR_o} \cdot \frac{\mu}{(1 + \mu)} \cdot \frac{1}{\rho^2} \tag{14}
\]

If one puts in the values from Fig. 4 and assumes \( \mu \gg 1 \) the limits of the axial ratio \( \dot{v}_a/\dot{v}_2 \) available with the circuit shown are 0.4 to 7.6.

/kt

PS/1500