A REMARK ON THE TRANSVERSE RESISTIVE INSTABILITY OF COASTING BEAMS.

1. Introduction.

A paper by Leslett et al.\textsuperscript{1)} which we shall refer to as LNS, investigates the interaction of a beam of particles with itself, including the effect of a resistive vacuum tank. They demonstrate the existence of an instability which is the result of two almost distinct processes acting in combination. These two processes can be broadly described as follows:

(a) The space charge fields of a beam, calculated in the approximation that the conductivity of the vacuum chamber is high, can annul the intrinsic betatron-frequency spread of the particles in the beam, in such a way that coherent betatron oscillations do not disappear (by becoming incoherent) in a time related to this frequency spread, but can persist indefinitely. For this to occur there is a threshold beam intensity which, other things being equal, is proportional to the intrinsic frequency spread.

(b) When a small resistivity of the vacuum chamber is included, this has a negligible effect on process (a), but introduces a term which can produce anti-damping of coherent betatron oscillations. In practical cases this term is very small, potentially capable of producing only a very slow growth of amplitude, and is likely to be completely swamped by the apparent damping associated with frequency-spread unless one is above the threshold for the process (a).

The value of the resistivity of the vacuum chamber material does not appear in the formula derived by LNS for the threshold: for any practical values of the parameters the (a) effect can be calculated to good approximation on the following assumptions:

The chamber is treated as a very good conductor, so that with one exception the fields due to the beam can be calculated by taking it to have zero resistivity, i.e., by assuming the boundary conditions

\[ E_{\parallel} = 0 \quad B_{\perp} = 0 \quad (1) \]
at the inside surface of the chamber wall. The exception is the beam's magneto-static field; this D.C., zero frequency, magnetic field is unaffected by a vacuum chamber of whatever conductivity and should be calculated subject to a boundary condition

\[ B_{\text{m}} = 0 \]  

(2)
on the surface of the magnet iron.

These boundary conditions imply a certain property of the fields, which is in conflict with the fields and the formula for the threshold given by LNS. In the next section we derive this property, and in section 3 we show where we believe that the derivation of the fields in LNS is in error.


We shall adopt the following terminology:

The "total field" is the field of the beam subject to the boundary conditions.

The "direct self-field" is the field that one would calculate for the same beam with all conductors and permeable materials removed to very large distances. It is therefore a field without boundary conditions (unless one counts, as boundary conditions, the usual convergence requirements at large distances).

The difference, total field minus direct self-field, we shall call the "image field". There are cases where it is even convenient to calculate it by means of the concept of images, but more importantly, it evidently satisfies, everywhere within the vacuum chamber, the homogeneous Maxwell's equations:

\[ \text{Div } E = 0 \]
\[ \text{Div } B = 0 \]
\[ c \text{ Curl } E - \mathbf{E} = 0 \]
\[ c \text{ Curl } H - \mathbf{E} = 0 \] 

(3)and this is a property characteristic of a field which has all its sources (charges, currents, dipoles etc.) outside the vacuum chamber.
Whatever its boundary condition, the magnetostatic part of an image field has this property and also is a D.C. field. One can, therefore, if the D.C. current and time-averaged distribution of the beam are taken as given, assimilate it into the properties of the field of the accelerator magnet, and disregard the fact that it is caused by the beam. It may change the $Q$-values, shift the equilibrium orbit $^*$, introduce non-linearity, but it is by definition unaffected by any small oscillatory motion of the beam, and therefore cannot have any influence on the instability dealt with by LNS; except, of course, influence of the same types that one could get, by way of static $Q$-changes or non-linearity, etc., from static changes in the properties of the accelerator magnet.

It ought therefore to be possible, when calculating the threshold for the LNS instability, to disregard the magnetostatic image force, or even to evaluate it with the wrong boundary condition, without affecting the result. Suppose then that we use this possibility and evaluate the magnetostatic image field with the boundary condition on the vacuum chamber wall

$$E_1 = 0$$  

(4)

instead of (2). Then all our beam fields are subject to the boundary condition (1).

There is a certain symmetry between electric and magnetic forces when we have the boundary conditions (1), and this can perhaps be most simply described in terms of the image concept: with (1), a positive charge has a negative charge as image, and a positive current parallel to the boundary has a negative current parallel to the boundary as image. Figure 1 illustrates the situation for the case of a single infinite plane boundary. Under these conditions one can extend the notion of magnetic cancellation to image fields, and therefore to total fields.

Consider two charges in empty space, both moving parallel to some straight line with velocity $\beta c$. As is well known, the ratio of the magnetic to the electric mutual transverse force between the charges is $-\beta^2$, so that the whole transverse force can be found by evaluating the electrical force and multiplying

$^*$

So necessitating a self-consistency method of calculation in order to find where the beam will be.
by \( \gamma^{-2} \), where as usual

\[ \gamma^{-2} = 1 - \beta^2 \]  

(5)

For our direct self-field one may say that like charges repel one another, this force being partially cancelled magnetically by the fact that like current attract. With the boundary conditions (1), the argument extends to the image forces; the (unlike) image charges attract the beam to the nearest boundary, the image currents repel it from the nearest boundary and the factor that results is again \( \gamma^{-2} \).

We shall not give a proof of this. The shortest general proof is to work in a coordinate system moving with the charges, so that there are no currents and only electrostatic fields, and then make a Lorentz transformation back to the laboratory system. The key point is that the boundary condition (1) is invariant for this transformation if the boundary is a cylinder with generators parallel to the direction associated with the transformation.

If the above arguments are accepted, the final expression for the force involved in the threshold considerations, process (a), of the LNS instability must depend on the particle velocity by way of a factor \( \gamma^{-2} \). In fact one finds in their equations (2.21) and (3.16a) that these terms do not contain a velocity-dependant factor at all: two magnetic terms independent of chamber resistivity appear, with the expected \( \beta^2 \) factor in front of them, in (2.20), but they cancel one another in the approximation \( k^2 \ll \eta^2 \), while there are two electric terms in (2.19) that do not.

3. Derivation of the Fields.

The method of investigation used by LNS can be summarised as consisting of the following steps.

(i) One supposes that the beam is displaced vertically from \( z = 0 \) by an amount

\[ z_b = \xi \exp \left[ i (n \theta - \omega t) \right] \]  

(6)

\footnote{Cylinder in the general sense, i.e., cross-section not necessarily circular.}

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and one calculates the fields, subject to the proper boundary conditions, for this configuration of current and charge.

(ii) On the basis of a certain unperturbed equilibrium distribution of the particles in phase space, valid \(^{\#}\) for \(\xi = 0\), combined with the first-order-in-\(\xi\) part of the field, one uses the Vlasov equation to calculate the first order perturbed distribution. The vertical displacement of the beam, say \(\langle z \rangle\), is calculated for this perturbed distribution, by averaging over all particles present at each \(\theta, t\).

(iii) This \(\langle z \rangle\) is equated with \(\delta\), thus requiring that the distribution calculated from the fields is consistent with the distribution from which the fields are calculated.

(iv) The equation which results (dispersion relation) is treated as an equation for \(\omega\) and solved \(^{\#\#}\), and the imaginary part of \(\omega\) interpreted in the usual way as a (positive or negative) build-up rate.

In LNS the average, \(F_a\), of the field just above the beam and the field just below the beam, is calculated in the approximation that the beam is a flat strip of zero vertical thickness; on condition that the beam is in fact thin \(^{\text{##}}\) this is an adequate approximation for calculating this average, but we disagree with the assumption of LNS that this average is the only relevant quantity. In detail, our calculation differs from theirs in two respects:

(a) We assume a beam with a vertical thickness \(d\). To avoid too much complication we suppose \(d\) is small in the sense of the last footnote, and we suppose a uniform vertical particle density distribution over this thickness.

(b) The calculated fields are functions of position and time, \(x, y, z, t\); and of the assumed beam configuration given by (6), so the expressions contain explicitly both \(z\), and \(z_b\) or \(\xi \exp \left[ i (n \theta - \omega t) \right]\). In LNS they substitute

\(^{\#}\) Note that this means that any part of the beam field which is independent of \(\xi\), rather than first or higher order in \(\xi\), is thus assumed to be taken into account in the unperturbed distribution.

\(^{\#\#}\) If this equation cannot be satisfied for any \(\omega\) then \(\xi = 0\) is the solution of the problem.

\(^{\text{##}}\) Thin compared with all vacuum chamber dimensions and compared with all wavelengths necessary to make adequate Fourier expansions of the horizontal distributions.
we assume that the distinction between \( z \) dependence and \( z_b \) dependence must be kept in these expressions in order to make a valid calculation of the perturbed distribution.

It should perhaps be remarked that in the calculation that follows we incorporate these two assumptions by a method intended to facilitate direct comparison with the work of LNS: the calculation could be less complicated if one were doing it independently.

We consider the vertical component of the electric field \( E_z \) in the region within the beam

\[
z_b - d/2 < z < z_b + d/2
\]

and look at its relationship to the average, \( E_{az} \), above and below, calculated in LNS.

One has by definition

\[
E_z (z_b + d/2) + E_z (z_b - d/2) = 2E_{az} \cdot (7)
\]

We shall put also

\[
E_z (z_b + d/2) - E_z (z_b - d/2) = 4 \pi \sigma' + E'_z (z_b) \cdot d \quad (9)
\]

where \( E'_z \) is that part of the vertical field gradient not associated with the charge density (so that (9) is effectively a definition of \( E'_z \)); we put it in because one can expect the image field to have such a gradient. From (8) and (9) one finds an expression for the field within the interval (7):

\[
E_z (z) = E_{az} + (E'_z (z_b) + \frac{4 \pi \sigma}{d}) (z - z_b) \quad (10)
\]

and there is a corresponding expression for \( E_x \).

The average field \( E_{az} \) depends of course on \( z_b \) (or \( \xi \) etc.) but contains no explicit dependence on \( z \) as such. \( E'_z \), besides being evaluated at \( z_b \), as indicated, also depends on the beam configuration, but it appears in (10) only multiplied by \((z - z_b)\), and we are going to work only to first order. 

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in $z$ and $\xi$, so it is sufficient to evaluate it for the unperturbed beam and
to zero order in $z_b$.

In LNS the average field $E_{az}$ is evaluated as the sum of two contributions,
the part due to the unperturbed beam, suffix zero, and the first order change due
to the beam displacement $z_b$, so we may rewrite (10):

$$E_z(z) = E_{oz} + E_{lz} + \left(E_{oz}' + \frac{4\pi \sigma}{d}\right)(z - z_b)$$  \hfill (11)

It is easy to see that, to first order in $z_b$,

$$E_{oz} = E_{oz}' \cdot z_b$$  \hfill (12)

for both may be obtained by taking the field of the unperturbed beam just out-
side it, differentiating with respect to $z$, keeping only terms of zero order
in $z$, and multiplying by $z_b$. So we have

$$E_z(z) = E_{lz} + z E_{oz}' + \frac{4\pi \sigma}{d}(z - z_b).$$  \hfill (13)

This contains two terms which are linear in $z$, rather than in $z_b$ and there-
fore $\xi$. Since we are trying to do a first order perturbation calculation in
$\xi$, this makes a certain difficulty, as we must decide how to deal with these terms.
In LNS they are dealt with by substituting $z_b$ for $z$; in section 2 we have
argued that they can be assimilated into the properties of the machine's static
field, and therefore ignored for calculation of the phenomenon that we are
studying. To decide between these views we shall put such a term through the
perturbation treatment and see what comes out.

The first order perturbed part of the distribution function is calculated
by eqn. (3.6) of LNS:

$$\psi_1 = -\int \mathbf{P}_z \cdot \frac{\partial \psi_0}{\partial \mathbf{P}_z} \, dt$$  \hfill (14)

with the integration carried out along the unperturbed orbit. Suppose $\mathbf{P}_z$
contains a term linear in $z$, say $Kz$; it will contribute to $\psi_1$ an amount

$$-K \int z \frac{\partial \psi_0}{\partial \mathbf{P}_z} \, dt.$$  \hfill (15)
We take from LNS, on the unperturbed orbit: -

\[ z = a \sin \phi \]

\[ \frac{\partial \psi_0}{\partial F_z} = \frac{\cos \phi}{m \nu_z \Omega} \frac{\partial \psi_0}{\partial a} \]

So (15) becomes

\[ -\frac{K_a}{m \nu_z \Omega^2} \frac{\partial \psi_0}{\partial a} \int \sin \phi \cos \phi \, dt \]

in which \( \phi \) is \( \nu_z \Omega t \). This gives

\[ -\frac{K_a}{4 m \nu_z^2 \Omega^2} \frac{\partial \psi_0}{\partial a} \cos 2\phi \]

(16).

The resulting electric dipole moment from this perturbation of the distribution is obtained by multiplying by \( e_z \) and integrating with respect to \( \phi \), \( a \), and \( \omega \). With \( z = a \sin \phi \), the \( \phi \)-integration yields zero, because

\[ \int_0^{2\pi} \sin \phi \cos 2\phi \, d\phi = 0 \]

(17).

So one sees that \( z \)-terms in the perturbing field do perturb the distribution, (16), but in such a way as to contribute nothing to the mean displacement and the dipole moment, and so nothing to the dispersion relation. The effective part of (13) is therefore

\[ B_{lz} = -\frac{4 \pi e}{d} a \]

(18)

and the corresponding magnetic field is

\[ B_{lx} = -\beta \frac{4 \pi e}{d} \]

(19).

With these modified fields the force used in the Vlasov equation should be

\[ \frac{F}{e} = (4 \pi e \xi /w) e^{i(ky-\omega t)} \sum_s \sigma_s^2 \left[ \eta (1-\beta^2) \text{ctnh} \left( \eta H/2 \right) \right. \]

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\[ -\frac{2}{d} (1-\beta^2) + i \beta^2 R (c \eta/w) \text{csch}^2 \left( \eta H/n \right) \]

(20)
in place of LNS (2.21).

It may be seen that both the real terms depend on the particle velocity by way of \((1 - \beta^2)\) factors, and that the direct self-field term proportional to \(1/d\) will dominate if \(d\) is small compared with \(H\) and with the horizontal beam width. The quantity \(U\), which enters the threshold consideration, can thus be written

\[
U = - \frac{\epsilon^2 N}{\gamma^2} \frac{e \omega c m_0}{d \Delta \beta \gamma} \mathcal{H} \left( \frac{d}{H}, \frac{H}{w}, \frac{\Delta}{H} \right)
\]  

(21)

where \(\mathcal{H}\) is a function of the geometry, close to 1 for \(d/H\) and \(d/\Delta\) small. The ratio of our expression (21) to their (3.16c) is approximately

\[
\frac{H^2}{d \Delta \gamma^2}
\]  

(22)

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Reference.


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Figure 1.

LINES OF FORCE WITH IDEAL METALLIC BOUNDARY.

Electric field, charge and image.

Magnetic field, current and image.

$E_{\parallel} = 0$

$B_{\perp} = 0$