Some Properties of Quadrupole Triplets

1. Introduction

Although quadrupole doublets have been widely described in a number of publications, few authors\textsuperscript{1) 2) 3) 4) 5)} have devoted their attention to quadrupole triplets, and it seems difficult to find in the literature any comprehensive set of formulae translating their optical properties.

An attempt is made below to put into evidence some properties of triplets, primarily in the frame of the thin lens approximation.

2. The General Triplet

2.1 The Transfer Matrices

We use the notation of Fig. 1 where $L_1$, $L_2$, $L_3$ represent the effective lengths of the quadrupoles, and $\Delta_1$, $\Delta_2$ the distances between their central planes.

Let $G_1$, $G_2$, $G_3$ be the gradients of the magnetic fields in the three quadrupoles. Using the notation

\[
K_1^2 = \frac{G_1}{Br}, \quad K_2^2 = \frac{G_2}{Br}, \quad K_3^2 = \frac{G_3}{Br}
\]

(1)

where $Br$ stands for the particle momentum divided by its charge, we can write for the three focal lengths

\[
\frac{1}{f_1c} = K_1 \sin K_1 L_1, \quad \frac{1}{f_2c} = K_2 \sin K_2 L_2, \quad \frac{1}{f_3c} = K_3 \sin K_3 L_3
\]

(2)

\[
\frac{1}{f_1d} = K_1 \sh K_1 L_1, \quad \frac{1}{f_2d} = K_2 \sh K_2 L_2, \quad \frac{1}{f_3d} = K_3 \sh K_3 L_3
\]

(3)

according to whether the quadrupoles are focusing or defocusing.

We shall use the notation \texttt{cde}

and \texttt{dcd}
for the two fundamental planes (convergent-divergent-convergent and divergent-convergent-divergent).

In the thin lens approximation we can then write for the transfer matrices

$$ M_{ocd} = \begin{bmatrix} 1 & 0 & x & 1 & \Delta_2 & x & 1 & 0 & x & 1 & \Delta_1 & x & 1 & 0 \end{bmatrix} $$

and

$$ M_{dco} = \begin{bmatrix} 1 & 0 & x & 1 & \Delta_2 & x & 1 & 0 & x & 1 & \Delta_1 & x & 1 & 0 \end{bmatrix} $$

We shall simplify the writing by introducing the symbols

$$ \frac{\Delta_1}{f_{1c}} = x_{11}, \quad \frac{\Delta_1}{f_{2c}} = x_{12}, \quad \frac{\Delta_1}{f_{3c}} = x_{13} $$

$$ \frac{\Delta_2}{f_{1c}} = x_{21}, \quad \frac{\Delta_2}{f_{2c}} = x_{22}, \quad \frac{\Delta_2}{f_{3c}} = x_{23} $$

and similarly

$$ \frac{\Delta_1}{f_{1d}} = y_{11}, \quad \frac{\Delta_1}{f_{2d}} = y_{12}, \quad \frac{\Delta_1}{f_{3d}} = y_{13} $$

$$ \frac{\Delta_2}{f_{1d}} = y_{21}, \quad \frac{\Delta_2}{f_{2d}} = y_{22}, \quad \frac{\Delta_2}{f_{3d}} = y_{23} $$

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In general \( x \) and \( y \) will be small quantities with respect to unity. Moreover the two arrays

\[
\begin{array}{ccc}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23}
\end{array}
\]

and

\[
\begin{array}{ccc}
  y_{11} & y_{12} & y_{13} \\
  y_{21} & y_{22} & y_{23}
\end{array}
\]

possess the property

\[ x_{ij} x_{kl} = x_{il} x_{kj} \]  \hspace{1cm} (8)

and

\[ y_{ij} y_{kl} = y_{il} y_{kj} \]  \hspace{1cm} (9)

With these notations the transfer matrices for the two fundamental planes of the triplet become

\[
M_{xyc} = \begin{vmatrix}
(1-x_{11})(1+y_{22}) - x_{21} & \Delta_1 + \Delta_2(1 + y_{12}) \\
(1-x_{23}) \left[ \frac{1}{P_{12}} - \frac{1}{P_{10}} (1+y_{12}) \right] - \frac{1}{P_{30}} (1-x_{11}) & (1 + y_{12})(1 - x_{23}) - x_{13}
\end{vmatrix}
\]  \hspace{1cm} (10)

and

\[
M_{ycd} = \begin{vmatrix}
(1+y_{11})(1-x_{22}) + y_{21} & \Delta_1 + \Delta_2(1 - x_{12}) \\
(1+y_{23}) \left[ \frac{1}{P_{12}}(1-x_{22}) - \frac{1}{P_{20}} \right] + \frac{1}{P_{30}}(1+y_{11}) & (1 - x_{12})(1 + y_{23}) + y_{13}
\end{vmatrix}
\]  \hspace{1cm} (11)

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2.2 Optical Properties

The matrix elements may be translated immediately in terms of optical properties.

We obtain

a) for the focal distances of the triplet lens

\[
\frac{1}{f_{\text{cdc}}} = (1 - x_{23}) \left[ \frac{1}{f_{1c}} (1 + y_{12}) - \frac{1}{f_{2d}} \right] + \frac{1}{f_{3c}} (1 - x_{11}) \tag{12}
\]

\[
\frac{1}{f_{\text{dd}}} = (1 + y_{23}) \left[ \frac{1}{f_{2c}} - \frac{1}{f_{1d}} (1 - x_{12}) \right] - \frac{1}{f_{3d}} (1 + y_{11}) \tag{13}
\]

b) for the position of the image-foci with respect to plane 0, (fig. 1)

\[
F_{\text{cdc}} = f_{\text{cdc}} \left[ (1 - x_{11})(1 + y_{22}) - x_{21} \right] \tag{14}
\]

\[
F_{\text{dd}} = f_{\text{dd}} \left[ (1 + y_{11})(1 - x_{22}) + y_{21} \right] \tag{15}
\]

c) for the position of the principal object and image planes

\[
z_{\text{cdc}} (H) = f_{\text{cdc}} \left[ 1 + x_{13} - (1 + y_{12})(1 - x_{23}) \right] \tag{16}
\]

\[
z_{\text{cdc}} (H') = f_{\text{cdc}} \left[ (1 - x_{11})(1 + y_{22}) - (1 + x_{21}) \right] \tag{17}
\]

\[
z_{\text{dd}} (H) = f_{\text{dd}} \left[ 1 - y_{13} - (1 - x_{12})(1 + y_{23}) \right] \tag{18}
\]

\[
z_{\text{dd}} (H') = f_{\text{dd}} \left[ (1 + y_{11})(1 - x_{22}) - (1 - y_{21}) \right] \tag{19}
\]
These abscissae are taken with respect to plane \( \Omega_1 \) for \( H \) and with respect to plane \( \Omega_3 \) for \( H' \) (Fig. 1).

\[ d \) An important criterion to decide whether a lens can be considered as thin or not is the distance between the principal planes. This distance can be written for the two fundamental directions

\[ \delta_{\text{odo}} = z_{\text{odo}} (H') - z_{\text{odo}} (H) + \Delta_1 + \Delta_2 \quad (20) \]

\[ \delta_{\text{dcd}} = z_{\text{dcd}} (H') - z_{\text{dcd}} (H) + \Delta_1 + \Delta_2 \quad (21) \]

Replacing equations (16) - (19) in (20) and (21) we find, by using (8) and (5) and reverting to the initial symbols

\[ \frac{\delta_{\text{odo}}}{F_{\text{odo}}} = \frac{1}{F_{2d}} \left( \frac{\Delta_1^2}{F_{1c}^2} + \frac{\Delta_2^2}{F_{3c}^2} \right) - \left( \frac{\Delta_1}{F_{1c}} + \frac{\Delta_2}{F_{3c}} \right)^2 - \frac{\Delta_1 \Delta_2 (\Delta_1 + \Delta_2)}{F_{1c} F_{2d} F_{3c}} \quad (22) \]

\[ \frac{\delta_{\text{dcd}}}{F_{\text{dcd}}} = \frac{1}{F_{2c}} \left( \frac{\Delta_1^2}{F_{1d}^2} + \frac{\Delta_2^2}{F_{3d}^2} \right) - \left( \frac{\Delta_1}{F_{1d}} + \frac{\Delta_2}{F_{3d}} \right)^2 + \frac{\Delta_1 \Delta_2 (\Delta_1 + \Delta_2)}{F_{1d} F_{2d} F_{3d}} \quad (23) \]

It should be noted that by putting \( \Delta_1 = \Delta, \Delta_2 = 0, \frac{1}{F_{3c,d}} = 0 \) we find immediately all formulae derived in doublet theory.

For example the distance between the principal planes in a doublet can be written

\[ \frac{\delta_{\text{cd}}}{F_{\text{cd}}} = \frac{\Delta^2}{F_{1c} F_{2d}} \quad (24) \]

\[ \frac{\delta_{\text{dc}}}{F_{\text{dc}}} = \frac{\Delta^2}{F_{1d} F_{2c}} \quad (25) \]
It is seen that in a doublet there is no way of compensation and \( \delta_{cd} \) and \( \delta_{dc} \) may be large as compared to \( f_{cd} \) and \( f_{dc} \). In the case of a triplet one can try to achieve coincidence of the principal planes. From equations (22) and (23) we find for the relevant conditions

\[
\Delta_1^2 f_{2c} + \Delta_2^2 f_{1c} - f_{2d} (\Delta_1 + \Delta_2)^2 - \Delta_1 \Delta_2 (\Delta_1 + \Delta_2) = 0
\]  
\[\text{(26)}\]

\[
\Delta_1^2 f_{2d} + \Delta_2^2 f_{1d} - f_{2c} (\Delta_1 + \Delta_2)^2 + \Delta_1 \Delta_2 (\Delta_1 + \Delta_2) = 0
\]  
\[\text{(27)}\]

For given excitations of the individual quadrupoles it should therefore be possible to determine \( \Delta_1 \) and \( \Delta_2 \) so as to achieve coincidence of the principal planes in both directions.

In the special case where \( \Delta_1 = \Delta_2 = \Delta \) the conditions (26) and (27) become

\[
f_{1c} + f_{3c} - 4f_{2d} - 2\Delta = 0
\]  
\[\text{(28)}\]

\[
f_{1d} + f_{3d} - 4f_{2c} + 2\Delta = 0
\]  
\[\text{(29)}\]

or, alternatively

\[
(f_{1c} + f_{1d}) + (f_{3c} + f_{3d}) = 4 (f_{2c} + f_{2d})
\]  
\[\text{(30)}\]

\[
\Delta = \frac{1}{2} (f_{1c} + f_{3c} - 4f_{2d})
\]  
\[\text{(31)}\]

e) The condition for equal focal lengths of a general triplet i.e., \( f_{odc} = f_{dcd} \) can be written from (12) and (13)
\[(1 - x_{23}) \left[ \frac{1}{F_{1c}}(1 + y_{12}) - \frac{1}{F_{2d}} \right] + \frac{1}{F_{3c}} (1 - x_{11}) = \]

\[(1 + y_{23}) \left[ \frac{1}{F_{2c}} - \frac{1}{F_{1d}} (1 - x_{12}) \right] - \frac{1}{F_{3d}} (1 + y_{11}) \]  \hspace{1cm} (32)

Carrying out the calculations we find

\[
\left( \frac{1}{F_{1c}} + \frac{1}{F_{1d}} \right) - \left( \frac{1}{F_{2c}} + \frac{1}{F_{2d}} \right) + \left( \frac{1}{F_{3c}} + \frac{1}{F_{3d}} \right) - \Delta_1 \Delta_2 \left( \frac{1}{F_{1c}} \frac{1}{F_{2d}} + \frac{1}{F_{1d}} \frac{1}{F_{2c}} \right)
\]

\[+ \Delta_1 \left[ \frac{1}{F_{1c}} \left( \frac{1}{F_{2d}} - \frac{1}{F_{3c}} \right) + \frac{1}{F_{1d}} \left( \frac{1}{F_{3d}} - \frac{1}{F_{2c}} \right) \right] + \Delta_2 \left[ \frac{1}{F_{2c}} \left( \frac{1}{F_{2d}} - \frac{1}{F_{1c}} \right) + \frac{1}{F_{3d}} \left( \frac{1}{F_{1d}} - \frac{1}{F_{2c}} \right) \right] = 0 \]  \hspace{1cm} (33)

For the special case \(\Delta_1 = \Delta_2 = \Delta\) the condition of equal focal lengths can be written

\[
\left( \frac{1}{F_{1c}} + \frac{1}{F_{1d}} \right) - \left( \frac{1}{F_{2c}} + \frac{1}{F_{2d}} \right) + \left( \frac{1}{F_{3c}} + \frac{1}{F_{3d}} \right)
\]

\[+ \Delta \left[ \frac{1}{F_{2d}} \left( \frac{1}{F_{1c}} + \frac{1}{F_{3c}} \right) - \frac{1}{F_{2c}} \left( \frac{1}{F_{1d}} + \frac{1}{F_{3d}} \right) + 2 \left( \frac{1}{F_{1d}} \frac{1}{F_{2c}} - \frac{1}{F_{1c}} \frac{1}{F_{3d}} \right) \right] \]  \hspace{1cm} (34)

\[- \Delta^2 \left( \frac{1}{F_{1c} F_{2d} F_{3c}} + \frac{1}{F_{1d} F_{2c} F_{3d}} \right) \]  \hspace{1cm} (35)

For given excitations of the individual lenses it is therefore possible to choose the distances between the quadrupoles so as to achieve equal focal lengths in both directions.

f) The conditions for afocality in both directions i.e.

\[
\frac{1}{F_{cde}} = 0 \hspace{1cm} \frac{1}{F_{dcd}} = 0
\]
can be obtained from (12) and (13).

\[ (1 - x_{23}) \left[ \frac{1}{f_{10}} (1 + y_{12}) - \frac{1}{f_{2d}} (1 - x_{11}) \right] + \frac{1}{f_{3c}} (1 - x_{11}) = 0 \]  
\[ (1 + y_{23}) \left[ \frac{1}{f_{2c}} - \frac{1}{f_{1d}} (1 - x_{12}) \right] - \frac{1}{f_{3d}} (1 + y_{11}) = 0 \]  

(36)  
(37)

Working out we find

\[ f_{3c} (f_{2d} - f_{10}) + f_{10} f_{2d} + \Delta_1 (f_{3c} - f_{2d}) + \Delta_2 (f_{10} - f_{2d}) - \Delta_1 \Delta_2 = 0 \]  
\[ f_{3d} (f_{1d} - f_{2c}) - f_{1d} f_{2c} - \Delta_1 (f_{3d} + f_{2c}) + \Delta_2 (f_{1d} - f_{2c}) + \Delta_1 \Delta_2 = 0 \]  

(38)  
(39)

For the more special case where \( \Delta_1 = \Delta_2 = \Delta \) these conditions become

\[ f_{3c} (f_{2d} - f_{10}) + f_{10} f_{2d} + \Delta (f_{10} + f_{3c} - 2f_{2d}) - \Delta^2 = 0 \]  
\[ f_{3d} (f_{1d} - f_{2c}) - f_{1d} f_{2c} + \Delta (f_{1d} + f_{3d} - 2f_{2c}) + \Delta^2 = 0 \]  

(40)  
(41)

It can be seen that the conditions of principal plane coincidence (26) and (27) are equivalent to those of afocality (40) and (41) if and only if

\[ \Delta_1 = \Delta_2 \]  
\[ f_{10} = f_{3c} \]  
\[ f_{1d} = f_{3d} \]  

(42)

In other words, in a symmetric triplet the principal planes can be brought into coincidence only for infinite focal lengths.
Using in this case the indices c (external) instead of 1 and 3 and i (internal) instead of 2, the sets (26), (27) and (40), (41) become with the conditions (42)

\[ f_{oc} + f_{cd} = 2(f_{ic} + f_{id}) \]  
\[ \Delta = f_{oc} - 2f_{id} = 2f_{ic} - f_{cd} \]

\[ f_{oc} + f_{cd} = 2(f_{ic} + f_{id}) \]  
\[ \Delta = f_{oc} - 2f_{id} = 2f_{ic} - f_{cd} \]

G) A triplet is generally weaker than a doublet. Let us look for the conditions under which a triplet \([f_1, f_2, f_3; \Delta_1, \Delta_2]\) can be made equivalent in overall focusing strength to a doublet \([f_1, f_2]\).

We can write the focal distances of the triplet in the form

\[ \frac{1}{f_{oc}} = \frac{1}{f_{1c}} + \frac{1}{f_{3c}} \left( 1 - \frac{\Delta_1}{f_{1c}} - \frac{\Delta_2}{f_{3c}} \right) \]  
\[ \frac{1}{f_{dc}} = \frac{1}{f_{1c}} - \frac{1}{f_{3d}} \left( 1 + \frac{\Delta_1}{f_{1c}} - \frac{\Delta_2}{f_{3d}} \right) \]

The conditions of equal focal lengths (i.e. \(f_{oc} = f_{cd}\) and \(f_{dc} = f_{do}\)) become therefore

\[ \frac{1}{f_{3c}} (1 - \frac{\Delta_1}{f_{1c}}) = \frac{\Delta_2}{f_{3c}} \frac{1}{f_{cd}} \]  
\[ \frac{1}{f_{3d}} (1 + \frac{\Delta_1}{f_{1d}}) = \frac{\Delta_2}{f_{3d}} \frac{1}{f_{dc}} \]

Discarding the obvious solution of an \( \infty \)focal lens \( \frac{1}{f_{3c}} = \frac{1}{f_{3d}} = 0 \) we find the conditions
\[ f_{10} f_{2+1} + \Delta_1 f_{2d} + \Delta_2 (f_{2d} - f_{10}) + \Delta_1 \Delta_2 = 0 \]  
\[ f_{1d} f_{20} + \Delta_1 f_{2c} + \Delta_2 (f_{2c} - f_{1d}) + \Delta_1 \Delta_2 = 0 \] 

which give \( \Delta_1 \) and \( \Delta_2 \). The focal lengths \( f_{30}, f_{3d} \) of the third quadrupole do not come into the picture.

3. The Symmetric Triplet

3.1 General Properties

We shall assume now

\[ \Delta_1 = \Delta_2 = \Delta \]
\[ L_1 = L_3 = L_0 \]
\[ L_2 = L_1 \]
\[ G_1 = G_3 = G_0 \]
\[ G_2 = G_1 \]

and therefore

\[ K_1 = K_3 = K_0 \]
\[ K_2 = K_1 \]  

The focusing strengths become

\[ \frac{1}{f_{1c0}} = K_0 \sin K_0 L_0 \]
\[ \frac{1}{f_{1c1}} = K_1 \sin K_1 L_1 \] 

\[ \frac{1}{f_{1c1}} = K_0 \sinh K_0 L_0 \]
\[ \frac{1}{f_{1c1}} = K_1 \sinh K_1 L_1 \]
We shall again introduce the symbols

\[
\frac{\Delta}{f_{e}} = x_e \quad \frac{\Delta}{f_{i}} = x_i \tag{55}
\]

\[
\frac{\Delta}{f_{cd}} = y_e \quad \frac{\Delta}{f_{id}} = y_i \tag{56}
\]

With these notations the transfer matrices can be written

\[
\mathbf{M}_{cdc} = \begin{bmatrix}
1 - 2x_e + y_i (1 - x_e) & \Delta(2 + y_i) \\
1 - \frac{x_e}{\Delta} \left[ 2x_e + y_i (1 - x_e) \right] & 1 - 2x_e + y_i (1 - x_e)
\end{bmatrix} \tag{57}
\]

\[
\mathbf{M}_{dcd} = \begin{bmatrix}
1 + 2y_e - x_i (1 + y_o) & \Delta(2 - x_i) \\
1 + \frac{y_e}{\Delta} \left[ 2y_e - x_i (1 + y_o) \right] & 1 + 2y_e - x_i (1 + y_o)
\end{bmatrix} \tag{58}
\]

The focal distances may now be written in the non-dimensional form

\[
\frac{\Delta}{f_{cdc}} = X = (1 - x_e) \left[ 2x_e - y_i (1 - x_e) \right] \tag{59}
\]

\[
\frac{\Delta}{f_{dcd}} = Y = (1 + y_o) \left[ -2y_e + x_i (1 + y_o) \right] \tag{60}
\]
and the same holds for the position of the foci

\[
\frac{\rho_{\text{cde}}}{\Delta} = \frac{1 - 2x_0 + y_1(1 - x_0)}{(1 - x_0)[2x_0 - y_1(1 - x_0)]} \tag{61}
\]

\[
\frac{\rho_{\text{dcd}}}{\Delta} = \frac{1 + 2y_0 - x_1(1 + y_0)}{(1 + y_0)[-2y_0 + x_1(1 + y_0)]} \tag{62}
\]

Calculating now the positions of the principal planes we find the remarkable relations

\[
\frac{z_{\text{cde}}(H)}{\Delta} = \frac{1}{1 - x_0} \quad \text{and} \quad \frac{z_{\text{dcd}}(H')}{\Delta} = -\frac{1}{1 - x_0} \tag{63}
\]

\[
\frac{z_{\text{dcd}}(H)}{\Delta} = \frac{1}{1 + y_0} \quad \text{and} \quad \frac{z_{\text{dcd}}(H')}{\Delta} = -\frac{1}{1 + y_0} \tag{64}
\]

This shows two properties

a) Provided \( x_0, y_0 \ll 1 \), i.e. provided the excitation of the outer quadrupoles is small, all principal planes coincide with the geometric center of the lens. Under these conditions the symmetric triplet can be considered as a thin lens of fixed position.

b) To adjust the optical properties of the triplet lens, one can vary \( x_1, y_1 \), i.e. the excitation of the inner quadrupole, without modifying the position of the principal planes.

These properties have no analogy in doublet theory.
In deriving equations (59) to (64) we have tacitly assumed
\[2x_o \neq y_1 (1 - x_o)\]
\[2y_o \neq x_1 (1 + y_o)\]

It is easy to check that the equalities
\[2x_o = y_1 (1 - x_o)\] \hspace{1cm} (65)
\[2y_o = x_1 (1 + y_o)\] \hspace{1cm} (66)
correspond precisely to the conditions of afocality as displayed by equations (43) and (44).

3.2 Stigmatic Operation of A Symmetric Triplet

If \( F \) denotes the position of the object and \( F' \) the position of the image, one generally has
\[F' = - \frac{M_{12} + M_{11}}{M_{22} + M_{21}} F\] \hspace{1cm} (67)

where the \( M \)'s are the elements of the triplet transfer matrix.

To deal with non-dimensional quantities we shall put
\[\frac{F}{\Delta} = F_0\] \hspace{1cm} (68)
\[\frac{F'}{\Delta} = F'_0\] \hspace{1cm} (69)

Writing out equation (67) in the two planes (ccd and dcd) we find
\[
F'_{\alpha}(cdo) = -\frac{2 + y_i + [1 - 2x_o + y_i(1 - x_o)]F_0}{1 - 2x_o + y_i(1 - x_o) + (1 - x_o)[-2x_o + y_i(1 - x_o)]F_0}
\]  
(73)

\[
F'_{\alpha}(cdo) = -\frac{2 - x_i + [1 + 2y_o - x_i(1 + y_o)]F_0}{1 + 2y_o - x_i(1 + y_o) + (1 + y_o)[2y_o - x_i(1 + y_o)]F_0}
\]  
(71)

The condition for stigmatic operation writes therefore

\[
\frac{2 + y_i + [1 - 2x_o + y_i(1 - x_o)]F_0}{1 - 2x_o + y_i(1 - x_o) + (1 - x_o)[-2x_o + y_i(1 - x_o)]F_0} =
\]

(72)

\[
\frac{2 - x_i + [1 + 2y_o - x_i(1 + y_o)]F_0}{1 + 2y_o - x_i(1 + y_o) + (1 + y_o)[2y_o - x_i(1 + y_o)]F_0}
\]

We shall now make use of the fact that the \(x\)'s and \(y\)'s are small compared to unity and neglect their products and powers; this procedure is especially justified in high energy work where the focal distances are large. Under these circumstances the condition for stigmatic operation becomes

\[
\frac{2 + y_i + (1 - 2x_o + y_i) F_0}{1 - 2x_o + y_i + (y_i - 2x_o) F_0} = \frac{2 - x_i + (1 + 2y_o - x_i) F_0}{1 + 2y_o - x_i + (2y_o - x_i) F_0}
\]  
(73)

We shall again discard the case of afoecality which now writes \(2x_o = y_i, 2y_o = x_i\). Equation (73) then becomes

\[
(T_i - 2T_o) F_0^2 + 2(T_i - 2T_o) F_0 + T_i - 4T_o = 0
\]  
(74)

where we have put
\[ T_o = x_o + y_o \] (75)
\[ T_i = x_i + y_i \] (76)

\( T_o \) can be taken as a measure of the focusing strength of the outer quadrupoles whereas \( T_i \) translates the focusing power of the inner quadrupole. Both quantities are always positive and generally small with respect to unity.

The solution of (74) is
\[ F_o = -1 + \sqrt{\frac{2T_0}{T_i - 2T_0}} \] (77)

and obviously \( F'o = F_o \), the tripot being symmetric.

For \( F_o \) to be real and positive we must have
\[ 2T_0 < T_i < 4T_0 \] (78)

If \( T_i = 2T_0 \) \( F_o \) is rejected to infinity. If \( T_i = 4T_0 \) \( F_o \) and \( F'o \) would coincide with the central planes of the outer lenses. Practically therefore the range is somewhat more limited than the interval expressed by the inequality (78). Any point situated between infinity and the lens entrance can therefore be made stigmatic when \( T_i \) varies from \( 2T_0 \) to somewhat less than \( 4T_0 \). In other words it is sufficient to vary the excitation of the central lens by a factor of somewhat less than 2 to obtain stigmatic operation for all points on an infinite line. The excitation of the outer quadrupoles can conveniently be kept constant to insure constant position of the principal planes.
3.3 Magnification

In general this is given by

\[ g = \frac{1}{M_{22} + M_{21} F} \]  

(79)

In the case of a symmetric triplet we find, using the matrix elements

\[ \frac{1}{g_{doc}} = 1 - 2x_o + y_1 (1 - x_o) + (1 - x_o) \left[ -2x_o + y_1 (1 - x_o) \right] F_0 \]  

(80)

\[ \frac{1}{g_{dod}} = 1 + 2y_o - x_1 (1 + y_o) + (1 + y_o) \left[ 2y_o - x_1 (1 + y_o) \right] F_0 \]  

(81)

Here it proves more convenient to use the angular magnifications which are the reciprocal values of the linear magnifications.

For a symmetric triplet operating under stigmatic conditions we find by replacing \( F_0 \) and using the same approximation as in the preceding paragraph

\[ \frac{1}{g_{doc}} - \frac{1}{g_{dod}} = \frac{T_1 - 2T_o + \sqrt{2T_o (T_1 - 2T_o)}}{2} \]  

(82)

We have shown that for stigmatic operation \( T_1 \) varies at most between \( 2T_o \) and \( 4T_o \). Consequently the difference

\[ \frac{1}{g_{doc}} - \frac{1}{g_{dod}} \]  

will vary at most between \( 0 \) and \( 4T_o \); it will therefore be small in general, contrary to what happens in a symmetric doublet where stigmatic operation implies large discrepancies in magnification. Moreover it is seen that

\[ g_{dod} > g_{doc}, \]  

(83)

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Writing

\[ \Theta_c = K_c L_c \] (84)

\[ \Theta_i = K_i L_i \] (85)

and denoting now by \( d \) the distance between the effective end planes of the quadruploes, we can write the transfer matrices for a symmetric triplet

\[
M_{edc} = \begin{bmatrix} \cos \Theta_c & \frac{1}{K_c} \sin \Theta_c \\ -K_c \sin \Theta_c & \cos \Theta_c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{ch} \Theta_i & \frac{1}{K_i} \text{sh} \Theta_i \\ K_i \text{sh} \Theta_i & \text{ch} \Theta_i \end{bmatrix} 
\]

\[ M_{dcd} = \begin{bmatrix} \text{ch} \Theta_c & \frac{1}{K_c} \text{sh} \Theta_c \\ -K_c \sin \Theta_c & \cos \Theta_c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta_i & \frac{1}{K_i} \sin \Theta_i \\ -K_i \sin \Theta_i & \cos \Theta_i \end{bmatrix} 
\]

\[ M_{edc} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{ch} \Theta_c & -\frac{1}{K_c} \text{sh} \Theta_c \\ K_c \sin \Theta_c & \cos \Theta_c \end{bmatrix} \begin{bmatrix} \text{ch} \Theta_i & \frac{1}{K_i} \text{sh} \Theta_i \\ K_i \text{sh} \Theta_i & \text{ch} \Theta_i \end{bmatrix} \]

The calculations can be somewhat simplified by using the reflection properties of symmetric matrices. The formulae one obtains for the optical properties of the lens are just manageable. For instance one finds for the two focal distances
\[
\frac{1}{F_{\text{ads}}} = -\frac{dK_o^2}{d\Theta_o} \left( \cosh \Theta_i + \frac{dK_1}{d\Theta_o} \sinh \Theta_i \right) \left( 1 - \cos 2\Theta_o - \frac{\sin 2\Theta_o}{dK_o} \right) \\
- \frac{\sinh \Theta_i}{2K_1} \left[ K_i^2 + K_o^2 - \frac{d^2}{dK^2} K_i^2 K_o^2 + (K_i^2 - K_o^2 + \frac{d^2}{dK^2} K_i^2 K_o^2) \cos 2\Theta_o \right]
\]

\[
\frac{1}{F_{\text{dcd}}} = \frac{dK_o^2}{d\Theta_o} \left( \cos \Theta_i - \frac{dK_1}{d\Theta_o} \sin \Theta_i \right) \left( 1 - \cosh 2\Theta_o - \frac{\sin 2\Theta_o}{dK_o} \right) \\
+ \frac{\sin \Theta_i}{2K_1} \left[ K_i^2 + K_o^2 - \frac{d^2}{dK^2} K_i^2 K_o^2 + (K_i^2 - K_o^2 - \frac{d^2}{dK^2} K_i^2 K_o^2) \cosh 2\Theta_o \right]
\]

By assuming for instance equal gradients and/or zero value for d one could considerably simplify these formulae. The physical justification of such simplifications is however questionable.
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