ON SOME PROBLEMS CONNECTED WITH THE ANALYSIS
OF SHOWERS IN THE PRODUCTION REGION
OF THE SPARK CHAMBER FOR THE NEUTRINO EXPERIMENT

by

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I. ANALYSIS OF SHOWERS IN SPARK CHAMBERS

1. Statement of the problem

In spark chambers made of a succession of metal plates, such as that used for the neutrino experiment or of similar type, electronic showers produce a number of sparks within a region of conical shape, the axis of which coincides, on the average, with the direction of the primary particle (an electron or a photon) which produced it (Figs. 1 and 2).

The determination of the axis, from the pictures of the event, may be of importance when the kinematics of the parent event is to be computed.

Moreover, when two showers are produced close to one another — as, for instance, in the case of the decay of a π⁰ or an η⁰ particle — it is necessary to measure the angle between the two showers to determine the energy of the parent particle. When the latter is of high energy the two showers may overlap and the discrimination becomes difficult.

Since the sparks produced by showers are, on the average, symmetrically distributed around the original direction of the initiating electron or photon, but their distribution in space fluctuates both in position and density from case to case, the problem is essentially one of statistical estimate. When the determination of the axis of the showers, especially in the case of two overlapping ones, is done by visual methods, on the print, by an observer, it becomes often a subjective question.

We have therefore tried to develop a statistical method to carry out the analysis independently of the observer, making it possible, at the same time, to establish limits of confidence on the determination of the relevant parameters, making use of all the information supplied by the photogrammes.
I.2 Determination of the shower axis.

Suppose we observe a shower in a spark chamber, at an angle \( \vartheta \) with respect to the perpendicular to the plates. In a picture this shower will appear projected at an angle \( \vartheta \) as indicated in Fig. 2.

It can be seen experimentally that the distribution of sparks, in each gap, around the axis, is approximately gaussian. In particular, it is very nearly gaussian in the region close to the axis, departing from it at larger distances.

The fact that such a distribution should be close to a gaussian could be expected on theoretical grounds; that it should be so in projection, considering also the overlapping of sparks, is not obvious. We take it as an experimental fact.

At different depths the apex of the distribution was found to fluctuate. The fluctuation, however, is not large and corresponds to less than 2\(^\circ\) at each depth. One should therefore expect, from best-fitting over-all depths, a far higher precision. This in fact seems to be the case as it will be proved later.

Such a distribution is observed strictly only for angles \( \vartheta \) close to zero. It does not depart appreciably from it for \( \vartheta < 30^\circ \). Thus, for the \( i \)-th gap, we expect a frequency distribution \( f_i \) given by the equation:

\[
f_i(y_{ij}) = e^{-\psi_{ij}^2},
\]

where

\[
\psi_{ij} = \frac{y_{ij} - y_0 - x_i \tan \vartheta}{\Delta y_i / \cos \vartheta};
\]

\( \Delta y_i \) being the spread expected at the depth of the \( i \)-th gap; the other parameters as indicated above and in Fig. 2.

Experience soon proved that the application of the standard Maximum-Likelihood method would lead to difficulties. In a spark chamber it is often impossible to establish whether or not a particular spark
System of coordinates used in the analysis of showers.

FIG. 2
belongs to a shower. With the above method a distant spark carries greater weight, in the equation, than those close to the core. Thus, a mistake, in associating with a shower, a spark which does not belong to it, leads to a comparatively great error in the determination of the axis.

This difficulty is avoided if one introduces the function 

\[ S_i(y_{ij} | \theta) = \frac{1}{A_i} \sum_{j} e^{-\frac{\gamma^2_{ij}}{\sigma_i}} \]  

(I.1)

(A\textsubscript{i} being a normalization factor) to describe the spark distribution in each gap. Introducing the variable

\[ \xi_{ij} = \frac{y_{ij} - y_0}{x_i}; \quad m = \tan \theta; \quad \sigma_i = \frac{\Delta y_i}{x_i} \sqrt{1 + m^2} \]

we get

\[ S_i(\xi_{ij} | m) = \frac{1}{n_i \sigma_i(m) \sqrt{\pi}} \sum_{j} e^{-\frac{(\xi_{ij} - m)^2}{\sigma_i}} \]  

(I.2)

Thus, a distant spark carries a weight, in the determination of the axis (i.e. of \( m \)), which falls rapidly with the distance from the core \( \ast \ast \).

We shall restrict the discussion to the case \( m^2 \ll 1 \).

The estimate of \( m \) is then the value \( m_1 \) of \( m \) which satisfies the equation

\[ \frac{\partial S_i}{\partial m} = 0. \]

\( \ast \) In what follows \( i \) denotes a gap and \( j \) a spark in any gap. Thus, \( y_{ij} \) denotes the co-ordinate of a \( j \)-th spark in an \( i \)-th gap.

\( \ast \ast \) It must be pointed out that the function \( S_i \) cannot be considered a 'probability distribution'.

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It can be shown (see Appendix A) that the value $m_1$ is asymptotically normal around the true (unknown) values $m_o$ with variance

$$\sigma_{m_1} = \frac{2}{3^{1/4}} \frac{\sigma_i}{\sqrt{n_i}},$$

having taken, for simplicity, $\sigma_i$ equal to the true variance of the spark distribution - which we can suppose to be known; $n_i$ is the total number of sparks in the $i$-th gap.

For example, for $\sigma_i = 2^2$, $n_i = 9$ we get $\sigma_{m_1} \approx 10$ mrad.

* * *

We can thus define a function $S_i$ for each gap. Let us consider then the product

$$P_s(\xi_{ij} | m) = \prod_{i=1}^{N} S_i = \prod_{i=1}^{i} \frac{1}{n_i^{1/2}} \sum_{j=1}^{n_i} e^{-\left(\frac{\xi_{ij} - m^2}{\sigma_i^2}\right)}.$$

The value of $m$ which maximizes $P_s$ satisfies the equation

$$\frac{\partial}{\partial m} \ln P_s = 0,$$

that is

$$\sum_i \frac{1}{S_i} \frac{\partial S_i}{\partial m} = 0.$$

The standard deviation of the estimate of $m$ around the true value is

$$\frac{1}{\sigma_m} = \sqrt{\sum \frac{1}{\sigma_{m_i}^2}}.$$
The $\sigma_m$ certainly differ from one another. Only to get an order of magnitude, taking $\sigma_m^i = 10 \text{ mrad}$ $N = 15$

$$\frac{1}{\sigma_m} = \sqrt{N} \frac{10}{9} \sigma_m = \frac{10}{3.9} = 2.6 \text{ mrad}.$$ 

I.3 Determination of the axis of two overlapping showers

The problem of determining the axis of showers becomes more difficult when two adjacent showers are being observed, especially if they overlap. In this case the safe assignment of an individual spark to one rather than the other shower is practically impossible.

Let us consider again the function $S_1$ related to the $i$-th gap only:

$$S_i = \frac{1}{n_i} \sum_{j} e^{-\left(\frac{\xi - \mu}{\sigma_i^j}\right)^2}.$$ 

(I.3)

Suppose we know the true distribution of sparks in the $i$-th gap. Let this be

$$f_1 = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sigma_2^j} e^{-\left(\frac{\xi - \mu'}{\sigma_2^j}\right)^2} + \frac{1}{\sigma_2^w} e^{-\left(\frac{\xi - \mu''}{\sigma_2^w}\right)^2} \right).$$

(I.4)

Then the number of terms in Eq. (I.3) which are inside the interval $d\xi$ will be $f_1 d\xi$. Thus, for $n_i$ large, $S_1$ tends to

$$S_1 \rightarrow \mathbb{E} = \frac{1}{\pi n_i \sigma_i} \int_{-\infty}^{\infty} \left( \frac{1}{\sigma_i^j} e^{-\left(\frac{\xi - \mu'}{\sigma_i^j}\right)^2} + \frac{1}{\sigma_i^w} e^{-\left(\frac{\xi - \mu''}{\sigma_i^w}\right)^2} \right) e^{-\left(\frac{\xi - m}{\sigma_i}\right)^2} d\xi.$$ 

(I.5)
The estimates of \( m \) will be those which satisfy the equation

\[
\frac{\partial \mathbb{S}}{\partial m} = 0,
\]

that is

\[
0 = \frac{1}{\pi n_1 \sigma_1} \left\{ \int_{-\infty}^{+\infty} \frac{1}{\sigma_1' \sigma_1''} (\xi - m)^2 \left( \frac{\xi - m}{\sigma_1'} \right)^2 - \frac{1}{\sigma_1' \sigma_1''} (\xi - m)^2 \left( \frac{\xi - m}{\sigma_1''} \right)^2 \right\} d\xi + \frac{1}{\sigma_1' \sigma_1''} \int_{-\infty}^{+\infty} (\xi - m)^2 \left( \frac{\xi - m}{\sigma_1'} \right)^2 d\xi.
\]

(II.6')

Explicitly, this means

\[
\frac{\sigma_1'(m_1 - m)}{\sigma_1^2 + \sigma_1'^2} e^{-\frac{(m - m')^2}{\sigma_1^2 + \sigma_1'^2}} + \frac{\sigma_1''(m_2 - m)}{\sigma_1^2 + \sigma_1''^2} e^{-\frac{(m - m'')^2}{\sigma_1^2 + \sigma_1''^2}} = 0.
\]

(II.7)

In the adjacency of \( m' \), the first exponential differs from units only for terms in \( (m - m')^2/(\sigma_1^2 + \sigma_1'^2) \). Then, putting \( \sigma_1^2 + \sigma_1'^2 = s_1'^2 \) and \( \sigma_1^2 + \sigma_1''^2 = s_1''^2 \)

\[
\frac{\sigma_1'(m - m')}{s_1'^2} = \frac{\sigma_1''(m'' - m')}{s_1''^2} e^{-\frac{(m' - m'')^2}{s_1''^2}}
\]

or

\[
m^* = m' + \frac{s_1'^3}{s_1''^3} \frac{\sigma_1'}{\sigma_1''} (m'' - m') e^{-\frac{(m' - m'')^2}{s_1''^2}}.
\]
where \( m^* \) is the estimate of \( m' \). Analogously

\[
m^{**} = m'' + \frac{s''^3}{s'_1} \frac{\sigma'_1}{\sigma''_1} (m' - m'') e^{-\left(\frac{m'' - m'}{s''_1}\right)^2}.
\]

Thus,

\[
\Delta m_o = m^{**} - m^* = (m'' - m') \left( 1 - \left[ \frac{\sigma'_1}{\sigma''_1} \left( \frac{s''}{s'_1} \right)^3 e^{-\left(\frac{m'' - m'}{s''_1}\right)^2} \right] + \frac{\sigma''_1}{\sigma'_1} \left( \frac{s''}{s'_1} \right)^3 e^{-\left(\frac{m' - m''}{s''_1}\right)^2} \right),
\]

(Eq. 1.8)

where \( m' - m'' \) is the true unknown separation and \( \Delta m_o \) the observed separation. From Eq. (1.8) we see that, as expected, the observed separation is often appreciably smaller than the true one. To simplify the discussion let us put \( \sigma'_1 = \sigma''_1 = \sigma_1 = \sigma; \quad s'_1 = s''_1 = 2\sigma^2 \).

Then

\[
\Delta m_o = (m'' - m') \left( 1 - 2\sigma \left( \frac{m' - m''}{\sqrt{2\sigma}} \right)^2 \right),
\]

(Eq. 1.9)

which indicates that \( \Delta m_o = 0 \) for \( m' - m'' = 2\sigma \ln 2 \), i.e. for such an angle corresponding to this value of \( m' - m'' \) there is no observable separation.

For example, taking \( \sigma = 2^\circ = 0.035 \) rad, the minimum detectable separation is \((m' - m'')_{\text{mds}} \approx 0.05 \sim 3^\circ\).
In the discussion we have implicitly assumed that the two showers are essentially equal, i.e. \( n'_1 = n''_1 \). This is rarely the case: but it has been observed that the angular spread varies little with the energy if only the part of the shower before the maximum (for sparks) is considered. At any rate for showers of different spread Eq. (I.8) reads

\[
\Delta \sigma_0 = (m'' - m') \left( 1 - \frac{\sigma'_1}{\sigma''_1} \right) n'_1 + \frac{\sigma''_1}{\sigma'_1} n''_1 - \frac{(m' - m'')^2}{\sigma''_1}
\]

where \( n'_1 + n''_1 = 1 \). For \( \sigma'_1 = \sigma''_1 = \sigma_1 = \sigma \)

\[
\Delta \sigma_0 = (m'' - m') \left( 1 - \frac{n'_1}{n''_1} \right) e^{-\left(\frac{m' - m''}{\sqrt{2\sigma}}\right)^2}
\]

Suppose, for example, \( n'_1 \ll n''_1 \)

\[
\Delta \sigma_0 \approx (m'' - m') \left( 1 - \frac{n''_1}{n'_1} \right) e^{-\left(\frac{m' - m''}{\sqrt{2\sigma}}\right)^2}
\]

then

\[
(m'' - m')_{\text{mds}} \approx 2\sigma \sqrt{\ln \left( \frac{n''_1}{n'_1} \right)} > 2\sigma \ln 2,
\]

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i.e. as expected the minimum detectable separation is larger than for equal showers.

* * *

As done in the previous section, we then redefine the function

$$P_s(\xi_{ij}|m) = \prod S_i$$

and look for the values of $m$ which maximize the function $\ln P_s$. These are given by the solution of the equation

$$\sum_{i}^{N} \frac{\partial}{\partial m} \ln S_i = 0,$$

where $N$ is the total number of gaps. On the average, for $n_i$ large, we get explicitly

$$\sum_{i} \left[ -\left( \frac{m-m'}{s_i} \right)^2 e^{\frac{m'}{s_i^3}} + \left( \frac{m-m''}{s_i} \right)^2 e^{\frac{m''}{s_i^3}} \right] = 0. \quad (I.11)$$

For $\sigma'_i = \sigma''_i = \sigma_i = \sigma$, both the numerator and the denominator become independent of $i$. Since the denominator is always $> 0$ and has no singularities, then the solution of Eq. (I.11) is given by

$$\left( m' - m \right) e^{-\left( m-m' \right)^2/s} + \left( m'' - m \right) e^{-\left( m-m'' \right)^2/s} = 0,$$
which gives the same result as Eq. (1.9) - the fluctuation is, however, reduced by a factor $1/\sqrt{N}$.

* * *

When the overlap is such that the two peaks are no longer distinguishable [see Eq. (1.9) or Eq. (1.10)], we can still try to discriminate single showers from double superimposed showers, by measuring the width of the $P_s$ function, defined as the difference between the values $m'_{1/2}$ and $m''_{1/2}$ at which

$$P_s(m'_{1/2}) = \frac{1}{2} P_s's maximum$$

So long as

$$e^{-\frac{4(m'' - m')^2}{s^2}} \ll 1,$$

then one finds

$$\Delta m_{1/2} = m''_{1/2} - m'_{1/2} = 2(m'' - m') \left(1 + \frac{s^2 \ln 2}{\sqrt{N(m'' - m')^2}}\right),$$

where $N$ is the total number of gaps and it has been taken $s_1' = s_1'' = s$. When $N(m'' - m')^2 >> s^2 \ln 2$, then

$$\Delta m_{1/2} \approx 4(m'' - m').$$

* It must be pointed out that the validity of this method of estimation rests, at the moment, only on the effectiveness as proved by experimental results (see next section). A discussion of it will be given elsewhere.
II. COMPUTER PROGRAM AND DISCUSSION  
OF THE EXPERIMENTAL RESULTS

II.1 Aim of the computer programme

The brightness of sparks of a shower in present day spark chambers contains no information about the physical processes in the shower development. All the useful information is therefore contained in the co-ordinates of the sparks. Their reconstruction in space is practically impossible, since, in general, one does not know which image of a spark in one view belongs to the one in the other view of a pair of stereo photographs. All the available information is therefore contained in the two sets of pairs of co-ordinates of the two stereo pictures, together with the knowledge of the properties of the spark chamber set-up. To a certain extent these two views can be analysed separately in parallel. In the following the analysis of one such view is discussed.

A statistical analysis in which up to a few hundred numbers are involved (such as the co-ordinates of the sparks, the co-ordinates of the gaps of the spark chamber and the different materials used), can possibly be done only by an electronic computer. We have therefore used the CERN Mercury computer to develop and test several methods of definition and determination of a shower axis. The most successful one is described below.

II.2 Computation of the shower axis

The shower, as seen on a spark chamber picture, is described by giving each co-ordinate \( y_{ij} \), measured from a reference line perpendicular to the plates, and the gap number \( i \) in which the spark occurred. We assume the shower develops with increasing \( i \), the first spark therefore belongs to the minimum value of \( i \). As is discussed later, the sparks after the "shower-maximum" (= gap with maximum number of sparks) are not useful for the determination of the axis. The maximum value of \( i \) is therefore given by the gap in which the largest number
of sparks occurs. All the following statements refer to the interval up to the maximum only.

Using pictures of electron showers of various energies up to 4 GeV in a set-up of brass chambers, it has been found or checked:

i) The distribution \( f(y_{ij}) \) of the co-ordinate in gap \( i \) can be represented by a Gaussian function (see Section I.2)

\[
f(y_{ij}) = \frac{1}{\sqrt{\pi} \Delta y_i} \exp \left[ -\frac{(y_{ij} - y_0 - x_i \tan \theta)^2}{\Delta y_i} \right];
\]

\( \Delta y_i \) depends on the radiation length \( L_i \) between gap \( N \) and the first gap of the shower and describes the width of the lateral distribution. \( y_0 + x_i \tan \theta \) is the co-ordinate of the shower axis in gap \( i \). The experimental dependence of \( \Delta y_i \) upon \( L_i \) has been fitted by the empirical curve

\[
\Delta y_i(L_i) = a + b \cdot L_i^{\\frac{3}{2}};
\]

\( a \) and \( b \) are independent of the shower energy up to the maximum of the shower. Of course, showers of higher energies reach their maximum at higher values of \( L_i \).

ii) The shower development is expressed by the distribution \( F[\nu, \bar{v}(L_i)] \), which indicates the frequency of finding \( \nu \) sparks in gap \( i \), i.e. after \( L_i \) radiation lengths of material traversed by the shower, \( \bar{v}(L_i) \) is the average number of sparks in gap \( i \).

We have chosen

\[
F[\nu, \bar{v}(L_i)] = \frac{\bar{v}^\nu}{\nu!} \cdot (\bar{v})^\nu,
\]
Single electron shower 145 GeV
Exploring width: 15 mr

FIG. 3 Angle between exploring shower and shower under test
with

\[ \tilde{\nu}(L_i) = c + d \cdot E \cdot L_i, \]

where \( E \) is the energy of the shower as given by the total number of sparks and an empirical calibration curve. The formula for \( \tilde{\nu}(L_i) \) is obviously not valid for showers with energy \( E \sim 0 \), but it is a fair approximation for \( E \gg 200 \text{ MeV} \).

We now let the computer calculate

\[ P_s = \max_{i} \sum_{j} \frac{1}{k} \sum_{i} \sum_{j} f(y_{ij'}) \cdot F(k(i), \tilde{\nu}), \]

where \( k(i) \) is the total number of sparks in gap \( i \) and the \( y_{ij'} \) are the measured spark co-ordinates.

The shower axis is then defined by the direction \( n \) which gives a maximum value for \( P_s \).

II.3 **Numerical values used in the computation**

\[ a = 0.155 \text{ cm} \]
\[ b = 0.077 \text{ cm}. \]

This corresponds to a width of the exploring "model shower" of approximately 15 mrad.

\[ c = 1 \]
\[ d = 0.26 \text{ GeV}^{-1} \text{ (radiation length)}^{-1}. \]

These values are derived from test pictures of showers in a spark chamber set-up consisting of three-plate modules, all brass \(^a\).

Plate thickness 0.5 cm.
Plate distance 1 cm.

\(^a\) For details see H. Feissner, The neutrino experiment, CERN 36-37, edited by C. Franzinetti.
II.4 Results of calculations of the direction of the shower axis

Figure 3 shows a plot of a typical \( P_s \) curve for an electron shower of 1.15 GeV. The shower shown in Fig. 1 was used. The interval between two different values of \( m \) is 5 mrad \( \approx 0.29^\circ \). From the appearance of the curve, we estimate an error in the definition of the shower axis of approximately \( \pm 2 \) mrad or \( \pm 0.12^\circ \).

The width (at \( P_s/P_{s,\text{max}} = 1/e \)) of the \( P_s \) curve in Fig. 3 corresponds to the width of the model shower (\( \sim 15 \) mrad) and has a fluctuation from shower to shower of \( \pm 3 \) mrad on the average.

II.5 Fukui bands, erratic sparks outside the shower

One often finds in a gap of the spark chamber, of the order of 10 sparks close together and clearly outside the main pattern of the shower, which arise mostly from a charged particle travelling along the gap between the plates. To test the behaviour of the \( P_s \) plot, we added such a Fukui band of 10 sparks to the shower of Fig. 1. The shape and position of the \( P_s \) curve remains — as expected — very well the same, however, the absolute value of \( P_{s,\text{max}} \) changes by approximately \( 2 \cdot 10^6 \).

From this and also from the structure of the \( P_s \) function, it is clear that sparks which are by some or several values of \( \Delta y \) outside the main shower, do contribute only weakly to the determination of the position of \( P_{s,\text{max}} \).

II.6 Overlapping showers initiated by pairs of \( \gamma \) rays

II.6.1 Construction of test events which simulate the symmetric decay of a neutral particle into two \( \gamma \) rays

The shower already analysed in Fig. 3 has been used as a basis for a number of constructed pairs of showers. Such a constructed shower consists of two original ones which have the first spark in common and of which the two axes form angles of 0, 3.3, 3.52, 3.96 and 4.4°. The one with 0°, of course, is an exact superposition of twice the original shower.
Two superimposed showers with different opening angles.
1.15 GeV each

FIG. 4
II.6.2 Results of the computation

Figure 4 shows the $P_a$ plots. The shapes of curves for angles between 0 and 1.5° (not shown in Fig. 4) cannot be distinguished. When the angle between the showers reaches $\sim 4^o$ two clearly distinct maxima appear at the correct positions. This value of 4°, for clear distinction, is by almost a factor two better than what simple visible observation provides*).

Details of the computer programme are given in Appendix C.

III. DETERMINATION OF THE NATURE OF THE PRIMARY RADIATION
($\gamma$ or $e$) PRODUCING A SHOWER IN A SPARK CHAMBER

III.4 Showers initiated by high-energy $\gamma$ ray.
Empty gap distribution

A $\gamma$ ray created at a point $x = x_A$ in a spark chamber, converts in a homogeneous medium into a pair of electrons at a distance $\Delta$ which is a statistically distributed variable according to the law

$$f(\Delta) = e^{-\Delta/x_0},$$

$x_0$ being the conversion length of the medium considered. Due to the uneven distribution of matter in the chamber, $f(\Delta)$ is no longer exponential.

In fact, let $x_A$ be the origin of the $\gamma$ ray (see Fig. 5) in the plate of thickness $\lambda_A$ and $x_V$ the vertex of the shower in the plate of thickness $\lambda_V$: $\lambda_1, \lambda_2, ..., \lambda_1$ the thickness of the plates traversed by the $\gamma$ without converting. Both $x_A$ and $x_V$ are unknown within $\lambda_A$ and $\lambda_V$, respectively.

The probability that a $\gamma$ created at $x_A$ converts at $(x_V, dx_V)$ is:

$$e^{-\sum \frac{\lambda_1}{x_01} - \frac{x_A}{x_0A} - \frac{x_V}{x_0V}} \cdot dx_V.$$

* H. Reisner, I. Ferrero, H.J. Gerber, H. Reinharz and J. Stein, Elastic charge exchange and $\eta$ production by negative pions of 4 GeV/c (to be published).
Since we require that the $\gamma$'s cross at least one gap without materializing, the factor $e^{-x_o/x_0}$ becomes unimportant. Then the probability of crossing plates $\lambda_1, \lambda_2, ..., \lambda_{14}$ and converting somewhere in $\lambda_y$ is

$$f = A \left[ \frac{\sum \lambda_i}{x_o} \int \left( e^{-\frac{x_V}{x_0}} \right) \frac{dx_V}{x_o V} \right]$$

(III.1)

$$= A \left[ \frac{\sum \lambda_i}{x_o} \left( 1 - e^{-\lambda y/x_0 V} \right) \right],$$

where $A$ is a normalization factor. Then the factor $(1 - e^{-\lambda y/x_0 V})$ being constant, the distribution function for the variable

$$\sum \frac{\lambda_i}{x_o} = \Delta$$

is given by

$$f(\Delta^*) = f \left( \frac{\Delta}{1 - e^{-\lambda y/x_0 V}} \right) = A \left[ \frac{\sum \lambda_i}{x_o} \right].$$

(III.2)

This distribution has been verified on 30 $\gamma$-initiated showers produced in neutrino interactions in the so-called 'production region'. Most of them originated in the brass set-up although a few also in the mixed brass-aluminium set-up. The values of the conversion length for $\gamma$'s used in the calculations for the practical use of formula (III.2) are:

- aluminium $x_o = 12$ cm $\quad 1 - e^{-\lambda y/x_0} = 0.0416$
- brass $x_o = 1.9$ cm $\quad 1 - e^{-\lambda y/x_0} = 0.23$.

The experimental results are shown in Fig. 6 and compared with the calculated distribution the agreement seems fairly good.
Distribution of distances $\Delta$ between the apex and the vertex of showers (in conversion length units)

30 showers

normalized exponential $e^{-\Sigma \frac{\lambda}{x_0}}$

FIG. 6
III.2 Showers initiated by electrons.

Single spark distribution

For parent electrons the problem is similar. In this case the electron travels over a certain length radiating photons. When a photon is created, the two travel together until the photon materializes. If the photon has sufficient energy to produce a pair which can be detected by the apparatus, then both the original electron and the pair will produce sparks. This is what we consider the 'vertex' of an electron-initiated shower.

In the neutrino set-up, at least a photon of energy \( k \geq 50 \) MeV is required. Since high-energy electrons have a high probability of radiating such a photon, it is practically only the conversion of the radiated photon which determines the length over which only the primary electron is visible.

Let the conversion length for a \( \gamma \) be indicated by \( x_0(k_\gamma) \) and the average length for an electron of energy \( E_0 \) to radiate a photon of energy \( k_\gamma \) be indicated by \( y_0(E_0, k_\gamma) \).

The cross-section for production of a photon of energy \( k_\gamma > k_0 = 50 \) MeV is then (see Heitler, Theory of Radiation, § 25)

\[
\Phi(k_0) = 4\varphi \ln(183 \times 10^3) \left\{ \frac{k}{3} \ln \frac{E_0}{k_0} - \frac{E_0 - k_0}{E_0} + \frac{E_0^2 - k_0^2}{2E_0^2} \right\},
\]

where \( \varphi = \alpha Z^2 r_e^2 \). If \( L_r \) is the so-called 'radiation length'

\[
\Phi(k_\gamma > k_0) = \frac{1}{L_r} \left\{ \frac{k}{3} \ln \frac{E_0}{k_0} - \frac{E_0 - k_0}{E_0} + \frac{E_0^2 - k_0^2}{2E_0^2} \right\}.
\]

For \( E_0 = 0.5 \) GeV and \( E_0 = 5 \) GeV we get for \( \Phi(E_0, k_\gamma) \)

\[
\Phi(0.5 \text{ GeV}; 50 \text{ MeV}) = \frac{2.7}{L_r} \quad \text{and} \quad \Phi(5 \text{ GeV}; 50 \text{ MeV}) = \frac{5.6}{L_r}.
\]
We then take a constant value at \( E_0 = 1.5 \text{ GeV} \) (which is near the average electron energy observed in neutrino interactions) which gives

\[
\psi(1.5 \text{ GeV}; 50 \text{ MeV}) = \frac{L_r}{L_r}.
\]

and

\[
\gamma_0 = \frac{1}{\phi} = \frac{L_r}{4} = \begin{cases} 
\text{for aluminium} & 2.2 \text{ cm} \\
\text{for brass} & 0.36 \text{ cm}.
\end{cases}
\]

The conversion length for a photon, \( x_0 \), varies considerably in the range between 5 and 100 MeV (about a factor 5), attaining for the latter energy practically its asymptotic value. Thus, the most effective photons will be those above 100 MeV. It will not introduce a large error to take \( x_0 = \text{const} = x_0(\infty) \), this being its asymptotic value used in the previous section.

Thus, \( \gamma_0 \ll x_0 \) as stated above.

To a first approximation we can assume that the process leading to the creation of a shower is that described in Fig. 7. This, of course, is not true as the electron may radiate more than a photon and then the probability of generating a pair at a given distance from the apex is higher. This will result in an apparent \( x_0 \), shorter than that calculated above. We shall see, however, that the experimental results agree with our rough estimate.

Thus, the distribution function for the variable \( \Delta \) defined in analogy to what stated in the previous section (\( \Delta = \Sigma \lambda/x_0 \), the sum not including either the plate where the electron was created, or that where multiplication started) is given approximately by

\[
f\left(\frac{\Delta - \lambda}{1 - e^{-\lambda V/x_0 V}}\right) \approx A e^{-\frac{\lambda}{x_0}}.
\]

In this case \( \Delta \) refers to the distance covered by the primary electron before multiplication takes place. In the gaps included in this distance, only a row of single sparks is visible. The experimental results are shown in Fig. 8.
Distribution of distances $\Delta$ between the apex and the vertex of showers (in conversion length units)

27 showers

normalized exponential $e^{-\sum \frac{A}{x_0}}$

FIG. 8
III.3 Experimental bias in the determination of the nature of the parent radiation

In the discussion of the previous section, it was implicitly admitted that an electron always produces a row of sparks while a γ ray always leaves a number of gaps empty, before either of them give rise to a shower.

This, of course, is not true. Both an electron and a γ ray can initiate a shower in the same plate where they have been created themselves, with a probability (approximately equal for both cases)

\[ p \approx \left(1 - e^{-\lambda_A/x_0A}\right) e^{\lambda_A/x_0A} . \]

When this happens γ or e showers are indistinguishable. For aluminium P is very small (~ 4%); for brass it could be as high as 40% (two adjacent brass plates).

On the other hand it has been observed that individual tracks, especially when originating from large disintegrations, associated with many other tracks, often do not trigger all gaps. Namely, an electron may start producing a row of sparks only when already at some distance from the apex of the disintegration. This effect is shown by the results plotted in Fig. 9. The 'empty gap' distribution both for γ showers and e showers associated with disintegrations, producing another additional track [(1s,1t) events] or n other additional tracks [1s,nt events]. This is probably due to 'robbing' effect near the apex where slow ionizing tracks may 'rob' sparks at the expense of fast low ionizing ones.

The slope of the distribution is greater for the events attributed to electrons than for those attributed to γ's - which means that the attribution has been, for the neutrino experiment, correct for most of the cases.

However, should a large γ component be present, the bias in the discrimination could be quite large. We notice in fact that the angle of a pair of electrons generated by a γ is of the order \( m_e/k_\gamma \).
i.e. very small for large $k\gamma$. That means that the two electrons of the pair could travel over some distance very close to each other, thus producing one single spark over several gaps. In this case the event (a $\gamma$ shower) would simulate an $e$ shower.

With the efficiency of the spark chamber used for the neutrino experiment, the fraction of events attributed to $e$ showers which could have been due to $\gamma$, was 3%.

* * *

8565/p/smg
FIG. 9
STATISTICAL PROPERTIES OF THE FUNCTION S FOR THE ESTIMATE OF
THE SHOWER AXIS. VARIANCE OF THE ESTIMATES

Let $f(x|m)$ be a frequency function of a continuous variable $x$ and of a parameter $m$. We define the function $S$ by the equation

$$S(x_1,x_2,...,x_n|m) = \frac{1}{n} \sum_{i=1}^{n} f(x_i|m), \quad (A.1)$$

where the $x_i$'s are a set of $n$ experimental data and $m$ is the unknown parameter, the value of which we want to determine.

We shall take as an estimate of $m$ the value $m^*$, which maximizes the function $S(x_i|m)$, i.e. satisfies the equation

$$\frac{\partial S}{\partial m} \bigg|_{m=m^*} = 0. \quad (A.2)$$

or

$$\frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\partial}{\partial m} f(x_i|m) \right]_{m=m^*} = 0. \quad (A.2')$$

We shall assume that:

a) At least a solution of Eq. (A.2) exists in a non-degenerate interval $a \rightarrow b$ of definition of $m$.

b) For every $m$ belonging to $a \rightarrow b$ and every $x$ in their interval of definition $x' \rightarrow x''$ the derivatives

$$\frac{\partial f}{\partial m}; \quad \frac{\partial^2 f}{\partial m^2} \quad \text{and} \quad \frac{\partial^3 f}{\partial m^3}$$

exist and satisfy the relations

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\[ \left| \frac{\partial f}{\partial m} \right| < F_1; \quad \left| \frac{\partial^2 f}{\partial m^2} \right| < F_2; \quad \left| \frac{\partial^3 f}{\partial m^3} \right| < F_3, \]

where \( F_1(x), F_2(x) \) are integrable in \( x' \rightarrow x'' \); and
\[
\int_{x'}^{x''} F_3 f(x|m) dx \quad \text{and} \quad \int_{x'}^{x''} F_1^2 f(x|m) dx
\]
are bounded for any value of \( m \).

Then, if \( m_0 \) is the true (unknown) value of \( m \)
\[
f'(x|m) = f'(x|m_0) + (m - m_0) f''(x|m_0) + \frac{1}{2} (m - m_0)^2 \Theta F_3, \quad (A.3)
\]
where the 'prime' indicates derivative with respect to \( m \) and \( |\Theta| < 1 \).

We assume Eq. (A.3) to be valid for every \( m \) in \( a \rightarrow b \). Thus, Eq. (A.2) reads
\[
\frac{1}{n} \sum_{i=1}^{n} f'(x_i|m_0) + (m - m_0) \frac{1}{n} \sum_{i=1}^{n} f''(x_i|m_0) + \frac{1}{2} (m - m_0)^2 \Theta \frac{1}{n} \sum_{i=1}^{n} F_3 = 0. \quad (A.4)
\]

For large values of \( n \) the sums tend asymptotically to the integrals:
\[
\frac{1}{n} \int_{x'}^{x''} f'(x|m_0) f(x|m_0) dx = E(f') = 0 \quad (A.5)
\]
(as it can be deduced by the fact that \( \int_{x'}^{x''} f^2 dx = \text{const} \))
\[
\frac{1}{n} \int_{x'}^{x''} f''(x|m_0) f(x|m_0) dx = E(f'') \quad (A.6)
\]
\[
= \frac{\partial}{\partial m} E(f') - \int_{x'}^{x''} f'^2 dx = - \int_{x'}^{x''} f'^2 dx \geq - \int_{x'}^{x''} F_1^2 dx
\]

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[since it follows from Eq. (A.5) that \( E(f') = 0 \); and from (b) that \( x' f'^2 \, dx \), always positive, is bounded for any value of \( m \). We can then write \( E(f'') = -h^2 \), \( h \) being a real number]. Furthermore,

\[
\frac{1}{n} \sum_{x'} f'^2 (x|m_0) \to \int f'^2 \, f \, dx = E(f'^2) = k^2,
\]

\( k \) being a real, finite number [see condition (c)]. From Eq. (A.4) we get

\[
m^*-m_0 = -\frac{1}{n} \sum_{x'} f'' (x|m_0) - \frac{1}{2} (m^*-m_0) \Theta \frac{1}{n} \sum F_3
\]

which we shall re-write as

\[
\frac{\sqrt{n}h^2}{k} (m^*-m_0) = \frac{1}{k/n} \sum_{x_1} f'(x|m_0)
\]

\[
- \frac{1}{nh^2} \sum f'' (x|m_0) - \frac{1}{nh^2} \frac{1}{2} (m^*-m_0) \Theta \sum F_3
\]

\( \Sigma f'(x|m_0) \), by the Central Limit Theorem, tends asymptotically to a normal distribution with mean value zero [see Eq. (A.5)] and variance \( k/n \) [see Eq. A.7)]. Thus, the numerator is an asymptotically normal variable with mean value zero and variance 1. \( \Sigma f'' (x|m_0) \) converges in probability to \(-nh^2\); thus the first term converges in probability to 1 with variance 1; the second converges in probability to zero.

Then, for the Convergence Theorem, the variable

\[
\frac{\sqrt{n}h^2}{k} (m^*-m_0)
\]
is asymptotically normal with mean value zero and variance 1. Consequently \( m^* \) is asymptotically normal with mean value \( m_0 \) and variance \( R/h^2 \sqrt{n} \), which means

\[
\sigma_m = \frac{\sqrt{\mathbb{E}(f'^2)}}{-\sqrt{n} \mathbb{E}(f'')}.
\]

For the frequency distribution used in Sections I.2 and II.1

\[
f(\xi | m) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\xi - m)^2}{2\sigma^2}}.
\]

we get \( \xi' = -\infty; \xi'' = +\infty \)

\[
\mathbb{E}(f'^2) = \frac{1}{\pi^{3/2} \sigma^2} \int_{-\infty}^{+\infty} \frac{-3(\xi - m)^2}{\sigma^2} \; d\xi = \frac{2}{3\sqrt{2\pi} \sigma^4}
\]

\[
\mathbb{E}(f'') = \int_{-\infty}^{+\infty} f'' \; d\xi = \int_{-\infty}^{+\infty} f'^2 \; d\xi = -\frac{1}{\pi \sigma^6} \int_{-\infty}^{+\infty} (\xi - m)^2 \; e^{-\frac{(\xi - m)^2}{2\sigma^2}} \; d\xi
\]

\[
= -\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma^3}
\]

\[
\sigma_{m^*} = \frac{2\sigma}{3^{3/4} \sqrt{n}} \frac{0.878}{\sqrt{n}} \sigma.
\]
APPENDIX B

PROBABILITY THAT TWO ADJACENT SHOWERS APPEAR AS OVERLAPPING ON THE PHOTOGRAPHS.
SINGLE VIEW AND STEREOSCOPIC VIEWS

B.1 Probability of overlapping in one single view

Two showers originating from the same interaction may appear as superimposed in a picture if their axes lie on or close to a plane which also contains the optical centre of the camera objective. In such a case a π⁰ may produce two showers which simulate a single γ shower.

Whenever the discrimination between two-shower from single-shower events is of importance, it is necessary to know what is the probability that the two showers, simultaneously produced, overlap to appear as one.

In spark-chamber pictures, high-energy showers manifest themselves by producing a large number of sparks in an almost conical region. As discussed and shown in Sections 1 and 2, the semi-aperture of this cone is practically independent of the shower energy: what varies with the energy is the longitudinal dimension of it.

Then we can describe the geometry of our events by considering the projection of two cones having the same vertex, equal aperture and different axis - as indicated in Fig. B.1. Using the notation indicated in the same figure, we shall consider two showers as overlapping when λ ≤ K·2r, K being a number which is chosen by the experimenter. Thus, for instance, if he chooses to consider two showers as distinct from one another when they appear in the picture clearly separated, K will be ≥ 2. If they overlap, but the angular spread appears 50% larger than that of a single shower, then K = 1.5, etc.
The discussion which follows refers to \( \pi^0 \) decays - but it can be applied to the decay of any other particle into two-shower producing rays, provided that the angular distribution of the secondary radiation in the system of reference of the parent particle is isotropic.

Then, (see Fig. B.1)

\[
\lambda = l_1 \sin \alpha + l_2 \sin \alpha + 2r.
\]

(B.1)

Substituting \( \lambda = 2k \eta \), \( l_1 = L \tan \Theta_1 \), \( l_2 = L \tan \Theta_2 \)

\[
\sin \bar{a} = \frac{2(\lambda - 1)\eta}{L} \frac{1}{\tan \Theta_1 + \tan \Theta_2}.
\]

(B.2)

\( \bar{a} \) is then the maximum value of \( \alpha \) at which the two showers appear as superimposed.

For \( \gamma \)'s arising from the decay of a \( \pi^0 \)

\[
tg \Theta_1 + tg \Theta_2 = \frac{2m_p \sin \Theta^*}{\eta^2 + m^2 \cos^2 \Theta^*},
\]

(B.3)

where \( m \) is the mass, \( \eta \) the momentum and \( E \) the energy of the \( \pi^0 \) in the lab. system and \( \Theta^* \) the decay angle in the c.m.s. For \( \pi^0 \)'s having a momentum of \( \gtrsim 500 \text{ MeV} \) \( \eta \approx E \) (in fact for \( \eta = 500 \text{ MeV} \) \( E = 517 \text{ MeV} \)).

We can then safely take \( \eta = E \), the error thus introduced being, at the most, \( \approx 1\% \);

\[
\sin \bar{a} = (\lambda - 1) \frac{E}{L \eta m} \sin \Theta^*
\]

\[
\bar{a} = \arcsin(h \sin \Theta^*) \quad h = (\lambda - 1) \frac{E}{L \eta m}.
\]

(B.4)

Since the plane of decay for \( \pi^0 \)'s is uniformly distributed over all azimuths, the probability that the two showers overlap in projection is \( 2\bar{a}/\pi \). Integrating over all \( \Theta^* \)'s, the overlap probability is:

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FIG. B1
\[ f(h) = 2 \int_{\pi/2}^{\theta^*} \arcsin(h \sin \Theta^*) \cos \Theta^* d \Theta^* = \frac{2}{\pi h} \left[ F(h) - (1 - h^2) E(h) \right], \] (B.5)

where \( F(h) \) and \( E(h) \) are the complete elliptic integrals of first and second kind, respectively (see Jahnke and Emde, Tables of functions, p. 52).

The definition of \( f(h) \) given in Eq. (B.4) is valid as long as \( h < 1 \). When \( h \geq 1 \) the integral has to be split into two parts to take into account that the integrand can - at the most - be equal to \( \pi/2 \) and once attained that the value remains constant.

\[
f(h) = \frac{2}{\pi} \left\{ \int_0^{\Theta_{\text{max}}} \arcsin(h \sin \Theta^*) \sin \Theta^* d\Theta^* + \frac{\pi}{2} \int_{\Theta_{\text{max}}}^{\pi/2} \sin \Theta^* d\Theta^* \right\} \]
\[
= \frac{2}{\pi} E \left( \frac{1}{h} \right),
\] (B.6)

where \( \sin \Theta_{\text{max}}^* = 1/h \). The function \( f(h) \) is plotted in Fig. B.3 (full line). The same function, for values of the parameters appropriate to \( \pi^0 \) decays in the neutrino brass chamber, i.e.

\[ L = 30 \text{ cm}; \ r = 3 \text{ cm}; \ m = 0.135 \text{ GeV}, \]

is plotted in Fig. B.4 as a function of the \( \pi^0 \) momentum \( p(\text{GeV}) \) and for different values of \( K \) (full line).

**B.2 Probability of overlapping in both views of a stereoscopic photographic recording**

When two views are available, at an angle \( 2\psi \) (see Fig. B.2), and the more restrictive condition is applied so that the visible angular spread of the two superimposed showers should not exceed \( 2K \rho \) in either, the number of cases of overlap obviously decreases.

In Fig. B.2 the circles centred in \( A \) and \( B \) mark the intersection of the two showers from a \( \pi^0 \) decay on the plane perpendicular to the \( \pi^0 \) direction. To all purposes we can assume that \( \alpha \) rotates
around A, rather than around the \( \pi^\circ \) line of motion, as this amounts to changing the origin of the \( y, z \) co-ordinates, which is irrelevant.

From Fig. B.2 one can easily see that:

1) For a given \( \ell_1 + \ell_2 \) the angular intervals, at which overlapping takes place in both views, are so distributed as to be symmetric with respect to the axis \( FF' \) and \( CC' \). Then, if \( \Delta \alpha \) refers to any of the four quadrants \( \hat{C}A\hat{F}, \hat{F}A\hat{C}', \hat{C}'A\hat{F}' \) or \( \hat{F}'A\hat{C} \). Thus, for a uniform azimuthal distribution of decays, the overlapping probability is

\[
f = \frac{\Delta \alpha}{2\pi} = \frac{2\pi}{\pi}.
\]

2) Selecting the quadrant \( \hat{C}A\hat{F}, \bar{\alpha} \) is determined by those values of \( \alpha \) which make the extreme \( B \) of the segment \( \bar{A}B = \ell_1 + \ell_2 \) touch the line \( CF \) while rotating around \( A \).

When \( \bar{A}B \) exceeds \( \bar{A}C \) \( [\ell_1 + \ell_2 \geq (\lambda - 2r)/\sin \psi] \), the overlap never occurs in both views.

When \( \bar{A}F \) is small, \( \bar{A}B \) < \( \bar{A}F \), the overlap occurs for \( \alpha \leq \bar{\alpha} \), where \( \bar{\alpha} \) satisfies the following condition (consider the triangle \( \hat{A}\hat{B}\hat{C} \))

\[
\sin(\psi + \bar{\alpha}) = \frac{\lambda - 2r}{\ell_1 + \ell_2}.
\]

Remembering Eqs. (B.1), (B.2) and (B.4) that means

\[
\bar{\alpha} = \arcsin(h \sin \Theta^*) - \psi.
\]

When \( \bar{A}B \) < \( \bar{A}B \) < \( \bar{A}F \), overlap occurs in two regions, corresponding to the angles (see Fig. B.3) \( \hat{C}A\hat{G} \) and \( \hat{G}'A\hat{F} \). The sum of these two intervals is

\[
\bar{\alpha} = 2 \arcsin(h \sin \Theta^*) - \frac{\pi}{2},
\]
independent of $\psi$, as it must be. (However, the limits of integration 
over $\Theta^*$ will depend on $\psi$.)

When $\overline{AB} < \overline{AE}$, overlap always occurs in both views. Thus, we 
have to subdivide the integration over $\Theta^*$ into four regions:

I. $l_1 + l_2 \geq \frac{2r(K-1)}{\sin \psi}$ \hspace{1cm} \text{i.e. } h \sin \Theta^* \leq \sin \psi \text{ (no overlapping)};

II. $\frac{2r(K-1)}{\cos \psi} \leq l_1 + l_2 \leq \frac{2r(K-1)}{\sin \psi}$ \hspace{1cm} \text{i.e. } \sin \psi \leq h \sin \Theta^* \leq \cos \psi;

III. $2r(K-1) \leq l_1 + l_2 \leq \frac{2r(K-1)}{\cos \psi}$ \hspace{1cm} \text{i.e. } \cos \psi \leq h \sin \Theta^* \leq 1;

IV. $l_1 + l_2 \leq 2r(K-1)$ \hspace{1cm} \text{i.e. } 1 \leq h \sin \Theta^* \text{ (constant overlapping)}.

Varying $\rho$, $h$ varies and may include either one, two, three or all the above 
regions. Accordingly, the integration over $\Theta^*$ yields:

i) for $h < \sin \psi$ \hspace{5cm} f(h,\psi) = 0 \text{ (no overlapping)};

ii) for $\sin \psi < h < \cos \psi$ \hspace{2cm} f(h,\psi) = f(h) - \frac{2}{\pi h} \left[ E(h, \arcsin \frac{\sin \psi}{h}) \right. \\
\hspace{4cm} \left. - (1-h^2) F(h, \arcsin \frac{\sin \psi}{h}) \right]

iii) for $\cos \psi < h < 1$ \hspace{5cm} f(h,\psi) = 2f(h) \\
\hspace{4cm} - \frac{2}{\pi h} \left[ E(h, \arcsin \frac{\sin \psi}{h}) - (1-h^2) F(h, \arcsin \frac{\sin \psi}{h}) \right] \\
\hspace{4cm} - \frac{2}{\pi h} \left[ E(h, \arcsin \frac{\cos \psi}{h}) - (1-h^2) F(h, \arcsin \frac{\cos \psi}{h}) \right]

iv) for $1 < h$ \hspace{5cm} f(h,\psi) = 2f(h) - \frac{2}{\pi} \left[ E\left(\frac{1}{h}, \psi\right) + E\left(\frac{1}{h}, \frac{\pi}{2} - \psi\right) \right],

where $f(h)$ is the single view function ($\psi = 0$) defined by the Eqs. (B.4) and 
(B.5); $F(h,\psi)$, $E(h,\psi)$ are the incomplete elliptic integrals of first and 
second kind (see Jahnke and Emde, loc.cit.).
For the neutrino production region the semi-aperture $\phi$
for any event in the whole chamber never exceeds the limits
$6^\circ \rightarrow 9^\circ$. (We observe, incidentally, that when the incidence of
the rays on the two cameras is not symmetric with respect to the
horizontal plane, the symmetry can be restored by a rotation of the
frame of reference for the angles $\alpha$ and $\phi$.) The function $f(h, \phi = \phi')$
has been calculated and is given in Fig. B.3 (dotted line). The same,
as a function of $p$, for the values of the parameters indicated in
Section B.2, is shown in Fig. B.4.

B.3 Overlapping probability in a definite angular interval in the
lab. system for showers originating from a $\pi^0$ of given momentum

(Single view, $0.5 \leq p_{\pi^0} \leq 4$, GeV)

It may be of interest to know the probability of overlapping
for showers originating from a $\pi^0$ of momentum $p$, making in the lab.
system an angle $\Theta_1 + \Theta_2 = \Theta$ included between two given values $\Theta' \rightarrow \Theta''$.
Thus, the integration over $\Theta^* \rightarrow$ is not to be extended to all possible
values of $\Theta^*$ but must be limited to those which produce an angular
separation $\Theta$ in the lab. system between the two showers inside the
above interval.

Expressing $\Theta^*$ as a function of $\Theta$ we get

$$\sin \tilde{\alpha} = \frac{2r(K-1)}{L} \cot \frac{\Theta}{2} = H \cot \frac{\Theta}{2},$$

where the notations are the same as those used in the previous sections
and again we have taken $p_{\pi^0} \approx E_{\pi^0}$. For values of $p_{\pi^0}$ not exceeding
the interval $0.5 \rightarrow 4$, GeV, for the parameter $H$ appropriate to the aluminium
production region of the neutrino set-up, the following formula is
approximate to better than 5%:

$$f(p|\Theta', \Theta'') = \frac{4r(K-1)}{3\pi L} \frac{m^2}{p^2} \left( \cot^3 \frac{\Theta'}{2} - \cot^3 \frac{\Theta''}{2} \right).$$

For $H = 0.2$ (as, for instance, using the values given in Section B.2
for the brass section of the production region) the same formula is approximate
to better than 10% only between $0.5 \rightarrow 1$ GeV for the $\pi^0$ energies. Outside
these limits the integration has to be carried out numerically.
PROBABILITY OF OVERLAPPING $f(p)$ FOR SHOWERS FROM $\pi^0$'S FOR DIFFERENT VALUES OF THE $\pi^0$ MOMENTUM $p$ AND OF THE TOLERANCE IN ANGULAR SPREAD $K-1$ IN THE $\rho$ - CHARROTS (BRASS)

$\rho$ - CHARIOTS (BRASS)

SINGLE VIEW
TWO VIEWS ($\gamma = 9^\circ$)

$\pi^0$ MOMENTUM

$10p$ (GeV/c)

FIG. B4
DESCRIPTION OF THE USE OF THE COMPUTER PROGRAMME
FOR THE DETERMINATION OF SHOWER AXIS

At present 1761 P₁ is the Mercury computer programme which computes the curves of Figs. 3 and 4. A 7090 version will become available later.

To run the calculation three tapes are needed: I, II, and III.

Tape I : is the programme. It contains the instructions and is called 1761 P₁. It is available from the Computer Reception.

Tape II : contains the relevant information about the spark chamber set-up. It has to be made by the user. It is described below in more detail.

Tape III : contains the information about the picture of the shower to be processed. It includes picture number, view, fiducials, and co-ordinates of shower origin and sparks.

Tape II

Tape II has to be called 1761 D₁₄, where A stands for a number A > 100.

A spark chamber set-up contains "plates" and "gaps". A "gap" is a space which can contain a spark. A "plate" is anything between two gaps. A set-up made of modules (Fig. C1) then has the following "plates"

A
B
C + D
E
F + G
H
I

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Each plate is characterized by a number, called $U_1 \ldots U_N$. N is the index of the last plate. N < 299.

The types of plates are labelled by numbers 1 \ldots 9.

$U_K$ is a number with some digits before the decimal point and three digits behind it. The last digit denotes the type of plate, the others denote the co-ordinate of the plate in cm. For the computation, the co-ordinates of the high-voltage plates are used to calculate the co-ordinates of the gaps. $U_K$ contains the co-ordinate and type of material which follows gap K.

G denotes the distance of two fiducial marks in real space in cm.

H denotes the distance of the location from the high-voltage plate where the Y co-ordinate is to be measured, in cm.

\[ \begin{align*}
W_1 \\
\vdots \\
W_7
\end{align*} \]

denote the thickness of the "plates" in radiation lengths.

\[ \begin{align*}
W_6 \\
W_{12}
\end{align*} \]

denote coefficients which describe the average shower, i.e. the lateral spread is given by $\bar{y} = W_{12} + W_{12} R_0 \sqrt{2}$, where $R_0$ is the total radiation length of the shower up to the sparks in question. Therefore one has:

\[ W_{12} = a \]
\[ W_{12} = b, \]

as used in Section II.3.

The shower development, i.e. the number of sparks after $R_0$ radiation lengths for a shower of energy $E$ in MeV, is given by

\[ U_0 = 1 - 0.001 E R_0 W_6. \]

$W_6$ is therefore a negative number. One has to identify $W_6 = -d$ of Section II.3. The other numbers $W_6, W_{11}, W_{13}, W_{14}, W_{15}$ are reserved numbers for future extensions. They can be set to zero at present.

F denotes the step in the direction during calculation. A good choice is $F = 0.005$. 
3-plate Sparkchamber Modules

FIG. C1
Tape II now is a row of the numbers and ends with ",>".
N
U_1
\vdots
U_N
G
H
W_1
\vdots
W_{\ell}
F
\rightarrow

Tape III is a row of numbers ending with ",>" and should be labelled 1761 D_{A+1}. B
C
X_1
Y_1
X_2
Y_2
X_3
Y_3
X_4
Y_4
L
Y_5
Y_6
\vdots
\vdots
Y_{n-1}
-1
Y_m
Y_{m+1}
B is the picture number.

C is any number which can be used for identification. B and C will be printed out. No calculation is done with B or C.

\( \frac{Y_1}{Y_2} \) denote the co-ordinates of the two fiducial marks as measured on the picture, in cm. Only \( Y_1 \) and \( Y_2 \) are used in the computation. Their difference corresponds to \( G \) of tape II.

\( X_1, Y_1, X_2, Y_2 \) denote the co-ordinates of the vertex of the two showers or the first spark of the shower, when a single shower is to be analysed. This point should be in a "gap".

\( X_3, Y_3 \) denote the co-ordinates of a point on a guessed shower axis.

\( T \) number of the first gap. \( T \) is a \( K \) value of tape II.

The rest of the tape contains \( Y \) co-ordinates of sparks. After the last measured spark of a gap there has to be the number "-1". After the "-1" of the last spark of the shower there has to be the total energy of the shower in MeV.

The number "-3" denotes the end of the shower. The next number on the tape is then considered to be the picture number of the next shower to be analysed.