AN INVESTIGATION OF SPACE CHARGE

EFFECTS FOR THE BOOSTER INJECTOR TO THE CPS

by

P.L. Morton*)

1. Notation
2. Incoherent space charge detuning
3. Longitudinal space-charge effects
4. Throbbing beam motion

*) On leave from Stanford Linear Accelerator Center.
I. NOTATION

It is the purpose of this note to investigate various space-charge effects that may be present in the booster injector for the CPS. The notation and values for the accelerator parameters used in this report are given below.

\[ r_p = \text{classical proton radius} = 1.53 \times 10^{-18} \text{ m} \]
\[ N = \text{number of protons per ring per pulse} = 2.5 \times 10^{12} \]
\[ h = \text{harmonic number} = 5 \]
\[ \nu_00 = \text{betatron oscillation frequency for the case of only one particle present} \approx 4.5 \]
\[ a_y = \text{"average" half height of the beam} = 1.5 \text{ cm} \]
\[ a_x = \text{"average" half width of the beam} = 2.4 \text{ cm} \]
\[ a = \text{"average" beam radius} = 2 \text{ cm} \]
\[ b = \text{vacuum chamber radius} = 5.3 \text{ cm} \]
\[ \beta_{\text{inj}} = v/c \text{ at injection (50 MeV)} = 0.314 \]
\[ \gamma_{\text{inj}} = (1 - \beta_{\text{inj}}^2)^{-\frac{1}{2}} = 1.053 \]
\[ R = \text{average radius} = 25 \text{ m} \]
\[ B = \text{bunching factor} = 1/2 \]
\[ L = \text{bunch length} = 15 \text{ m} \]
\[ \sigma = \text{conductivity of stainless steel vacuum chamber} = 10^{16} \text{ sec}^{-1} \]
\[ \Omega = \text{angular revolution frequency at injection} = 3.77 \times 10^6 \text{ rad/s} \]
\[ \text{eV} = \text{peak voltage gain per turn} = 12 \text{ keV} \]
\[ \phi_s = \text{synchronous phase angle} = 5^\circ \]
II. INCOHERENT SPACE CHARGE DETUNING

The space-charge forces can change the betatron oscillation frequency until a resonance is reached. This incoherent tune shift has been calculated by Laslett\(^1\), in all of its glory, for a beam of uniform density taking into account the image effects due to the vacuum chamber walls and the magnetic poles. For the case of the booster the value of \( \gamma \) is low so that the image effects may be neglected (5\%), and one obtains for the shifted betatron oscillation frequency of a uniform elliptical beam,

\[
\nu_0^2 = \nu_0^2 - 2\nu_0 \Delta \nu
\]

with

\[
\Delta \nu = \frac{N r_R}{\pi \nu_0 P} \beta^2 \gamma^2 \frac{a_y}{(a_x + a_y)} = 0.18
\]

The classical limit for \( \Delta \nu \) is usually taken equal to 0.25. Thus the designed value of the intensity of the booster is below the intensity limit due to the effect. If it is necessary to increase significantly the intensity of the Booster above the designed value something must be done to overcome this space charge detuning. Laslett\(^2\) has studied the possibility of introducing passive elements around the beam in the straight sections in order to change the image terms in the space-charge formula; however, these are only helpful for larger values of \( \gamma \).

Thus at present there is no known practical way to increase the space-charge limit of the Booster significantly except by increasing the injection energy or the transverse beam dimensions.
III. LONGITUDINAL SPACE-CHARGE EFFECTS

The space-charge forces tend to weaken the focusing force that arises from the r.f. acceleration of the beam, and thus decreases the stable longitudinal phase area. If one assumes that the phase area is uniformly filled with particles then Nielsen and Sessler\(^3\) have obtained curves that demonstrate the decrease of phase area with phase density. The total number of particles inclosed in the stable phase area is given by

\[
N = \frac{ReY}{4\pi\hbar E_0} \Lambda A(\Lambda, \varphi_s)
\]

where \(E_0\) is the rest energy of the proton

\( g \) is a constant equal to \([1 + 2\ln(b/a)]\) for a round beam in a round vacuum chamber,

\( \Lambda \) is proportional to the phase density,

\( A \) is the stable phase area.

From Nielsen and Sessler\(^3\) one can obtain the following curves*)

for \( A \) vs \( \Delta A \)

Fig. 2 of ref. 3 is in error and the ordinate axis should read \((\text{Area}/2 \text{ instead of Area})\).

PS/6679
With \( N = 2.5 \times 10^{12} \) protons we obtain from eqn. (3) \( \Delta A = 2.14 \).

For this value of \( \Delta A \), one obtains from Fig. 1 \((A/A_0) = 0.85\) for \( \varphi_s = 11.5^\circ \) and \((A/A_0) = 0.89\) for \( \varphi_s = 0^\circ \) where \( A_0 \) is the stable phase area for no particles \((A = 0)\). Thus for \( \varphi_s = 5^\circ \) we would expect a decrease in the stable phase space of 13%.

This result agrees with computer calculations of U. Bigliani\(^4\) which determine the space-charge forces from calculated positions of many "macro-particles".

IV. THROBBING BEAM MOTION

a) Review

The possibility of unstable coherent transverse oscillations, due to the finite conductivity of the vacuum chamber walls, has been demonstrated for both uniform beams\(^5\)\(^6\) and tightly bunched\(^6\)\(^7\) beams. In order to understand the assumptions that are used in the two cases a short review is given below. This review will be helpful for the next section where the present theory is extended in an attempt to study these instabilities for the Booster.

For simplicity only the case of the vertical dipole oscillation is reviewed here for a round beam in a round chamber. The essential ingredient for studying the transverse motion is a knowledge of the local and wake forces on the particle. It will always be assumed that the variation of density in the longitudinal direction is small over a distance equal to \( b/\gamma \), with \( b \) the pipe radius and \( \gamma \) due to the Lorentz contraction of the longitudinal dimensions.

For a beam with a uniform transverse density inside a cylinder of radius \( a \), centered in the vacuum chamber, we obtain for the electric and magnetic space-charge field

\[
E_{rs} = \frac{2e\lambda}{a^2} r \quad \text{and} \quad E_{\theta s} = \beta E_{rs}
\]
which yields the gradient type of space charge force

$$\frac{1}{m'F_{YS}} = \left(\frac{2r_\lambda \sigma^2}{\nu^2a^2}\right)y$$

(4)

where the vertical direction is the y axis, $\lambda_s \sim (N/2\pi RB)$ is the longitudinal linear particle density at positions $z_s$, $F_{YS}$ is the vertical force on the "s" particles.

If the center of the beam is displaced vertically by an amount $Y_s$, which is small compared to $a$, we obtain the additional constant force given by

$$\frac{1}{m'F_{YS}} = -\frac{2r_\lambda Y_s \sigma^2}{\nu^2a^2} \left(\frac{b^2 - a^2}{b^2}\right).$$

(5)

The external force due to the magnetic focusing elements is given by

$$\frac{1}{m'F_{EXT}} = \nu_s \sigma^2 \Omega^2 \ y.$$  

(6)

The wake force on an "s" particle due to an "r" particle is obtained by defining the longitudinal positions of the particles by

$$z_s = z_{s_0} + \beta c t, \quad z_r = z_{r_0} + \beta c t, \quad z_r \geq z_s.$$ 

The wake force at position $z_s$ due to an "r" particle is proportional to the transverse displacement of the "r" particle at the time when it passed the point $z_s$. In addition the force decays in time inversely proportional to the square root of the time since the particle
passed by the observation point. One obtains for the wake force at \( z_s \)

\[
\frac{1}{m_Y} F_{ws}(z_s, t) = \sum_{r,n} \left\{ \frac{2r \beta^2 c^3}{m_Y b^3 (\beta c) V_s} \right\} \frac{N_r Y_r \left[ t - \frac{(z_s - z + 2\pi n R)}{\beta c} \right]}{\left| z_s - z + 2\pi n R \right| V_s} \tag{7}
\]

where \( N_r \) is the number particles at position \( r \) and the sum over \( n \) is to take into account the position of the \( r \)th particle on previous revolutions.

The assumptions present in deriving this force are: the distance between the \( s \) and \( r \) particle, \( (z_s - z_r) \), is large compared to \( b/\gamma \), the conductivity is high so that the conduction current is large compared to displacement current, the transverse oscillation frequency of the \( r \) particle is below the cut off for the propagating modes in the vacuum chamber, and that the wake fields have not yet diffused through the vacuum chamber wall.

The equation of motion for the center of the beam is thus

\[
y_s + \left( \frac{\nu_0^2 \Omega^2 - 2r \beta \frac{c^2}{\gamma^2 b^2}}{\gamma^2 b^2} \right) y_s = \frac{1}{m_Y} F_{ws} . \tag{8}
\]

We now look for eigenmodes such that

\[
y_s = \xi_s e^{i\nu t} \quad y_r = \xi_r e^{i\nu t} \tag{9}
\]

where \( \nu \) is the frequency of the particular eigenmodes and obtain

\[
(\nu_0^2 - E \lambda_s - \nu^2) \xi_s - (A/R) \sum_{r,s} N_r G_{r,s} \xi_r = 0 \tag{10}
\]
where

\[ \begin{align*}
B &= \frac{2r \frac{R^2}{P}}{\beta^2 \gamma^3 b^2}, \\
A &= \frac{r \frac{R}{P}}{\pi(\eta \beta \rho)^2} \gamma \frac{b^2}{b^2} \\
\end{align*} \]

(11)

\[ G_{r,s} = \sum_{n} \frac{2\pi^{1/2} e^{-i\nu(\theta_r - \theta_s)}}{|\theta_r - \theta_s + 2\pi|^2}, \]

(12)

and \( \theta = z/R \).

By changing to the azimuthal coordinate \( \theta \) and denoting \( \xi_s \) by \( \xi(\theta_s) \), we obtain

\[ [(\nu^2 - B\lambda(\theta_s) - \nu^2)\xi(\theta_s)] - \text{A Limit} \int_{\epsilon \to 0}^{2\pi + \theta_s + \epsilon} \lambda(\theta_r) \xi(\theta_r) G(\theta_r - \theta_s, \nu) d\theta_r. \]

(13)

Strictly speaking the limit \( \epsilon \) should not be allowed to go below \( b/\gamma R \), however, we will neglect this point here since \( b/\gamma R \ll 2\pi \) and very few particles are in this region.

In order to determine the stability of this type of oscillation one must solve either eqn. 10 or eqn. 13 for the various eigenvalues \( \nu \). If the imaginary part of \( \nu \) is positive the oscillations will be damped, and a negative imaginary part of \( \nu \) will indicate an unstable mode.

An alternate form of eqn. 13 can be obtained by introducing Fourier series for the azimuthal dependent quantities in eqn. 13:

\[ \lambda(\theta) = \sum \lambda_n e^{in(\theta - \Omega t)}, \quad \xi(\theta) = \sum \xi_n e^{in(\theta - \Omega t)}, \]

* The quantity \( G_{r,s} \) is discussed in great detail in the Appendix of ref. 7.
and

\[ G(\theta) = \sum_{n} g_n e^{-i(n(\theta - \Omega t)}, \quad \text{with} \quad g_n = \frac{1}{\sqrt{2}} \left[ 1 + i \text{Sign}(n - \nu) \right] \frac{1}{|n - \nu|^{1/2}}, \]

to obtain

\[ (\nu_{00}^2 - \nu^2) \xi_m^2 - \sum_n (B + 2\pi A g_n) \lambda_m^2 \xi_{n-m}^2 = 0. \] (14)

Similar expressions can be obtained for other throbbing beam motion beside the dipole motion presented here and these types of motion will be discussed shortly. It is instructive to solve eqns. 13 or 14 for two special cases.

**Case 1:** the case of an azimuthal uniform beam. Thus one substitutes for \( \lambda_m^2 = \lambda_{00}^2 \delta_{0,m} \) into eqn. 14 to obtain

\[ \nu_{00}^2 = \nu_{00}^2 - B\lambda_0 - 2\pi A g_n \lambda_0 \]

or

\[ \nu^2 = \nu_{00}^2 - \frac{2r R^2 \lambda_0}{\beta^2 \gamma^3 b^2} - \left( \frac{2\pi b \lambda_0}{\beta^2 \gamma^3 b^2} \right) \left[ \frac{1 + i \text{Sign}(n - \nu_{00})}{|n - \nu_{00}|^{1/2}} \right] \] (15)

This result agrees with ref. 5 and demonstrates that modes with \( n > \nu_{00} \) are unstable and modes with \( n < \nu_{00} \) are stable.

**Case 2:** a single bunch of length \( L \) such that \( \nu L / R << 1 \). One substitutes for \( \lambda(\theta) = N / L \) for \( -L / (2R) < \theta < L / (2R) \) and \( \lambda(\theta) = 0 \) for other values of \( \theta \). Then assuming that the bunch moves as a unit one expands the function \( G(\theta) \) for small values of \( \theta \) in terms of \( \nu L / R \) to obtain to first
order in $\nu L/R$

$$\nu_{oo}^2 = \frac{FN}{L} - \frac{AN}{R} G(2\pi, \nu_{oo}) - \frac{AN}{R} \left( \frac{R}{L} \right)^2 \left[ \frac{8\pi/3}{3} - \frac{8\pi/15}{R} \left( \frac{\nu_{oo} L}{R} \right) \right] - \nu^2 = 0 , \quad (16)$$

or

$$\nu \approx \nu_{oo} - \frac{i}{2\nu_{oo}} \left\{ \frac{2\pi R^2 N}{\beta^2 \gamma^3 L \pi^2} - \frac{\beta \gamma^3}{\pi (\pi \beta \alpha)^2} \right\} \left[ G(2\pi, \nu_{oo}) + \left( \frac{\pi R}{L} \right)^{1/2} \left( \frac{3}{15R} - \frac{8

\nu_{oo} L}{15R} \right) \right] , \quad (17)$$

Except for an error of $(2\pi)^{1/2}$ in ref. 7 the result agrees with that obtained in ref. 7 for a single bunch oscillating as a unit. The imaginary part of $\nu$ is given by

$$\nu_1 = \frac{1}{2\nu_{oo}} \frac{\beta \gamma^3}{\pi (\pi \beta \alpha)^2} \left[ G(2\pi, \nu_{oo}) + \left( \frac{\pi R}{L} \right)^{1/2} \left( \frac{3}{15R} - \frac{8\nu_{oo} L}{15R} \right) \right] , \quad (18)$$

The first term in the bracket is due to the wake fields of particles at the front of the bunch influencing particles later in the bunch on the same turn. The second term is due to the wake fields of the particles on previous turns. It can be shown that

$$\text{Im} \ G(2\pi, \nu_{oo}) < 0 \quad \text{for} \quad n < \nu_{oo} < n + \frac{1}{2}$$

where $n$ is an integer, and the motion is stable. However, there exists a value $\nu_t$ such that $n + 1/2 < \nu_t < n + 1$ defined by

$$\text{Im} \ G(2\pi, \nu_t) = \frac{3\gamma^2}{15L^{1/2}} \left( \frac{\nu_t L}{R} \right) , \quad (19)$$

PS/6679
The motion will be unstable for values of $v_{oo}$ such that

$$v_t < v_{oo} < n + 1,$$  \hspace{1cm} (20)

and the motion will be stable for values of $v_{oo}$ such that

$$n < v_{oo} < v_t.$$  \hspace{1cm} (21)

In the limit that $L/R \to 0$ the value of $v_t \to (n + 1/2)$ and one obtains the usual stability condition.

Eqn. 10 or one of the alternate eqns. 13 or 14 is the essential starting point to study the dipole oscillation. Since in many accelerators several bunches are accelerated at one time, it is necessary to establish when the motion of one bunch is influenced by the motion of another bunch. In order to understand the derivation of such a criterion we return to eqn. 10 and consider two very short bunches equally spaced around the accelerator. We obtain the two coupled eqns.

\[ \begin{align*}
[v_{oo}^2 - &B\lambda_1 - \frac{AN_1}{R} G(2\pi, v_{oo}) - \nu^2] \xi_1 - \frac{AN_2}{R} G(\pi, v_t) \xi_2 = 0 \\
- \frac{AN_1}{R} G(\pi, v_{oo}) \xi_1 - [v_{oo}^2 - B\lambda - \frac{AN_2}{R} G(2\pi, v_{oo}) - \nu^2] \xi_2 = 0
\end{align*} \]

which yields the eigenvalues.

\[ \nu^2 = v_{oo}^2 - \frac{1}{2} \left[ B(\lambda_1 + \lambda_2) + \frac{A(N_1 + N_2)}{R} G(2\pi, v_{oo}) \right] \]

\[ \pm \sqrt{\left[ B(\lambda_2 - \lambda_1) + \frac{A(N_1 - N_2)}{R} G(2\pi, v_{oo}) \right]^2 + 4 \frac{A^2 N_1 N_2}{R^2} G^2(\pi, v_{oo})} \]

FS/6679
We see that for the case where

\[
\left| B(\lambda_2 - \lambda_1) + \frac{AN(N - N_2)}{N} \right| G(2\pi, \nu_{oo}) \gg \left| 2 \frac{AN}{R} G(\pi, \nu_{oo}) \right| \tag{25}
\]

that the motion of the bunches is uncoupled. The first term on the left side of eqn. 25 is usually the dominate term so that the criterion for independent or single bunch motion is

\[
\frac{\Delta N}{N} > \left( \frac{\beta^2 \gamma^2 L}{\pi b} \right) \left( \frac{2\pi}{\nu_{oo} \sigma_R} \right)^{\frac{1}{2}} G(\pi, \nu_{oo}) \quad . \tag{26}
\]

b) Application to the Booster

We have seen that for the case where the number of particles per bunch is sufficiently different for different bunches that the bunches will behave independently. For the case of \( h \) bunches the criterion for independent motion is

\[
\frac{\Delta N}{N} > \left( \frac{\beta^2 \gamma^2 L}{\pi b} \right) \left( \frac{2\pi c}{\nu_{oo} \sigma_R} \right)^{\frac{1}{2}} \tag{27}
\]

Thus at injection for the Booster the bunches will behave independently if

\[
\frac{\Delta N}{N} > 10^{-3} . \tag{28}
\]

This condition will certainly be fulfilled and thus for the case where a bunch moves as a unit it will be sufficient to study single bunch motion.

So far, the single particle betatron oscillation frequency \( \nu_{oo} \) has been taken equal to a constant for all particles. In general, due to non-linearities and energy spreads, \( \nu_{oo} \) is not the same for all particles and there exists a spread in \( \nu_{oo} \) equal to \( \Delta \nu_{oo} \). If this frequency spread
is large enough it is possible that coherent oscillations, that would be
unstable otherwise, may be Landau damped.

For the single bunch throbbing beam modes the approximate spread
in the betatron oscillation frequency $\Delta v_0$ necessary to Landau damp coherent
oscillations is given below for several modes where $\Delta v$ is the incoherent
tune shift (eqn. 2).

<table>
<thead>
<tr>
<th>Type of Mode</th>
<th>Approx. Freq. of Mode</th>
<th>Approximate spread $\Delta v_0$ necessary for Landau damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole</td>
<td>$\nu \sim v_0$</td>
<td>$\Delta v$</td>
</tr>
<tr>
<td>Quadrupole</td>
<td>$\nu \sim 2v_0$</td>
<td>$\frac{1}{4} \Delta v$</td>
</tr>
<tr>
<td>Sextupole</td>
<td>$\nu \sim v_0$</td>
<td>$\frac{1}{4} \Delta v$</td>
</tr>
<tr>
<td>Sextupole</td>
<td>$\nu \sim 3v_0$</td>
<td>$\frac{1}{12} \Delta v$</td>
</tr>
<tr>
<td>Octupole</td>
<td>$\nu \sim 2v_0$</td>
<td>$\frac{1}{6} \Delta v$</td>
</tr>
<tr>
<td>Octupole</td>
<td>$\nu \sim 4v_0$</td>
<td>$\frac{1}{32} \Delta v$</td>
</tr>
</tbody>
</table>

It appears that several of the modes with frequencies approximately
equal to $v_0$ or $2v_0$ must be stabilized by other means than Landau
damping, while modes with higher frequencies will probably be Landau
damped. As was discussed before for the dipole oscillation there exists
values $\nu_t$ and $\nu_t'$ such that oscillations with frequencies approximately
$v_0$ and $2v_0$ are stable if

\[
\begin{align*}
    n &< v_0 < \nu_t \\
    n' &< 2v_0 < \nu_t'
\end{align*}
\]  

(29)

and

(30)
with $n$ and $n'$ integers. For the case of a bunch of length $L$ with a Gaussian density distribution in the longitudinal direction the same equation defining $v_t$ and $v_t'$ is

$$\text{Im } G(2\pi, v_t) = \left( \frac{R}{L} \right)^{1/2} e^{-(\frac{v_t L}{2R})^2} \sum_n \frac{\Gamma(n + 3/4)}{\Gamma(2n + 2)} \left( \frac{v_t L}{R} \right)^{2n+1} ,$$

(31)

with $\Gamma(x)$ the complete gamma function.

Thus for oscillation of a bunch as a single unit with frequency $\nu_0$ in the Booster the stability region is

$$4 < \nu_0 < 4.85 ,$$

(32)

and for the oscillation of frequency $2\nu_0$ the stability region is

$$4 < \nu_0 < 4.39 , \text{ or } 4.5 < \nu_0 < 4.89 .$$

(33)

Since the betatron oscillation frequencies of the Booster are designed to be within these limits the Booster should not suffer from the type of throbbing beam instability where the bunch moves as a single unit.

In the case of the Booster the length of a bunch is of the order of the oscillation wavelength and hence the possibility of unstable internal coherent oscillations exists. An attempt to study this type of motion is presented in the next section.

b) Internal motion in a bunch

Let us consider the following particle density function as shown
in Fig. 2

\[ \lambda(\theta) = \lambda_0 \sum_{n=0}^{N} \left[ 1 - \frac{(\theta - \frac{2\pi n}{5})^2}{a^2} \right] H \left[ a^2 - (\theta - \frac{2\pi n}{5})^2 \right] \]

where

- $H(x)$ is the Heaviside unit step function equal to unity for positive $x$ and zero for negative $x$.
- For the Booster $a = \pi/10$.

Next we will divide each bunch into slices as shown in Fig. 3.
where the width of each slice, $\delta \theta$, is the same and

$$\delta \theta = \frac{b}{\gamma R} = 1.9 \times 10^{-3} \text{ radian.} \quad (35)$$

Since the azimuthal extent of the strong local fields is of the order of $\delta \theta$, we assume that all of the particles inside of one slice are strongly coupled together by the local fields while one slice is coupled to another slice only by the wake fields. We will of course expect the wake fields to couple slice A to slice A', slice B to slice B', etc. because of the equal particle densities and hence equal eigenvalues for the oscillation frequency. However, as we will see below, slices with different densities are not coupled by the wake fields. To demonstrate this consider only the two adjacent slices A and B, of all the slices with different densities, these slices will have the strongest coupling because the density variation between them is the smallest and because they are adjacent to each other and the wake fields decrease with distance. Returning to eqn. 10 we have for the motion of these two slices, ignoring the other slices, the following coupled eqns.

\[ [\nu_0^2 - B\lambda(B) - AG(2\pi, \nu)\lambda(B)\delta \theta - \nu^2] \xi(B) - [AG(\delta \theta, \nu)\lambda(A)\delta \theta] \xi(A) = 0, \quad (36) \]

\[ - [AG(2\pi - \delta \theta, \nu)\lambda(B)\delta \theta] \xi(B) + [\nu_0^2 - B\lambda(A) - AG(2\pi, \nu)\lambda(A)\delta \theta - \nu^2] \xi(A) = 0. \quad (37) \]

The notation here is poor and the reader is reminded that $\lambda(A)$, $\xi(A)$, $\lambda(B)$ and $\xi(B)$ refer to the densities and displacements of slices A and B while the coefficient A and B are defined by eqn. 11.
we use the fact the $\delta \Theta << 2\pi$ to obtain $G(2\pi - \delta \Theta, \nu) = G(2\pi, \nu)$ and $G(\delta \Theta, \nu) \approx 2(\pi/\delta \Theta)^{1/2}$. The criterion that slices A and B are uncoupled is

$$B^2[\lambda(A) - \lambda(B)]^2 \gg 2\Lambda^2\lambda^2(\delta \Theta)^2\left(\frac{\pi}{\delta \Theta}\right)^{1/2} |G(2\pi, \nu)|, \quad (38)$$

or

$$|\frac{\Delta \lambda}{\lambda}| \gg \left(\frac{\Theta^2\gamma^2}{2\pi b}\right) \left(\frac{BC}{\pi \beta a}\right)^{1/2} \left(\frac{L_{\pi}}{\delta \Theta}\right)^{1/4} \Theta |G(2\pi, \nu)|^{1/2}. \quad (39)$$

Since $|G(2\pi, \nu)|^{1/2}$ is of the order of unity this condition becomes for the Booster

$$|\frac{\Delta \lambda}{\lambda}| \gg 5.18 \times 10^{-6}. \quad (40)$$

We use eqn. 34 to calculate the variation of particle density in the distance $\delta \Theta$ to obtain:

$$|\frac{\Delta \lambda}{\lambda}| \approx \left(\frac{\delta \Theta}{\alpha}\right)^2 \approx 3.67 \times 10^{-5} >> 5.18 \times 10^{-6}. \quad (41)$$

Thus for the Booster, slices with unequal densities will be uncoupled, and it is necessary only to study the coupling between slices with equal densities.

Since slices A and A' have the largest number of particles and they are closest together we will study the coupling between them next. The coupling between slices B and B', C and C' and etc. will always be less
than the coupling between $A$ and $A'$. In addition slices $A$ and $A'$ of one bunch are coupled to slices $A$ and $A'$ of all of the other bunches, however, the coupling is strongest between slices in the same bunch so that in order to obtain the largest growth rate of the unstable mode we will consider only the coupling of the two slices in one bunch. For these two slices we obtain the following two eqns.

\[
\begin{align*}
[v_0^2 - BA - AG(2\pi, \nu) \lambda \delta \nu - v^2] \xi(A) - AG(\delta \theta, \nu) \lambda \delta \nu \xi(A') = 0 \quad (42) \\
-[AG(2\pi - \delta \theta, \nu) \lambda \delta \nu] \xi(A) + [v_0^2 - BA - AG(2\pi, \nu) \lambda \delta \nu - v^2] \xi(A') = 0. \quad (43)
\end{align*}
\]

Again we use the fact that for $\delta \theta << 2\pi$, $G(2\pi - \delta \theta, \nu) \approx G(2\pi, \nu)$ and $G(\delta \theta, \nu) \approx 2(\pi/\delta \theta)^{1/2}$ to obtain for the imaginary part of $\nu$

\[
\text{Im} \nu \approx \pm \frac{1}{v_0} A\lambda_0 \delta \theta \left(\frac{4\pi}{\delta \theta}\right)^{1/4} \text{Im}[G(2\pi, \nu_0)]^{1/2}. \quad (44)
\]

For the case of the Booster for $N = 2.5 \times 10^{12}$ we have $\lambda_0 = 4.77 \times 10^8$/cm and

\[
\text{Im} \nu = \pm 3.67 \times 10^{-7} \text{Im}[G(2\pi, \nu_0)]^{1/2}. \quad (45)
\]

Thus we see that we always have a stable and an unstable mode. For most values of $\nu_0$ the $\text{Im}[G(2\pi, \nu_0)]^{1/2}$ is close to unity so that the growth time of the unstable mode is

\[
\tau = \left|\frac{1}{\text{Im} \nu \Omega}\right| \approx 1 \text{ sec.}
\]

It must be emphasized that this is shortest growth time $\tau$ (lower limit) and that the synchrotron oscillations will move particles from slices $A$ and $A'$ into slices that are further apart and less dense, so that for part of time the particles are in slices that are not as strongly coupled together.
as they are in slices A and A'.

At present a computer program has been written to study the eigenvalues of eqn. 14 and the results of the work will be written later. Preliminary computer results seem to support the present model.
REFERENCES


2) L.J. Laslett; Private Communication.


4) U. Bigliani; Private Communication.


