BEAM BLOW-UP AND LUMINOSITY REDUCTION DUE TO LINEAR COUPLING

by

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Abstract

The solutions of the linearly coupled motions are written as functions of the initial coordinates of a single particle. Using the subsequent form of the coupled motions one can analyse the coherent oscillations of a kicked beam and it is shown how these oscillations can be used for measuring the real and imaginary parts of the complex coupling coefficient, which characterises the skew quadrupole and axial field excitation. Furthermore, using the motion solutions one can establish formulae for the relative change of the beam sizes and the luminosity loss in the absence of shaving. When the beams are shaved vertically, the vertical beam size change is given in an integral form, which has been solved numerically.

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LIST OF FREQUENTLY USED SYMBOLS AND THEIR MEANINGS

x  horizontal transverse coordinate
z  vertical transverse coordinate
y  general transverse coordinate, either horizontal or vertical
y' derivative of the general transverse coordinate
R  average machine radius
θ  azimuthal angle
t  time
I  beam current
B_x, B_z transverse components of the magnetic induction
B_s  longitudinal component of the magnetic induction
B_C  magnetic rigidity
δ_y  transverse betatron amplitude function in plane (y, s)
α_y  Twiss parameter associated with the derivative of δ_y
μ_y  phase of betatron oscillation in plane (y, s)
Q_y  number of betatron oscillations per revolution in transverse plane (y, s)
C  linear coupling complex coefficient
Δ  distance from the resonance
u  horizontal normalised coordinate
v  vertical normalised coordinate
w  general transverse normalised coordinate
f_{rev}  revolution frequency
σ_y  general transverse r.m.s. value of the beam dimension
E_y  general transverse emittance
L  luminosity

A prime denotes differentiation with respect to θ and a star denotes differentiation with respect to the phase μ_y. The sign - on top of a complex variable indicates its complex conjugated value.
1. INTRODUCTION

In a recent report\(^1\), the theory of resonances has been extended to three-dimensional magnetic fields. In this context, the linear coupling resonance \(Q_x - Q_z = p\) has been treated in some detail. The explicit solution of the coupled motions and the motion invariants are given\(^1\) as functions of a complex coupling coefficient and of the distance from the resonance line.

Starting from these results summarised in chapter 2, it looked interesting to investigate the dependence of the coupled motions on the initial conditions and this is done in chapter 3. The consequent analytical form of the coupled motions gives the possibility of analysing the coherent oscillations of a kicked beam. This analysis is given in chapter 4 and it is shown how these coherent oscillations can be used for measuring the coupling coefficient. The beam blow-up due to linear coupling, which was the principal reason of this study, is treated in chapter 5. The change in the two beam dimensions is given analytically for an unshaved beam in presence of linear coupling. For a shaved beam with Gaussian profiles, the change in beam height is numerically calculated. Since the luminosity is inversely proportional to the beam height, the vertical beam blow-up due to coupling gives the luminosity reduction directly.

2. MAIN RESULTS OF THE LINEAR COUPLING THEORY

In reference 1, linear coupling phenomena are described in detail for machines with any form factor. The perturbed motion is characterised by a complex coupling coefficient \(C\), which is, for the second order difference resonance\(^1\)

\[
C = \frac{1}{2\pi R} \int_0^{2\pi} \sqrt{\frac{\beta_x}{\beta_z}} \left[ K + \frac{MR}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) - \frac{MR}{2} \left( \frac{1}{\beta_x} + \frac{1}{\beta_z} \right) \right] \exp \left[ i[(\mu_x - \mu_z) - (\Delta - p)\theta]\right] d\theta,
\]

(2.1)

where

\[
K(\theta) = \frac{R^2}{2Bo} \left( \frac{\partial B}{\partial x} \frac{\partial^2 z}{\partial^2 z} - \frac{\partial B}{\partial z} \frac{\partial^2 z}{\partial^2 x} \right),
\]

\[
M(\theta) = \frac{R}{Bo} \frac{B_y}{s},
\]

\[\Delta = Q_x - Q_z.\]

(2.2)

\(p\) is the harmonic number associated with the resonance \(Q_x - Q_z = p\).
The meanings of the other symbols appearing in this expression are given in the list at the beginning of this report. The real and imaginary parts of \( C \) will be noted throughout \( C_1 \) and \( C_2 \), respectively.

The explicit solution of the coupled motions can be derived from the equations (1.3.3) and (1.5.17) in reference 1. By putting the explicit form of the Floquet's functions and by replacing \( \kappa \) by \( C/2 \) the solutions become,

\[
x = \frac{\gamma}{2} \left( \frac{A_{+}}{\omega_{+}} e^{i \omega_{+} \theta} + \frac{A_{-}}{\omega_{-}} e^{i \omega_{-} \theta} \right) \sqrt{\frac{B}{2R}} e^{i \mu x} + c.c.
\]

\[
z = \left( A_{+} e^{-i \omega_{-} \theta} + A_{-} e^{-i \omega_{+} \theta} \right) \sqrt{\frac{B}{2R}} e^{i \mu z} + c.c.,
\]

where

\[
\omega_{\pm} = \frac{1}{2} \left[ -\Delta \pm \sqrt{\Delta^2 + |C|^2} \right].
\]

\( \Delta \) and \( C \) being defined above,

\( A_{+} \) and \( A_{-} \) are complex constants of the motion, which depend on the initial coordinates,

\( c.c. \) means complex conjugate.

For the following developments, it is also necessary to have the expressions for the derivatives of the motions,

\[
x' = \frac{\gamma}{2 \omega_{+}} \left\{ i \left( A_{+} e^{i \omega_{+} \theta} + A_{-} e^{i \omega_{-} \theta} \right) \sqrt{\frac{B}{2R}} e^{i \mu x}
\right.
\]

\[
+ \left( \frac{A_{+}}{\omega_{+}} e^{i \omega_{+} \theta} + \frac{A_{-}}{\omega_{-}} e^{i \omega_{-} \theta} \right) (i - \alpha_{x}) \sqrt{\frac{B}{2R}} e^{i \mu x} \right\} + c.c.
\]

\[
z' = \left\{ -i \left( \omega_{-} A_{+} e^{-i \omega_{-} \theta} + \omega_{+} A_{-} e^{-i \omega_{+} \theta} \right) \sqrt{\frac{B}{2R}} e^{i \mu z}
\right.
\]

\[
+ \left( A_{+} e^{-i \omega_{-} \theta} + A_{-} e^{-i \omega_{+} \theta} \right) (i - \alpha_{z}) \sqrt{\frac{B}{2R}} e^{i \mu z} \right\} + c.c.
\]

where ' denotes the derivative with respect to \( \theta \). The following relations have been used in this derivation:

\[
\mu' = \frac{R}{\beta_y}
\]

\[
\beta' = -2R \alpha_y
\]
3. Solution of the Coupled Motions in Terms of the Initial Conditions

3.1. Expressions of the Complex Amplitudes

The motion solutions given in chapter 2 contain two complex amplitudes \( A_+ \) and \( A_- \) which have to be expressed in terms of the initial coordinates. The relations (2.3) and (2.5) can be re-written using a matrix formalism,

\[
\begin{bmatrix}
  x \\ x' \\ z \\ z'
\end{bmatrix}
= T
\begin{bmatrix}
  A_+ \\ A_- \\ \Lambda_+ \\ \Lambda_
\end{bmatrix}.
\tag{3.1}
\]

The coefficients of the matrix \( T \) are directly given by the relations (2.3) and (2.5). Initially, i.e. at \( \theta = 0 \), the relation (3.1) is simply,

\[
\begin{bmatrix}
  x_0 \\ x'_0 \\ z_0 \\ z'_0
\end{bmatrix}
= T_0
\begin{bmatrix}
  A_+ \\ A_- \\ \Lambda_+ \\ \Lambda_
\end{bmatrix}
\tag{3.2}
\]

where \( T_0 \) represents the matrix \( T \) at \( \theta = 0 \). For reasons of simplicity, it has been assumed in what follows that \( \mu_y = 0 \) at \( \theta = 0 \). The inversion of the matrix \( T_0 \) gives the complex amplitudes in terms of the initial coordinates. The explicit forms of the coefficients of \( T_0 \) and the long algebraic development associated with the inversion of \( T_0 \) are summarised in the Appendix I.

The final results are:

\[
A_+ = i \frac{\omega_+}{\eta} \frac{1}{\frac{R}{B_{x,0}} + \frac{R^2}{B_{x,0}} - C_1^2}
\]

\[
\left\{ \begin{array}{l}
C_1 \left( \frac{R}{B_{x,0}} \left( \alpha_{x,0} + i \omega_+ \right) \right) - i \omega_+ - \frac{R C}{\omega + B_{x,0}} - \frac{R^2 C}{\omega + B_{x,0}} (\alpha_{x,0} + i) \sqrt{\frac{2R}{B_{x,0}}} x_0 \\
\left[ C_1 - \frac{R C}{\omega + B_{x,0}} \right] \sqrt{\frac{2R}{B_{x,0}}} x'_0 \\
\left[ C_1 \frac{1}{2} \left( \frac{R}{B_{z,0}} \left( \alpha_{z,0} + i \right) + i \right) - i \frac{R C}{\omega + B_{z,0}} + \frac{R^2 C}{\omega + B_{z,0}} (\alpha_{z,0} + i) \sqrt{\frac{2R}{B_{z,0}}} z_0 \\
+ \frac{C}{2\omega_+} \left[ C_1 + \frac{R C}{\omega + B_{x,0}} \right] \sqrt{\frac{2R}{B_{z,0}}} z'_0 \right\}
\end{array} \right.
\tag{3.3}
\]

\[
A_- = -A_+ (\omega_+ \leftrightarrow \omega_-)
\]
where the last relation means that $A_\pm$ has a sign opposed to the one of $A_\mp$ and verifies the same algebraic form, in which the parameters $\omega_+$ and $\omega_-$ are exchanged.

For reasons of convenience, a new parameter $\eta$ has been introduced in (3.3),

$$\eta = \sqrt{\Delta^2 + |C|^2}.$$  
(3.4)

The parameters $C_1$, $C_2$, $C_i$, $\Delta$, $\omega_-$ and $\omega_+$ are defined in chapter 2 and the other functions (taken here at $\theta = 0$) have their normal meanings which are given in the list at the beginning of this report.

3.2. Analysis of the transverse motions

Substituting the expressions (3.3) for the complex amplitudes back into (2.3), gives the solution of the motions in terms of the initial conditions, but first, it is interesting to re-write the solution (2.3) using the sinusoidal functions,

$$x = \frac{\sqrt{\Delta}}{\eta G} \left[ a_+ \cos (\mu + \omega_+ \theta) - b_+ \sin (\mu + \omega_+ \theta) \\
- a_- \cos (\mu + \omega_- \theta) + b_- \sin (\mu + \omega_- \theta) \right]$$

$$z = \frac{\sqrt{\Delta}}{\eta G} \left[ f_+ \cos (\mu - \omega_+ \theta) - g_+ \sin (\mu - \omega_+ \theta) \\
- f_- \cos (\mu - \omega_- \theta) + g_- \sin (\mu - \omega_- \theta) \right].$$

Consequently, the definitions of the newly introduced coefficients are,

$$a_\pm = \pm \frac{nG}{\omega_{\pm} \sqrt{2R}} \text{Re} \,(C_{A_{\pm}})$$

$$b_\pm = \pm \frac{nG}{\omega_{\pm} \sqrt{2R}} \text{Im} \,(C_{A_{\pm}})$$

$$f_\pm = \pm \frac{\sqrt{2}}{\sqrt{R}} \frac{nG}{\eta} \text{Re} \,(A_{z_{\pm}})$$

$$g_\pm = \pm \frac{\sqrt{2}}{\sqrt{R}} \frac{nG}{\eta} \text{Im} \,(A_{z_{\pm}})$$

$$G = \frac{4R^2}{G_{x,0} G_{z,0}} - C_1^2.$$
Using now the expressions (3.3) of the complex amplitudes $A_{\pm}$, it is possible to write the coefficients $a$, $b$, $f$ and $g$ in terms of the initial coordinates,

\[ a_\pm = \left[ \frac{C_1^2}{\beta_{x,0}} \left( \frac{R}{\beta_{x,0}} + \omega_{x} \right) - \frac{R}{\beta_{x,0}} \frac{C_2^2}{\omega_{x,0} - \frac{R^2}{\beta_{x,0} \beta_{z,0}}} \right] \frac{x_0}{\sqrt{\beta_{x,0}}} \]

\[ + \ C_1 C_2 \frac{1}{\beta_{x,0}} \left( \frac{x_0'}{\omega_{x,0}} + \frac{R a_{x,0}}{\beta_{x,0}} x_0 \right) \]

\[ + \ C_1 \left( \frac{4R^2}{\beta_{x,0} \beta_{z,0}} - |C|^2 \right) \frac{z_0}{\sqrt{\beta_{z,0}}} \frac{1}{\sqrt{\beta_{x,0}}} \left( z_0' + \frac{R a_{z,0}}{\beta_{z,0}} z_0 \right) \]

\[ b_\pm = -C_1 C_2 \left( \frac{2}{\beta_{x,0}} \frac{R}{\beta_{x,0}} + \omega_{x} \right) \frac{x_0}{\sqrt{\beta_{x,0}}} - \left( \frac{R}{\omega_{x,0} \beta_{z,0}} |C|^2 - C_1^2 \right) \frac{1}{\sqrt{\beta_{x,0}}} \left( x_0' + \frac{R a_{x,0}}{\beta_{x,0}} x_0 \right) \]

\[ - \ C_2 \left( \frac{4R^2}{\beta_{z,0} \beta_{x,0}} - \frac{R}{\omega_{x,0} \beta_{z,0}} |C|^2 \right) \frac{z_0}{\sqrt{\beta_{z,0}}} \frac{1}{\sqrt{\beta_{x,0}}} \left( z_0' + \frac{R a_{z,0}}{\beta_{z,0}} z_0 \right) \]

\[ f_\pm = \frac{C_1}{\beta_{x,0}} \left( \frac{4R^2}{\beta_{x,0} \beta_{z,0}} + \frac{4R}{\beta_{x,0}} \omega_{x} - |C|^2 \right) \frac{x_0}{\sqrt{\beta_{x,0}}} + \frac{R C_2}{\beta_{x,0}} \frac{1}{\sqrt{\beta_{x,0}}} \left( x_0' + \frac{R a_{x,0}}{\beta_{x,0}} x_0 \right) \]

\[ - \left[ C_1^2 \left( \omega_{x} + \frac{R^2}{\beta_{x,0} \beta_{z,0}} \right) + C_2^2 \left( \frac{R}{\beta_{z,0}} + \frac{R^2}{\beta_{x,0} \beta_{z,0}} \right) \right] \frac{z_0}{\sqrt{\beta_{z,0}}} \]

\[ - \ C_1 C_2 \frac{1}{\sqrt{\beta_{z,0}}} \left( z_0' + \frac{R a_{z,0}}{\beta_{z,0}} z_0 \right) \]

\[ g_\pm = -2C_2 \left( \frac{R}{\beta_{x,0}} \omega_{x} - \frac{R^2}{\beta_{x,0} \beta_{z,0}} \right) \frac{x_0}{\sqrt{\beta_{x,0}}} - \frac{2C_1}{\beta_{x,0}} \left( \frac{R}{\beta_{z,0}} - \omega_{x} \right) \frac{1}{\sqrt{\beta_{x,0}}} \left( x_0' + \frac{R a_{x,0}}{\beta_{x,0}} x_0 \right) \]

\[ + \ C_1 C_2 \left( \frac{R}{\beta_{z,0}} - \omega_{x} \right) \frac{z_0}{\sqrt{\beta_{z,0}}} \frac{1}{\beta_{x,0}} \left( z_0' + \frac{R a_{z,0}}{\beta_{z,0}} z_0 \right) \frac{1}{\sqrt{\beta_{z,0}}} \left( z_0' + \beta_{x,0} \right) \frac{z_0}{\sqrt{\beta_{z,0}}} \frac{1}{\beta_{x,0}} \left( z_0' + \beta_{x,0} \right). \]

The terms appearing on the right hand side of the relations (3.7) have been grouped in such a way that the normalised variables appear spontaneously. The definitions of the normalised variables are,
\[ w = \frac{y}{\sqrt{g_y}} \]
\[ w^k = \frac{d\omega}{du} = \frac{\sqrt{g_y}}{R} \left( \gamma' + \frac{R_y}{g_y} y \right), \]

verifying
\[ w^2 + w^k^2 = \text{constant}. \]

It is obvious from (3.5) that the single particle motions contain fast and slow oscillations associated with the phases \( \varphi_y \) and \( \varphi_z \), respectively. It is now interesting to factorize the signal into a slowly oscillating envelope component and a fast oscillating component. This has been done in Appendix 2 and the result is:

\[ x(\theta) = \frac{\sqrt{g_x}}{n_x} A_x \cos \left( u_x - \frac{\Delta}{2} \theta - \phi_x \right) \]
\[ z(\theta) = \frac{\sqrt{g_z}}{n_z} A_z \cos \left( u_z + \frac{\Delta}{2} \theta - \phi_z \right), \]

where

\[ A_x = \frac{1}{6} \sqrt{(a_+ - a_-)^2 + (b_+ - b_-)^2 + 4(a_+ a_- + b_+ b_-) \sin^2 \frac{n}{2} \theta + 4(a_+ b_- - a_- b_+ \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta} \]

\[ \phi_x = \arctan \frac{(b_+ - b_-) \cos \frac{n}{2} \theta - (a_+ a_-) \sin \frac{n}{2} \theta}{(a_+ - a_-) \cos \frac{n}{2} \theta - (b_+ - b_-) \sin \frac{n}{2} \theta} \]

and

\[ A_z = \frac{1}{6} \sqrt{(f_+ - f_-)^2 + (g_+ - g_-)^2 + 4(f_+ f_- + g_+ g_-) \sin^2 \frac{n}{2} \theta + 4(f_+ g_- - f_- g_+ \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta} \]

\[ \phi_z = \arctan \frac{(g_+ - g_-) \cos \frac{n}{2} \theta - (f_+ f_-) \sin \frac{n}{2} \theta}{(f_+ - f_-) \cos \frac{n}{2} \theta - (f_- + f_+ \sin \frac{n}{2} \theta} \]

It is obvious from (3.10) and (3.11) that the period of the envelope oscillations is given by the parameter \( n \) (3.4),

\[ T = \frac{1}{n f_{rev}}, \]

where \( f_{rev} \) is the revolution frequency verifying \( \theta = 2\pi f_{rev} t \).

The frequencies of the fast oscillating components can be deduced from the equations (3.9). These frequencies differ from the frequencies existing in absence of linear coupling, and the frequency perturbations are,
\[ \frac{\delta f_x}{f_{\text{rev}}} = -\left( \frac{\Delta}{2} + \frac{d\phi_x}{d\phi} \right) \]
\[ \frac{\delta f_z}{f_{\text{rev}}} = \frac{\Delta}{2} - \frac{d\phi_z}{d\phi}. \]

\( \delta f_x \) and \( \delta f_z \) are the frequency changes in the horizontal and vertical plane, respectively,
\( \phi_x \) and \( \phi_z \) are the phase factors given in (3.10) and (3.11).

The consequent tune changes caused by coupling are,
\[ \delta Q_x = -\frac{\Delta}{2} - \frac{1}{2\pi} [\phi_x (2\pi + \theta) - \phi_x (\theta)] \]
\[ \delta Q_z = \frac{\Delta}{2} - \frac{1}{2\pi} [\phi_z (2\pi + \theta) - \phi_z (\theta)]. \]

4. TRANSVERSE COHERENT OSCILLATIONS OF A KICKED BEAM

4.1. Oscillations of a horizontally kicked beam

The single particle solutions (3.5) and (3.9) can also describe the coherent oscillations of a pulse which is excited by a kicker. In the simplest case, the beam is excited in one of the transverse planes. The following development concerns a horizontal kick, but the results would be equivalent for a vertical kick provided the roles of the horizontal and vertical variables are exchanged.

The initial conditions associated with a horizontal kick are,
\[ x_0 = z_0 = z^0 = 0 \]
\[ x^1 \neq 0 \] (4.1)

Introducing these boundary conditions into (3.7) and using the consequent expressions in (3.5) gives the explicit form of the coherent oscillations following a horizontal kick,
\[ x = \sqrt{\frac{\beta_x}{\beta_{x,0}}} \frac{x_0}{\eta_0} \left[ C_1 C_2 \cos (\nu_x + \omega_\theta) + \left( \frac{R}{\omega_\theta B_{z,0}} |C|^2 - C_1^2 \right) \sin (\nu_x + \omega_\theta) \right. \]
\[ - C_1 C_2 \cos (\nu_x + \omega_\theta) - \left( \frac{R}{\omega_\theta B_{z,0}} |C|^2 - C_1^2 \right) \sin (\nu_x - \omega_\theta) \left] \right. \]
\[ z = \sqrt{\frac{\beta_z}{\beta_{z,0}}} \frac{x^1}{\eta_0} \left[ 2 R C_2 \cos (\nu_z - \omega_\theta) + 2 C_1 \left( \frac{R}{\beta_{z,0}} - \omega_\theta \right) \sin (\nu_z - \omega_\theta) \right. \]
\[ - 2 R C_2 \cos (\nu_z - \omega_\theta) - 2 C_1 \left( \frac{R}{\beta_{z,0}} - \omega_- \right) \sin (\nu_z - \omega_\theta) \left], \right. \]
where $\beta_x$ and $\beta_z$ are the betatron amplitudes at any position $s$ where the coherent oscillation is observed.

$\beta_{x,0}$ and $\beta_{z,0}$ are the betatron amplitudes at the kicker position, all the other parameters being defined in chapters 2 and 3.

The explicit solutions (4.2) have been written voluntarily in the non-factorized form, in order to compare them with the solutions given by A.W. Chao and M. Month\textsuperscript{2}) for the case of skew fields only, i.e. $M \equiv 0$ (2.2). The motion solutions corresponding to a horizontally kicked beam are given by the relations (20) in reference 2, and contain only the sinus terms appearing in (4.2). This means that the solutions of A.W. Chao and M. Month are strictly valid only if $C_2 \equiv 0$, i.e. when $C$ is real. This is in fact restrictive, since $C$ will almost always be complex, even if $M \equiv 0$, because of the phase terms in (2.1). This limitation in reference 2 comes from the assumption that a change of origin (for $\theta$) can always reduce $C$ to being real and positive. The effect of a change of origin is limited by the maximum difference between the betatron phase advances from the old to the new origin and cannot always be made large enough to make $C$ real. The relations (4.2) are valid for any origin with longitudinal fields as well as skew fields.

Factorizing now the signals (4.2) by using (3.9) gives the slowly oscillating envelopes and the phase shifts,

$$A_x^2 = \left( \frac{4nRx_0'}{G^0_{x,0} \sqrt{\beta_x}} \right)^2 \left[ 1 + \left( \frac{C_1' C_2 + C_1^2 \beta_z^2}{4n^2R^2} - \frac{|C|^2}{n^2} - \frac{C_1^2 \beta_{z,0}^2}{R^2} \right) \sin^2 \frac{n}{2} \theta \right.$$

$$\left. - \frac{C_1 C_2 \beta_{z,0}}{nR} \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta \right]$$

$$A_z^2 = \frac{2nC_1' x_0'}{G \sqrt{\beta_x}} \left[ 1 + 4 \left( \frac{R^2 |C|^2}{n^2 \beta_z^2} + \frac{\Delta R}{C_1' \beta_z \eta^2} - \frac{|C|^2}{4n^2} \right) \sin^2 \frac{n}{2} \theta \right.$$

$$\left. + \frac{RC_2}{C_1' \eta^2 \beta_z} \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta \right]$$

$$\phi_x = \arctg \left[ \frac{2nR \cos \frac{n}{2} \theta - C_1' C_2 \beta_{z,0} \sin \frac{n}{2} \theta}{(2R\Delta - C_1^2 \beta_{z,0}) \sin \frac{n}{2} \theta} \right]$$

$$\phi_z = \arctg \left[ \frac{C_1' \beta_{z,0} \cos \frac{n}{2} \theta + 2RC_2 \sin \frac{n}{2} \theta}{C_1 (2R + \beta_{z,0} \Delta) \sin \frac{n}{2} \theta} \right].$$
These equations are equivalent to the relations (4.11) and (4.12) given in reference 3. They are identical only if the dominant terms of (4.3) are retained using the inequality R/εz₀ ≫ |C|. In this case, (4.3) becomes,

\[ A_x^2 = \frac{\eta^2 \beta_{x,0} x_{0}^2}{R^2} \left[ 1 - \frac{|C|^2}{\eta^2} \sin^2 \frac{n}{2} \theta \right] \]

\[ A_z^2 = \frac{\eta^2 \beta_{z,0} x_{0}^2}{R^2} \frac{|C|^2}{\eta^2} \sin^2 \frac{n}{2} \theta, \]

in agreement with reference 3. It is evident from (4.4) that the fraction F of the energy (taken here as the square of the envelope amplitudes) interchanged between the two signals and the ratio S of the minimum to the maximum of the horizontal envelope are

\[ F = \frac{|C|^2}{\eta^2} = \frac{|C|^2}{\Delta^2 + |C|^2} \]

\[ S = \frac{\Delta}{\eta} = \frac{\Delta}{\sqrt{\Delta^2 + |C|^2}}. \]

It is interesting to note that the maxima and minima of the envelopes (4.4) appear for \( \eta \theta = \eta \pi \), \( n \) being an integer, and that these envelopes are independent of the phase of \( C \) in the imaginary plane if the assumption \( R/\varepsilon z_0 \gg |C| \) is verified.

4.2. Beam oscillations following an inclined kick

In this section, the coherent oscillations of a pulse which is excited by an inclined kicker will be described. The boundary conditions associated with such a kick are,

\[ x_0 = z_0 = 0 \]

\[ x_0' \neq 0 \quad z_0' \neq 0 \]

Using these boundary conditions in (3.7) and using the consequent expressions in (3.9) gives the slowly oscillating envelopes and the phase shifts. First the substituion of (4.6) into (3.7) gives,
\[ a_\pm = C_1 C_2 \frac{x_0'}{\sqrt{\beta_{x,0}}} - \frac{2RC_2}{\sqrt{\beta_{x,0}}} \frac{x_0'}{\sqrt{\beta_{z,0}}} \]
\[ b_\pm = \left[ C_1^2 - \frac{R}{\omega_{z,0}} |C|^2 \right] \frac{x_0'}{\sqrt{\beta_{x,0}}} + \frac{C_1}{2} \left[ \frac{|C|^2}{\beta_{x,0}} - \frac{4R}{\beta_{x,0}} \right] \frac{z_0'}{\sqrt{\beta_{z,0}}} \]
\[ f_\pm = \frac{2RC_2}{\beta_{z,0}} \frac{x_0'}{\sqrt{\beta_{x,0}}} - C_1 C_2 \frac{z_0'}{\sqrt{\beta_{z,0}}} \]
\[ g_\pm = 2C_1 \left( \frac{R}{\beta_{x,0}} \right) \frac{x_0'}{\sqrt{\beta_{x,0}}} + \left( C_1^2 + \frac{R}{\beta_{x,0}} \right) \frac{|C|^2}{\sqrt{\beta_{z,0}}} \frac{z_0'}{\sqrt{\beta_{z,0}}} \]

In order to reduce the algebraic calculations, it is interesting to assume already in (4.7) that \( R/\beta_{x,0} >> |C| \) and \( R/\beta_{z,0} >> |C| \). In this way, the number of terms is reduced by a factor of 2 and the final expressions for \( A_y \) and \( \phi_y \) (3.9) are,

\[ A^2_x = \frac{n^2}{R^2} \beta_{x,0} x_0'^2 - \frac{1}{R^2} \left[ |C|^2 (\beta_{x,0} x_0'^2 - \beta_{z,0} z_0'^2) - 2C_1 \sqrt{R} \beta_{x,0} x_0' z_0' \right] \sin^2 \frac{n}{2} \theta \sin \frac{n}{2} \theta \]
\[ + 2 \frac{nC_2}{R^2} \sqrt{\beta_{x,0}} \beta_{z,0} x_0' z_0' \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta \]

\[ A^2_z = \frac{n^2}{R^2} \beta_{z,0} z_0'^2 + \frac{1}{R^2} \left[ |C|^2 (\beta_{x,0} x_0'^2 - \beta_{z,0} z_0'^2) - 2C_1 \sqrt{R} \beta_{x,0} x_0' z_0' \right] \sin^2 \frac{n}{2} \theta \cos \frac{n}{2} \theta \]
\[ - 2 \frac{nC_2}{R^2} \sqrt{\beta_{x,0}} \beta_{z,0} x_0' z_0' \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta \]

\[ \phi_x = \arctg \left[ \frac{n \sqrt{\beta_{x,0}} x_0' \cos \frac{n}{2} \theta + C_2 \sqrt{\beta_{z,0}} z_0' \sin \frac{n}{2} \theta}{(\Delta \sqrt{\beta_{x,0}} x_0' - C_1 \sqrt{\beta_{z,0}} z_0') \sin \frac{n}{2} \theta} \right] \]
\[ \phi_z = \arctg \left[ \frac{n \sqrt{\beta_{z,0}} z_0' \cos \frac{n}{2} \theta - C_2 \sqrt{\beta_{x,0}} x_0' \sin \frac{n}{2} \theta}{(C_1 \sqrt{\beta_{x,0}} x_0' - \Delta \sqrt{\beta_{z,0}} z_0') \sin \frac{n}{2} \theta} \right] \]

Exactly the same oscillating terms appear with opposed signs in the expressions of the envelope components (4.8). This is nothing more than the so-called interchanged energy and hence there is no intrinsic difference between the signals of one plane and of the other. The signals have the following form in agreement with (4.8),

\[ A^2_y = d_1 + a_1 \sin^2 \frac{n}{2} \theta + b_1 \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta, \]

(4.9)
and this function has minima and maxima for

\[ \eta \theta = n \pi - \arctg \left( \frac{b_1}{a_1} \right). \]  

(4.10)

It has been shown in section 4.1 that the oscillation envelopes of a horizontally kicked beam have minima and maxima for \( \eta \theta = n \pi \). Hence, the oscillation envelopes following an inclined kick are shifted with respect to the envelopes following a horizontal kick by

\[ \psi = 2\pi \frac{\delta t}{T} = \arctg \frac{b_1}{a_1} \]  

(4.11)

where

\begin{align*}
\frac{b_1}{a_1} & = \pm \frac{2 \eta C_2}{R} x_0' z_0' \sqrt{\frac{\beta_{x,0}}{\beta_{z,0}}} \\
& = \frac{1}{R} \left[ |C|^2 (x_0'^2 \beta_{x,0} - z_0'^2 \beta_{z,0}) - 2 C_1 \Delta x_0' z_0' \sqrt{\frac{\beta_{x,0}}{\beta_{z,0}}} \right],
\end{align*}

the sign depending upon the plane considered (4.8). \( \delta t \) is the time shift of the envelope and \( T \) is its period (3.12). The shift (4.11) gives a relation between \( C_1, C_2 \) and \( |C|^2 \) and gives directly the ratio \( \eta C_2 / C_1 \Delta \) in the special case where \( x_0'^2 \beta_{x,0} = z_0'^2 \beta_{z,0} \). It is important to note that this shift not only depends upon the modulus \( |C| \), but also upon the phase of \( C \) in the imaginary plane.

4.3. **Beam oscillations following two transverse kicks**

In this section, the coherent oscillations of a pulse which is excited by two separate kickers will be described. Instead of considering an inclined kick as in section 4.2, it may be interesting for practical reasons to consider a horizontal kicker followed by a vertical one. The boundary conditions associated with these kickers are at the position of the second kicker.

\begin{align*}
z_0 & \neq 0 \quad z_0' & \neq 0 \\
x_0 & \neq 0 \quad x_0' & \neq 0.
\end{align*}

(4.12)

\( z_0' \) is the vertical kick given by the second kiever, whereas \( x_0 \) and \( x_0' \) are the calculable horizontal coordinates at the second kicker position due to a horizontal kick \( \bar{x}_0 \) given by the first kicker. It is assumed that the kickers are close together and no appreciable coupling has occurred by the time the beam reaches the second kicker.

Using these boundary conditions in (3.7) and (3.9) gives the envelopes of the coherent oscillations. Assuming once more that \( R/\beta_{y,0} >> |C| \), the expression for \( A_2 \) becomes,
\[ A_z^2 = \frac{\beta x_0 \beta z_0}{R^2} \left[ z_0^{1/2} \frac{\beta x_0}{R^2} \frac{n^2 |C|^2}{z_0^{1/2}} + x_0^{1/2} \frac{\beta z_0}{4R^2} \frac{n^2 |C|^2}{x_0^{1/2}} \frac{x_0^{1/2}}{x_0} \frac{C_0^2 n^2 \beta z_0}{2R} \right] + x_0 x_0^{1/2} \frac{\sqrt{R}}{x_0^{1/2}} \frac{C_1 \eta^2}{x_0^{1/2}} \right]

+ \frac{\beta x_0 \beta z_0}{R^2} \left[ z_0^{1/2} \frac{\beta x_0}{x_0^{1/2}} \frac{R^2 (1 + \beta^2 x_0^{1/2})}{\beta x_0^{1/2}} |C|^2 + x_0 x_0^{1/2} \frac{\beta x_0^{1/2}}{x_0} |C|^2 + 2x_0 x_0^{1/2} \frac{R y_0}{\beta x_0^{1/2}} \frac{C_1 \Delta}{x_0} \right. \sin^2 \frac{n}{2} \theta 

- 2x_0 x_0^{1/2} \frac{R(y_0 \beta x_0^{1/2} - C_2 \Delta)}{\beta x_0^{1/2}} \sin \frac{n}{2} \theta \cos \frac{n}{2} \theta \]

\[ (4.13) \]

The vertical envelope has again the form given in (4.9) and the positions of its minima and its maxima are also given by (4.10). Hence, the vertical oscillation envelope following two separate kicks is shifted with respect to the envelope following a horizontal kick by \( \psi \), given in (4.11). Note that the coefficients \( a_i \) and \( b_i \) are different here from those written in (4.11) and are explicitly given in (4.13).

The consequent time shift \( \delta_t = \frac{T \psi}{2 \pi} \) depends on the modulus \( |C| \) and on the phase of \( C \) in the imaginary plane, as in section 4.2.

4.4. Analysis of coherent oscillations and coupling measurement

Two methods for measuring the complex coupling coefficient for the second order difference resonance have already been described in some detail\(^7\). One of them is based on the relations developed in this report. We do not want to repeat here the detailed analysis of the signals and of the measuring performance, but only summarize the most important equations which are necessary in order to deduce \( \Delta, |C|, C_1 \) and \( C_2 \) from the measurements done on the coherent oscillation of a kicked pulse. The basic equations for this purpose are in (3.12), (4.5) and (4.11).

In agreement with (3.12) and with the second relation (4.5), the coherent oscillation of a horizontally kicked beam gives the moduli of \( \Delta \) and \( C \),

\[ |\Delta| = \frac{S}{T \text{rev}} \]

\[ |C| = \frac{1}{T \text{rev}} \sqrt{1 - S^2} \]

\[ (4.14) \]
is the period of the coherent oscillation envelope,

\( f_{\text{rev}} \) is the revolution frequency,

\( S \) is the ratio of the minimum to the maximum of the horizontal envelope.

In order to get some information about the phase of \( C \), it is necessary to observe the coherent oscillation following an inclined kick (section 4.2) or two transverse kicks (section 4.3). In agreement with (4.11), the measurement of the time shift \( \delta_t \) of the envelope gives,

\[
\delta_t = a_1 \tan 2\pi \frac{t}{T}.
\]  

(4.15)

The coefficients \( b_1 \) and \( a_1 \) are given in (4.11) for an inclined kick and can be deduced from (4.13) for two transverse kicks. If we neglect in (4.13) the small term \( z_0^2 C_1 C_2/n/R \), the coefficients \( a_1 \) and \( b_1 \) are linear functions of \( C_1 \) and \( C_2 \) in both of the cases, so that (4.15) takes the following form,

\[
a_2 C_1 + b_2 C_2 = d_2 |C|^2,
\]

(4.16)

with

\[
C_1^2 + C_2^2 = |C|^2,
\]

\( |C| \) being calculated from (4.14).

Solving these two equations gives \( C_1 \) and \( C_2 \). The simplest solution \(^4\) is obtained for an inclined kick satisfying,

\[
\frac{x_0'}{\sqrt{E}} = \frac{z_0'}{\sqrt{E} x_0},
\]

\( E \) being the energy.

i.e.

\[
\frac{C_2}{C_1} = \pm S \tan 2\pi \frac{\delta_t}{T},
\]

\( S \) being the ratio defined in (4.5) for a horizontal kick.

5. CHANGE IN BEAM DIMENSIONS AND LUMINOSITY REDUCTION

5.1. Change in beam size and luminosity for an unshaved beam

The solutions of the coupled motions are given in (3.5). The beam sizes in the presence of coupling are determined by the average of \( x^2 \) and \( z^2 \). At first, the time-average, which is equivalent to an average over \( \theta \), gives,
\[ \bar{x}^2 = \frac{\beta_x}{2\eta^2 G^2} \left( a_+^2 + b_+^2 + a_-^2 + b_-^2 \right) \]
\[ \bar{z}^2 = \frac{\beta_z}{2\eta^2 G^2} \left( f_+^2 + g_+^2 + f_-^2 + g_-^2 \right), \]

\( \eta \) and \( G \) are defined in (3.4) and (3.6), respectively.

The next average has to be done over the particles in the beam,
\[ \langle x^2 \rangle = \frac{\beta_x}{2\eta^2 G^2} \left( \langle a_+^2 \rangle + \langle b_+^2 \rangle + \langle a_-^2 \rangle + \langle b_-^2 \rangle \right) \]
\[ \langle z^2 \rangle = \frac{\beta_z}{2\eta^2 G^2} \left( \langle f_+^2 \rangle + \langle g_+^2 \rangle + \langle f_-^2 \rangle + \langle g_-^2 \rangle \right). \]

(5.2)

The expressions of \( a_\pm, b_\pm, f_\pm \) and \( g_\pm \) in terms of the normalised variables (3.8) are given in (3.7). For a matched beam, the averages satisfy,
\[ \langle x^2 \rangle = \langle y^2 \rangle = \frac{\langle y^2 \rangle}{\beta_y}, \]

(5.3)

and the cross terms like \( \langle xy \rangle \) are zero. These relations (5.3) have to be introduced into the squares of the coefficients (3.7). Keeping the dominant terms by using the inequalities \( R/\beta_y \gg |C| \), the averages over the particles are,
\[ \langle a_\pm^2 \rangle = \langle b_\pm^2 \rangle = \frac{R^4}{\beta_x \beta_y \beta_z \omega} \left( \frac{|C|^4}{\beta_x \beta_y \beta_z} \langle x_0^2 \rangle + 4 \frac{|C|^2}{\beta_z} \langle z_0^2 \rangle \right) \]
\[ \langle f_\pm^2 \rangle = \langle g_\pm^2 \rangle = \frac{R^4}{\beta_x \beta_y \beta_z \omega} \left( 4 \frac{|C|^2}{\beta_x \beta_y \beta_z} \langle x_0^2 \rangle + \frac{|C|^4}{\beta_z} \langle z_0^2 \rangle \right), \]

(5.4)

where \( \omega_\pm \) is the parameter defined in (2.4).

The forms of the averages (5.4) are independent of the transverse distributions of the particles in the beam. Hence, no assumptions about the beam shape have been made so far.

The next algebraic step consists of introducing (5.4) into (5.2), taking into account that,
\[ \frac{|C|^2}{\eta^2} \left( \frac{1}{\omega_+^2} + \frac{1}{\omega_-^2} \right) = 16 \left( 1 - \frac{|C|^2}{2\eta^2} \right). \]

(5.5)
The result of this operation is,

\[
\langle x^2 \rangle = \beta_x \left[ \frac{1 - |C|^2}{\Delta^2} \frac{\langle x^2 \rangle}{\beta_x,0} + \frac{|C|^2 \langle x_0^2 \rangle}{\Delta^2} \right]
\]

\[
\langle z^2 \rangle = \beta_z \left[ \frac{|C|^2 \langle x_0^2 \rangle}{\Delta^2} \frac{\langle x_0^2 \rangle}{\beta_z,0} + \left( 1 - \frac{|C|^2}{\Delta^2} \right) \frac{\langle z_0^2 \rangle}{\beta_z,0} \right].
\] (5.6)

It is possible to re-write the results (5.6) by using the following definition of the r.m.s. value,

\[
\sigma_y = \sqrt{\langle y^2 \rangle},
\] (5.7)

and the corresponding definition of the emittance,

\[
E_y = 4\pi \frac{\sigma_y}{\beta_y}.
\] (5.8)

For reasons of convenience, let us introduce the ratio of the initial uncoupled emittances,

\[
r = \frac{E_{x,0}}{E_{z,0}}.
\] (5.9)

Putting (3.4), (5.7), (5.8) and (5.9) into (5.6) gives,

\[
\frac{\sigma_x}{\sigma_{x,0}} = \sqrt{\frac{\beta_x}{\beta_{x,0}}} \sqrt{1 + \frac{1}{2} (1 - \frac{r}{r - 1}) \frac{|C|^2}{\Delta^2 + |C|^2}}
\]

\[
\frac{\sigma_z}{\sigma_{z,0}} = \sqrt{\frac{\beta_z}{\beta_{z,0}}} \sqrt{1 + \frac{1}{2} (r - 1) \frac{|C|^2}{\Delta^2 + |C|^2}}
\] (5.10)

Since the effective beam-width and the effective beam-height are proportional to the r.m.s. values \(\sigma_x\) and \(\sigma_z\) respectively, the expressions (5.10) give directly the changes in the effective dimensions for two identical beams in head-on collision. In these expressions \(\sigma_z\) and \(\beta_z\) as well as \(\sigma_{x,0}\) and \(\beta_{x,0}\) have to be taken at the same azimuthal position and for effective dimensions the values at the beam intersection must be used.

The two basic expressions for the luminosity with zero and non-zero colliding angle respectively are\(^3\),

\[
L = (\beta_1 + \beta_2) \frac{I_1 I_2 d}{e^2 \sigma_1 \sigma_2 \text{w}_{\text{eff}} \text{h}_{\text{eff}}}
\]

\[
L = \sqrt{\beta_1^2 + \beta_2^2 + 2\beta_1 \beta_2 \cos \psi - \beta_1^2 \beta_2^2 \sin^2 \psi} \frac{I_1 I_2}{e^2 \sigma_1 \sigma_2 \sin \psi \text{w}_{\text{eff}} \text{h}_{\text{eff}}}
\] (5.11)
\( B_1, B_2 \) are the relative speeds of the two beams,

\( I_1, I_2 \) are the average beam currents,

d is the interaction length,

\( \psi \) is the collision angle,

\( w_{\text{eff}} \) is the effective width of two steered beams,

\( h_{\text{eff}} \) is the effective height of two steered beams.

The relative luminosity reduction due to coupling is easy to deduce from (5.10) and (5.11) for zero and non-zero colliding angle respectively,

\[
\text{Zero angle: } \quad \frac{L - L_0}{L} = 1 - \sqrt{1 + \frac{1}{2} \frac{|C|^2}{\Delta^2 + |C|^2} \left( 1 - \frac{1}{2} \frac{|C|^2}{\Delta^2 + |C|^2} \right) \left( r + \frac{1}{r} - 2 \right)}
\]

\[
\text{Non-zero angle: } \quad \frac{L - L_0}{L} = 1 - \sqrt{1 + \frac{1}{2} (r - 1) \frac{|C|^2}{\Delta^2 + |C|^2}},
\]

(5.12)

C, \( \Delta \) and r are defined in (2.1), (2.2) and (5.9), respectively.

In a previous work done by H.G. Hereward\textsuperscript{6}) and briefly summarised in Appendix 3, an expression for the vertical size increase was derived as a function of a coupling coefficient c, which is different from the C given in (2.1). This expression and the second relation (5.10) are not identical but can be compared provided it is assumed that,

\[
|c| = \frac{1}{2} \frac{|C|}{\sqrt{\Delta^2 + |C|^2}}.
\]

(5.13)

Both expressions have a term in \( |c|^2 \), but only the expression given in Appendix 3 contains a term in \( |c|^4 \). This difference is negligible when c is small, i.e. if \( \Delta \) is large compared with \( |C| \). However, this difference becomes more important when \( \Delta \) is of the same order of magnitude as \( |C| \). In this specific case, (5.10) gives a beam height increase 6 to 15 % larger than the formula in Appendix 3.

The paper of A.W. Chao and M. Month\textsuperscript{2}) is the only other work known to the author dealing with the effect of coupling on the beam distribution. This reference does not give the change in the sizes of the beam, but the time evolution of beam distribution profiles, assuming that the unperturbed initial beam distribution is gaussian. Nevertheless, starting with the distribution (43) of reference 2, it is possible to average on the time and to obtain an expression for the change in the vertical beam dimension for instance. This development has been done and the resulting expression is of a basically different form from that given in (5.10). In this expression the ratio \( \sigma_z /\sigma_{z,0} \) can never be greater than \( \sqrt{2} \) whatever the values of \( |C| \) and r (5.9). This limit on the vertical blow-up is very surprising and it
is felt that the final distribution (43) in reference 2 is based on a false
assumption. The new distribution is obtained by replacing the variables in the
initial distribution directly by their expressions in terms of the new variables.
The new distribution should in fact be found by using a convolution product of
the type described in the next section. It is perhaps interesting to note again
that it is not necessary to know the analytical form of the distribution, when
the beam is not shaved.

5.2. Change in beam size and luminosity for a vertically shaved beam

The normalised coordinates \((u, u^*)\) in the horizontal plane and \((v, v^*)\) in
the vertical plane will be used throughout this section, their definitions being
given in (3.8). Since the beam is shaved vertically in the ISR and the crossing
angle is horizontal, only the vertical motion and the change in the beam height
will be considered.

By virtue of (3.5), (3.7) and (3.8), the vertical motion in the normalised
coordinates can be written as follows,

\[
v = d \, u_0 + e \, u^*_0 + p \, v_0 + q \, v^*_0,
\]
where the coefficients are \(\theta\)-functions.

\[
d = \frac{C_1}{2n} \left[ \cos (u_z - \omega \theta) - \cos (u_z - \omega \theta) \right] + \frac{C_2}{2n} \left[ \sin (u_z - \omega \theta) - \sin (u_z - \omega \theta) \right]
\]

\[
e = \frac{C_2}{2n} \left[ \cos (u_z - \omega \theta) - \cos (u_z - \omega \theta) \right] - \frac{C_1}{2n} \left[ \sin (u_z - \omega \theta) - \sin (u_z - \omega \theta) \right]
\]

\[
p = \frac{1}{n} \left[ \omega_+ \cos (u_z - \omega \theta) - \omega_- \cos (u_z - \omega \theta) \right]
\]

\[
q = \frac{1}{n} \left[ \omega_- \sin (u_z - \omega \theta) - \omega_+ \sin (u_z - \omega \theta) \right].
\]

The definitions of the parameters involved are given in chapters 2 and 3.

As soon as beam shaving is considered, it is necessary to assume some trans-
verse distribution of the particles, whereas this was not necessary for unshaved
beams. Assuming transverse Gaussian distributions of the particles, the initial
beam density can be written,

\[
\rho_0(u_0, u^*_0, v_0, v^*_0) = \frac{1}{4\pi^2 \sigma_{u_0} \sigma_{u^*_0} \sigma_{v_0} \sigma_{v^*_0}} \exp \left[ -\frac{1}{2} \left( \frac{u_0^2}{\sigma_{u_0}^2} + \frac{v_0^2}{\sigma_{v_0}^2} + \frac{u^*_0^2}{\sigma_{u^*_0}^2} + \frac{v^*_0^2}{\sigma_{v^*_0}^2} \right) \right].
\]
In order to estimate the effective height or the vertical r.m.s. value in the presence of the coupling, it is necessary to know the new vertical beam density. This density is given by a convolution product of the initial densities, in which the relation (5.14) is used. Taking into account that the r.m.s. values of \( u_0^* \) and \( v_0^* \) and \( u_0^* \) and \( v_0^* \) respectively, this density can be written,

\[
\rho(v) = \frac{1}{4\pi^2 \sigma_{u_0}^2 \sigma_{v_0}^2} \int_{\Omega_0} \exp \left\{ -\frac{1}{2} \left[ \frac{u_0^2 + u_0^* 2}{\sigma_{u_0}^2} + \frac{v_0^2}{\sigma_{v_0}^2} + \frac{1}{q} \frac{v}{\sigma_{v_0}^2} - (v - du_0^* - pv_0^* - eu_0^* \sigma_{v_0}^2) \right] \right\} \, du_0^* dv_0^* du_0^* dv_0^*. \tag{5.17}
\]

The volume of integration \( \Omega_0 \) depends upon the shaver position. When there is no shaving, the volume \( \Omega_0 \) becomes infinite and the integral (5.17) can be solved. The integration is rather long so only the final result is given,

\[
\rho(v) = N \exp \left( -\frac{1}{2} \frac{v^2}{\sigma_v^2} \right), \tag{5.18}
\]

where \( N \) is a normalisation factor and

\[
\sigma_v^2 = \frac{d^2 \sigma_{u_0}^2 + e^2 \sigma_{v_0}^2 + p^2 \sigma_{v_0}^2 + q^2 \sigma_{v_0}^2}{v_v^2}. \tag{5.19}
\]

Using the expressions (5.15) and averaging them over \( \theta \), the second relation (5.10) is obtained, as would be expected since this is a particular case of the problem treated in section 5.1.

As shown in section 5.1, it is necessary to average the vertical amplitude of the motion \( z \) (or the normalised quantity \( v \)) over time, i.e. over \( \theta \), and over the particles inside the beam, in order to estimate the square of the standard deviation \( \sigma_z^2 = \langle z^2 \rangle \) (or the square of the normalised standard deviation \( \sigma_v^2 \)). Furthermore, in the specific case of beam shaving, it is also necessary to normalise \( \sigma_z^2 \) or \( \sigma_v^2 \) with respect to the remaining fraction \( F_{s,z} \) of the shaved beam.

This problem can be formulated analytically as follows,

\[
F_{s,z} = \frac{n}{4\pi} \int_0^{4\pi/\eta} \frac{d\theta}{q} \int_{\Omega} \rho(v) \, dv_0^* dv_0^* dv_0^* dv_0^*. \tag{5.20}
\]

\[
\sigma_v^2 = \frac{n}{4\pi F_{s,z}} \int_0^{4\pi/\eta} \frac{d\theta}{q} \int_{\Omega} v^2 \rho(v) \, dv_0^* dv_0^* dv_0^* dv_0^*. \]
Hence, we have to solve two integrals of the fifth order. The next problem consists of defining the volume of integration $\Omega$. The shaver will remove the particles which at any time exceed the vertical position of the shaver blade $z_s$ if we assume that the coupling was present before the shaving started. Thus, the condition for a particle to remain in the beam can be written,

$$\bar{z} \leq z_s \sqrt{\frac{E}{z_s}},$$

(5.21)

where $\bar{z}$ is the maximum vertical position reached by a particle and $z_s$ is the vertical betatron amplitude at the shaver. Using the factorization (3.9), the condition (5.21) becomes,

$$\frac{\hat{z}}{\bar{z}} \leq \frac{z_s}{\sqrt{\beta_z}},$$

(5.22)

where $\bar{z}$ is the normalised position of the shaver blade and $\hat{z}$ is the maximum of the slowly oscillating envelope. This envelope can be expressed in terms of the normalised initial coordinates by using the relations (3.7), (3.8) and (3.11),

$$\frac{\bar{z}}{\bar{z}} = v_0^2 + v_0^2 + \frac{2A}{\bar{z}} \sin^2 \frac{n}{\bar{z}} \theta + 2B \sin \frac{n}{\bar{z}} \theta \cos \frac{n}{\bar{z}} \theta,$$

(5.23)

where,

$$A = \frac{C_1}{\bar{z}} \left( u_0^2 + u_0^2 - v_0^2 - v_0^2 \right) - \frac{C_1}{\bar{z}} \left( u_0 v_0 + u_0 v_0 \right) + \frac{C_2}{\bar{z}} \left( u_0 v_0 - u_0 v_0 \right),$$

$$B = -\frac{C_2}{\bar{z}} \left( u_0 v_0 + u_0 v_0 \right) - \frac{C_1}{\bar{z}} \left( u_0 v_0 - u_0 v_0 \right).$$

We have to look for the maxima of (5.23). We know already from (4.10) that these maxima occur at,

$$n \theta = n \pi - \arctg \frac{B}{A}.$$  (5.24)

Putting (5.24) into (5.23), it is easy to show that, whatever the signs of $A$, $B$ and the sinusoidal functions are, the maxima of (5.23) satisfy a unique relation,

$$\left( \frac{\hat{z}}{\bar{z}} \right)^2 = v_0^2 + v_0^2 + A + \sqrt{A^2 + B^2},$$

(5.25)

$A$ and $B$ being defined above.

Under these conditions, it is easy to define the volume of integration $\Omega$, i.e. the initial volume containing the particles which will remain in the beam after shaving, by using the following inequality [(5.22) and (5.25)].
\[ v_0^2 + v_{\theta}^2 + A + \sqrt{A^2 + B^2} \leq v_{\theta}^2. \]  
\[ \text{(5.26)} \]

A and B are given in (5.23). In these expressions, we have to eliminate \( v_{\theta}^2 \) since the integrals (5.20) are not functions of \( v_{\theta}^2 \). This is easily done by using (5.14),

\[ v_{\theta}^2 = \frac{1}{q} (v - d u_0 - c u_0^2 - p v_0). \]  
\[ \text{(5.27)} \]

Since the volume \( \Omega \) (5.26) is independent of \( \theta \), the first integral on \( \theta \) in (5.20), i.e. the averaging over the independent variable, can be done beforehand. Having the form of the coefficients \( d \), \( e \), \( p \) and \( q \) (5.15), this average is easily done,

\[ \bar{d} = \bar{e}^2 = \frac{|C|^2}{4n^2}. \]
\[ \bar{p}^2 = \bar{q}^2 = \frac{1}{2n^2} (\omega_i^2 + \omega_r^2). \]
\[ \bar{d} = 0 \]  
\[ \bar{d} = \bar{e} = \bar{p} = \bar{q} = \frac{C_1 \Delta}{4n^2}. \]  
\[ \text{(5.28)} \]

By virtue of the averages (5.28), the integrals (5.20) become two integrals of the fourth order in a volume \( \Omega \) given by (5.26), (5.23) and (5.27). Because of the complexity of the inequality (5.26), these integrals cannot be solved analytically. It is nevertheless easy to solve them numerically as is explained in Appendix 4 and a computer program HEBUC has been written for that purpose.

The second integral (5.20) gives the vertical beam size \( \sigma_v = \sigma_z / \sqrt{\sigma_z^2} \) in the presence of coupling as a function of the first integral (5.20) which gives the fraction of particles left after shaving. Typical curves giving the remaining fraction of the shaved beam \( F_{s,z} \) as a function of the ratio of the final to the initial beam heights \( \sigma_z / \sigma_{z,0} \) are represented in figs. 1 and 2 for different emittance ratios (5.9) and different values of \( F \) (4.5).

Again comparing the results obtained with the integrals (5.20) and the unpublished results of H.G. Hereward in the presence of shaving, the remark made in section 5.1 is still valid. Both results are close, but not identical, if the equivalence (5.13) is assumed to be true and the difference can reach 15% if \( |c| \) is large enough.
The curves of figs. 1 and 2 directly give the relative change in effective height for two identical beams in head-on collision in the presence of coupling and vertical shaving. The second relation (5.11) then gives the loss of luminosity associated with this height change, when the crossing angle is horizontal.

Note that the preceding development can be repeated in the horizontal plane, in order to estimate the change in the horizontal beam size and the luminosity loss for zero crossing angle.

6. CONCLUSIONS

This report treats in some details the coherent oscillations of a kicked beam in the presence of linear coupling. It gives easy-to-use formulae for the relative change of the beam sizes and the luminosity loss in the presence of linear coupling, but in the absence of shaving. In the specific case where the beams are shaved vertically, the vertical beam size change is given in an integral form, which has been solved numerically for different emittance ratios and different coupling strengths.

REFERENCES

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2) A.W. Chao and M. Month, Observables with linearly coupled oscillators, BNL-19329, CRISP 74-16 (1974).


6) H.G. Hereward, Private communication.

7) B. Schorr, Private communication.
Fig. 1 - Vertical blow-up due to linear coupling for different rates of vertical shaving and low emittance ratios.
Fig. 2 - Vertical blow-up due to linear coupling for different rates of vertical shaving and high emittance ratios.
DETERMINATION OF THE COMPLEX AMPLITUDES

The relations between the initial coordinates and the complex amplitudes are written in a matrix form in (3.2), where $T_0$ is a 4 x 4 matrix with the following coefficients,

\[ T_{011} = \frac{C}{2\omega_+} \sqrt{\frac{x,0}{2R}} \]
\[ T_{012} = \frac{C}{2\omega_-} \sqrt{\frac{x,0}{2R}} \]
\[ T_{013} = \frac{C}{2\omega_+} \sqrt{\frac{x,0}{2R}} \]
\[ T_{014} = \frac{C}{2\omega_-} \sqrt{\frac{\beta_{x,0}}{2R}} \]
\[ T_{021} = \frac{C}{2} \left[ i \sqrt{\frac{x,0}{2R}} + \frac{i - \alpha_x}{\omega_+} \sqrt{\frac{R}{2g_{x,0}}} \right] \]
\[ T_{022} = \frac{C}{2} \left[ i \sqrt{\frac{x,0}{2R}} + \frac{i + \alpha_x}{\omega_-} \sqrt{\frac{R}{2g_{x,0}}} \right] \]
\[ T_{023} = -\frac{C}{2} \left[ i \sqrt{\frac{x,0}{2R}} + \frac{i + \alpha_x}{\omega_+} \sqrt{\frac{R}{2g_{x,0}}} \right] \]
\[ T_{024} = -\frac{C}{2} \left[ i \sqrt{\frac{x,0}{2R}} + \frac{i - \alpha_x}{\omega_-} \sqrt{\frac{R}{2g_{x,0}}} \right] \]
\[ T_{031} = T_{032} = T_{033} = T_{034} = \frac{\beta_{z,0}}{2R} \]
\[ T_{041} = -i\omega_- \sqrt{\frac{z,0}{2R}} + (i - \alpha_z) \sqrt{\frac{R}{2g_{z,0}}} \]
\[ T_{042} = -i\omega_+ \sqrt{\frac{z,0}{2R}} + (i - \alpha_z) \sqrt{\frac{R}{2g_{z,0}}} \]
\[ T_{043} = i\omega_- \sqrt{\frac{z,0}{2R}} - (i - \alpha_z) \sqrt{\frac{R}{2g_{z,0}}} \]
\[ T_{044} = i\omega_+ \sqrt{\frac{z,0}{2R}} - (i + \alpha_z) \sqrt{\frac{R}{2g_{z,0}}} , \]

$C$ and $\omega_+$ are defined in Chapter 2.
The determinant of this matrix which has one line with identical terms is,

\[
D = \left( \frac{4B^2}{\bar{\mathcal{E}}_0 \bar{\mathcal{E}}_0 \bar{\mathcal{z}}_0 \bar{\mathcal{z}}_0} - C_1^2 \right) \frac{\bar{B}^2 \bar{\mathcal{E}}_0 \bar{\mathcal{z}}_0 \bar{\mathcal{z}}_0 \eta^3}{4R^4 \omega_0 |C|^2}
\]  

(Al.2)

where \( \eta \) is given in (3.4).

Using the properties of the matrix \( T_0 \), the equations (3.2) can be combined linearly in order to eliminate one parameter, i.e. \( A_4 \). Slightly changing the notation in using \( A_1, A_2, A_3 \) and \( A_4 \) instead of \( A_-, A_-, \bar{A}_-, \bar{A}_- \), the three remaining equations are,

\[
\sum_{k=2}^{4} (T_{021}T_{01k} - T_{011}T_{02k})A_k = T_{021}x_0 - T_{011}x_0'
\]

(Al.3)

\[
T_{031} \sum_{k=2}^{4} (T_{041} - T_{04k})A_k = T_{041}z_0 - T_{031}z_0'
\]

\[
T_{031} \sum_{k=2}^{4} (T_{01k} - T_{011})A_k = T_{031}x_0 - T_{011}z_0
\]

Repeating the same operation on the equations (Al.3), \( A_2 = A_- \) can be eliminated,

\[
\sum_{k=3}^{4} a_k A_k = y_1
\]

(Al.4)

\[
\sum_{k=3}^{4} b_k A_k = y_2
\]

with

\[
a_k = T_{031} \left[ (T_{021}T_{012} - T_{011}T_{022})(T_{01k} - T_{011}) + (T_{011} - T_{012})(T_{021}T_{01k} - T_{011}T_{02k}) \right]
\]

\[
b_k = T_{031} \left[ (T_{041} - T_{042})(T_{01k} - T_{011}) + (T_{011} - T_{012})(T_{041} - T_{04k}) \right]
\]

\[
y_1 = (T_{021}x_0 - T_{011}x_0')T_{031}(T_{011} - T_{012}) + (T_{031}x_0 - T_{011}z_0)(T_{021}T_{012} - T_{011}T_{022})
\]

\[
y_2 = T_{031} \left[ (T_{041} - T_{042})(T_{031}x_0 - T_{011}z_0) + (T_{011} - T_{012})(T_{041}z_0 - T_{031}z_0') \right].
\]
The solution of (A1.4) can easily be written explicitly,

\[
A_3 = \overline{A}_+ = -\frac{1}{D} (b_4 y_1 - a_4 y_2)
\]

\[
A_4 = \overline{A}_- = \frac{1}{D} (b_3 y_1 - a_3 y_2),
\]

where D is the determinant given in (A1.2).

Replacing in \(a_k, b_k, y_1 \) and \( y_2 \) (A1.4) the coefficients \( \Gamma_{0;ij} \) by their algebraic form (A1.1), we obtain,

\[
a_3 = i \frac{n}{4R^2\omega_+} \beta^{3/2} \frac{\sqrt{\beta}}{z_{x,0}} \left( C_1 + \frac{RC}{\omega \beta x_{x,0}} \right)
\]

\[
a_4 = i \frac{n}{4R^2\omega_+} \beta^{3/2} \frac{\sqrt{\beta}}{z_{x,0}} \left( C_1 + \frac{RC}{\omega \beta x_{x,0}} \right)
\]

\[
b_3 = i \frac{n}{4R^2\omega_+} \beta^{3/2} \frac{\sqrt{\beta}}{z_{x,0}} \left( C_1 - \frac{RC}{\omega \beta z_{x,0}} \right)
\]

\[
b_4 = i \frac{n}{4R^2\omega_+} \beta^{3/2} \frac{\sqrt{\beta}}{z_{x,0}} \left( C_1 - \frac{RC}{\omega \beta z_{x,0}} \right)
\]

\[
y_1 = i n \left( \frac{\beta z_{x,0}}{2R} \right) \frac{3/2}{2} x_0 - 2 \frac{n}{C_n} \frac{\beta z_{x,0} \sqrt{\beta}}{x_{x,0}} \left[ \frac{R}{\beta z_{x,0}} (a_{z,0} - i) z_0 + z'_0 \right]
\]

\[
y_2 = -\frac{n}{C_n} \frac{\beta x_{x,0} \sqrt{\beta}}{2R} \frac{3/2}{2} \frac{1}{\omega_+ |C|^2} z_0.
\]

The introduction of the expressions (A1.2) and (A1.6) into the solution (A1.5) gives the explicit form of \( \overline{A}_+ \) and \( \overline{A}_- \), as functions of the initial coordinates \( x_0, x'_0, z_0 \) and \( z'_0 \). Taking finally the complex conjugate of the expressions obtained gives the result in (3.3).
APPENDIX 2

FACTORIZATION OF THE TRANSVERSE SIGNALS

The solutions of the motion equations in the presence of linear coupling are written in (3.5). Only the horizontal motion will be considered here, since the solutions (3.5) are very similar in both planes. Using the explicit form (2.4) of \( \omega_x \), the horizontal motion solution becomes,

\[
x = \frac{\sqrt{\eta} x}{n_0} \left[ a_+ \cos \left( \frac{\omega_x - \Delta \theta + n_0 \theta}{2} \right) - b_+ \sin \left( \frac{\omega_x - \Delta \theta + n_0 \theta}{2} \right)
- a_- \cos \left( \frac{\omega_x - \Delta \theta - n_0 \theta}{2} \right) + b_- \sin \left( \frac{\omega_x - \Delta \theta - n_0 \theta}{2} \right) \right],
\] (A2.1)

\( \eta \) is given in (3.4) and \( G \) in (3.6). \( \Delta \) is the distance from the resonance.

The coupling coefficient \( C \) appears in \( \eta \) (3.4) and since \( \eta \) is usually small, \( \eta \theta/2 \) is a slowly varying function. For these two reasons, it is interesting to factorize the signal (A2.1) into a slowly oscillating component depending on \( C \) and a component independent of \( C \), which is a fast oscillating component since \( \omega_x \) is a rapidly varying function. For this purpose, it is necessary to use the well-known formulae of angle additions,

\[
\begin{align*}
\cos \left( \frac{\omega_x - \Delta \theta \pm \frac{n_0}{2} \theta}{2} \right) &= \cos \left( \frac{\omega_x - \Delta \theta}{2} \right) \cos \frac{n_0}{2} \theta \pm \sin \left( \frac{\omega_x - \Delta \theta}{2} \right) \sin \frac{n_0}{2} \theta, \\
\sin \left( \frac{\omega_x - \Delta \theta \pm \frac{n_0}{2} \theta}{2} \right) &= \sin \left( \frac{\omega_x - \Delta \theta}{2} \right) \cos \frac{n_0}{2} \theta \pm \cos \left( \frac{\omega_x - \Delta \theta}{2} \right) \sin \frac{n_0}{2} \theta.
\end{align*}
\] (A2.2)

Introducing (A2.2) into (A2.1) and re-grouping the terms gives for the horizontal motion,

\[
x = \frac{\sqrt{\eta} x}{n_0} \left\{ \left[ (b_- - b_+) \cos \frac{n_0}{2} \theta - (a_+ + a_-) \sin \frac{n_0}{2} \theta \right] \sin \left( \frac{\omega_x - \Delta \theta}{2} \right)
+ \left[ (a_+ - a_-) \cos \frac{n_0}{2} \theta - (b_- + b_+) \sin \frac{n_0}{2} \theta \right] \cos \left( \frac{\omega_x - \Delta \theta}{2} \right) \right\}.
\] (A2.3)

In order to get the expression for the modulated envelope, it is necessary to finish with a product of two sinusoidal functions with different frequencies. This can easily be done by applying to (A2.3) the well-known formula,

\[
d \cos \psi + h \sin \psi = \sqrt{d^2 + h^2} \cos \left( \psi - \arctan \frac{h}{d} \right).
\] (A2.4)

Comparing and identifying the contents of the largest brackets in (A2.3) with the left hand side of (A2.4), gives the relations,
\[
\psi = \nu_x - \frac{\Delta}{2} \theta \\
\tan h = (b_- - b_+) \cos \frac{\pi}{2} \theta - (a_+ + a_-) \sin \frac{\pi}{2} \theta \\
d = (a_+ - a_-) \cos \frac{\pi}{2} \theta - (b_- + b_+) \sin \frac{\pi}{2} \theta,
\]
which give consequently,
\[
\sqrt{d^2 + h^2} = \sqrt{(a_+ - a_-)^2 + (b_- - b_+)^2 + 4(a_+ a_+ + b_+ b_-) \sin^2 \frac{\pi}{2} \theta + 4(a_- b_- - a_+ b_+) \sin \frac{\pi}{2} \theta \cos \frac{\pi}{2} \theta}
\]
\[
\frac{h}{d} = \frac{(a_+ + a_-) \sin \frac{\pi}{2} \theta - (b_- - b_+) \cos \frac{\pi}{2} \theta}{(b_- + b_+) \sin \frac{\pi}{2} \theta - (a_+ - a_-) \cos \frac{\pi}{2} \theta}.
\]
These two relations give directly the slowly oscillating envelopes within the factor $G$ and the tangent of the phase shift of the fast oscillating component resulting from the coupling. These results are equally valid for the vertical motion, the only difference being the sign of the $\Delta$-contribution to the fast oscillation.
NORMAL MODES AND LUMINOSITY LOSS\textsuperscript{6)}

Let us suppose a particle enters the machine with betatron coordinates $x_0, x'_0, z_0$ and $z'_0$. It is convenient to assume that $x$ and $z$ are referred to the $\beta_z$ of the intersections and that $x'$ and $z'$ are in such units that both $\beta$-values are 1.

Assuming a coupling which is constant in time and independent of the azimuth, there may be two normal modes which are predominantly vertical and horizontal, respectively. The complex amplitudes of these normal modes will be noted $v$ and $u$, respectively. In this case, the transverse motions can be written,

$$
\begin{align*}
-x + ix' &= v e^{-i\omega t} + u e^{-i\omega t} \\
z + iz' &= v e^{-i\omega t} - u e^{-i\omega t}.
\end{align*}
\tag{A3.1}
$$

$c$ is a complex coupling ratio, small compared with 1, and $c'$ is introduced only to give a convenient normalisation. $c'$ is a little smaller than 1 and satisfies,

$$
|c'|^2 + |c|^2 = 1. \tag{A3.2}
$$

The normal modes can be written in terms of the initial betatron coordinates, by putting $t = 0$ in (A3.1). This gives,

$$
\begin{align*}
u &= c'(x_0 + ix'_0) - c(z_0 + iz'_0) \\
v &= c'(x_0 + ix'_0) + c'(z_0 + iz'_0). \tag{A3.3}
\end{align*}
$$

The incoming beam is assumed to be matched, so that,

$$
\begin{align*}
\langle x_0^2 \rangle &= \langle x_0' \rangle, \quad \langle x_0 x'_0 \rangle = 0, \\
\langle z_0^2 \rangle &= \langle z_0' \rangle, \quad \langle z_0 z'_0 \rangle = 0. \tag{A3.4}
\end{align*}
$$

and it is also assumed not to show the consequences of any previous coupling, so that all the cross, second order moments such as $\langle x_0 z_0 \rangle$ are zero.

Then, averaging (A3.3) over the beam gives,

$$
\begin{align*}
\langle |u|^2 \rangle &= 2|c'|^2\langle x_0^2 \rangle + 2|c|^2\langle z_0^2 \rangle \\
\langle |v|^2 \rangle &= 2|c|^2\langle x_0^2 \rangle + 2|c'|^2\langle z_0^2 \rangle. \tag{A3.5}
\end{align*}
$$
This gives the circulating beam emittances in terms of the machine normal modes. It is now necessary to find the beam height. Averaging over time the vertical amplitude given in (A3.1) gives,

\[ \overline{z^2} = \frac{1}{2} |v|^2 |c^2 |^2 + \frac{1}{2} |u|^2 |c^2| . \]  

(A3.6)

Averaging (A3.6) over the beam and using the relations (A3.5), we get,

\[ \langle z^2 \rangle = \langle z_0^2 \rangle (|c|^2 + |c^4|^2) + \langle z_0^2 \rangle (2 |c|^2 |c^4|^2). \]  

(A3.7)

Defining an emittance ratio \( r = \langle z_0^2 \rangle / \langle z_0^2 \rangle \), equivalent to the one introduced in (5.9), and taking into account the normalisation relation (A3.2), the mean square of the beam height in the presence of coupling becomes,

\[ \langle z^2 \rangle = \langle z_0^2 \rangle [1 + 2(r - 1) |c|^2 - 2(r - 1) |c|^4], \]  

(A3.8)

and this is the relation we have to compare with the square of the second equation (5.10).

This theory can also be extended to the case of vertical shaving. It is then necessary to assume an incoming distribution, which may be taken to be gaussian in \( x_0, x_0', z_0, z_0' \). Since (A3.3) is linear, the distribution in real and imaginary parts of \( u \) and \( v \) will also be gaussian. Having the distribution in the normal modes, it is necessary to determine which part of the beam is removed by the shaver. This part is given by the condition that all the particles which have at any time an amplitude \( z \) (A3.1) larger than the shaver position \( z_s \) are removed. Integrating the distribution in \( u \) and \( v \) in the volume given by this condition gives the remaining fraction of the beam. Calculating the mean squares \( \langle |u|^2 \rangle \) and \( \langle |v|^2 \rangle \) with the same distribution and the same volume gives the beam height.
NUMERICAL CALCULATION OF THE 5-DIMENSIONAL INTEGRALS

The integrals to be solved are given in (5.20). At first, it is convenient to change the variables by putting

\[ u_0 = \sigma_{u_0} x_1, \quad v_0 = \sigma_{v_0} x_3 \]

\[ u_0' = \sigma_{u_0} x_2, \quad v_0' = \sigma_{v_0} x_4 \]

\[ v = \sigma_{u_0} (dx_1 + ex_2) + \sigma_{v_0} (px_3 + qx_4). \]

(A4.1)

With this notation, the first integral (5.20) becomes,

\[
F_{s,z} = \frac{n}{4\pi} \int_0^{4\pi/n} \int_i^{4\pi} \prod_{i=1}^4 \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) dx_i.
\]

(A4.2)

The volume \( \Omega \) is defined by the inequality (5.26) and can be written as a function of \( x_i \), by using (A4.1). Since \( \Omega \) is independent of \( \theta \), the integral (A4.2) can be simplified,

\[
F_{s,z} = \prod_{i=1}^4 \int_0^{4\pi} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) dx_i.
\]

(A4.3)

The last integral (A4.3) can easily be calculated by a crude Monte-Carlo method. Let us suppose that,

\[
H(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(A4.4)

Let \( (x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, \ldots, x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}) \), \( \ell = 1, 2, \ldots, n \), be a sample from a 4-dimensional uncorrelated standardized normal distribution. We then have for \( F_{s,z} \),

\[
F_{s,z} = \frac{\bar{F}_{s,z}}{n} = \frac{1}{n} \sum_{\ell=1}^n H \left[ \frac{v_0^2}{s^2} - \sigma_{v_0}^2 (x_{3,\ell}^2 + x_{4,\ell}^2) - A_\ell - \sqrt{A_\ell^2 + B_\ell^2} \right]
\]

(A4.5)

where \( A_\ell \) and \( B_\ell \) are the expressions given in (5.23) and written as functions of \( x_{i,\ell} \), by using (A4.1).

Using again the notation (A4.1), the second integral (5.20) becomes,
\[ \sigma_v^2 = \frac{n}{4\pi F_{s,z}} \int_0^{4\pi/n} \prod_{i=1}^4 \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) dx_i. \]  

(A4.6)

Developing the square term, (A4.6) becomes,

\[ \sigma_v^2 = \frac{1}{F_{s,z}} \sum_{j \neq k} a_{jk} \prod_{i=1}^4 x_j^x \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) dx_i, \]  

(A4.7)

where the new coefficients \( a_{jk} \) are the averages over \( \theta \) of the squares or of the cross-products of the parameters \( d, e, p \) and \( q \), multiplied by \( \sigma_{u_0} \) or \( \sigma_{v_0} \). For instance,

\[ a_{11} = \sigma_{u_0}^2 \frac{n}{4\pi} \int_0^{4\pi/n} d^2d\theta. \]  

(A4.8)

\[ a_{13} = \sigma_{u_0} \sigma_{v_0} \frac{n}{4\pi} \int_{j=0}^{4\pi/n} dp d\theta. \]

The coefficients \( a_{jk} \) can directly be deduced from the averages over \( \theta \) given in (5.28). Knowing the forms of \( a_{jk} \), the remaining integrals

\[ S_{jk} = \frac{1}{F_{s,z}} \prod_{i=1}^4 x_j^x \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) dx_i. \]  

(A4.9)

can again be evaluated by a crude Monte-Carlo method, using the function \( H(x) \) (A4.4),

\[ S_{jk} = \frac{1}{n} \sum_{i=1}^n x_{j_i}^x \prod_{i=1}^4 \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x_i^2}{2} \right) \]  

(A4.10)

where \( (x_{j_i}^x), i = 1, 2, \ldots, n, \) is a sample from a 4-dimensional uncorrelated standardized normal distribution. \( A_k \) and \( B_k \) are the expressions (5.23) written as functions of \( x_{j_k}^x \), by using (A4.1).

A Fortran computer program has been written for the 7600 CDC computer in order to calculate numerically the integrals (5.20). It is based on the two relations (A4.5) and (A4.10) and its name is HEBUC.