LONGITUDINAL STABILITY OF STACK DENSITY MODULATIONS

IN THE PRESENCE OF A "LUMPED" IMPEDANCE

by

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Geneva - 30th August, 1970
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The value of a formalism lies not only in the range of problems to which it can be successfully applied, but equally in the degree to which it encourages physical intuition in guessing the solution of intractable problems.

A.N. Pippard
1. **INTRODUCTION**

The study of longitudinal instabilities in synchrotrons and storage rings is well established. The work described below was undertaken initially for educational reasons, so that the author might understand the earlier studies and therefore be in a position to explain to his somewhat perplexed colleagues in the Vacuum Group some of the reasons behind the dictates of the Beam Equipment Interaction Committee.

The results of this work are presented in the form of a note for closed distribution inside the ISR department; this because the author does not fully understand the results but thought that they might be of some limited usefulness to colleagues with more expertise in the field of instabilities.

The model examined below is believed to be applicable to the CERN ISR, and storage rings of that general type, wherein the circulating stacked beams contain already some vestiges of a density modulation perhaps left behind by the bucket-bunched beams of the stacking process.

The problem posed is: does the amplitude of an existing density modulation grow **indefinitely** in the presence of a lumped impedance $Z$ across which the beam current $I$ induces a voltage $-IZ$?

The not unexpected answer to this question is no, not indefinitely, but what was not expected by the author is that under certain circumstances the amplitude may itself be modulated and increases by at most a factor of about two. That is to say that after growing to a certain maximum it then decreases and eventually ends up in a presumably stable but throbbing mode. However, this result is suspect since according to Liouville the local density in phase space cannot increase — thus one has to interpret the result as an indication of what might happen to a particular bunch shape, namely, the latter is likely to be deformed in such a manner that the "bunching factor" increases and
decreases in accord with the increase and decrease of the amplitude of the density modulation.

2. GENERAL FORMALISM

Consider a phase space in canonical coordinates \((w, \theta, t)\) of the usual bucket theory for circular accelerators. Let \(\psi(w, \theta, t)\) be the charge density of particles and therefore Liouville's theorem yields:

\[
\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \psi}{\partial w} \frac{dw}{dt},
\]

(1)

where

\[
w = \int \frac{dE}{\Omega(E)}; \quad \frac{dw}{dt} = \frac{dE}{d\theta} \frac{d\theta}{dt} = \frac{1}{\Omega} \frac{dE}{dt}
\]

(2)

and \(E\) is the energy, \(\Omega(E)\) the revolution frequency, i.e. \(d\theta/dt\), of a particle having this energy.

Let the lumped impedance \(Z\) be situated at \(\theta_1\) on the azimuth such that the beam current \(I(t)\) at \(\theta_1\) produces the voltage

\[
V(t) = -I(t) Z.
\]

(3)

Thus if the particle transit time across this impedance is short compared with both the revolution period and the periodic changes in \(V(t)\), then one may write

\[
\frac{dE}{dt} = \Omega e V(t) \delta(\theta - \theta_1 - 2\pi),
\]

(4)

where \(\delta(x)\) is the Dirac delta function which may be expanded in terms of a Fourier series, i.e.
\[ \delta(\theta - \theta_1 - 2m\pi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos m(\theta - \theta_1). \]  

(5)

The "force" term in equation (1) then becomes

\[ \frac{dW}{dt} = -e I(t) Z \left[ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos m(\theta - \theta_1) \right]. \]  

(6)

Furthermore, and by definition

\[ I(t) = \int_{-\infty}^{\infty} dw \psi(w, \theta_1, t) R\Omega, \]  

(7)

where \( R \) is the mean radius of the particle of energy \( E \), so that \( R\Omega \) is the velocity and a function of \( w \) alone.

A fundamental assumption concerning the distribution function must now be made in order to continue this analysis; this is that we may write,

\[ \psi(w, \theta, t) = \left[ \psi_0(w) + \psi_1(w) e^{i(n\theta - \omega t)} \right]. \]  

(8)

Accordingly, the beam contains both a density modulation in \( \theta \) and \( t \), and a time independent part. The latter does not contribute to the "force" term, thus

\[ I(t) = I_1 e^{i(n\theta_1 - \omega t)}, \]  

(9)

where

\[ I_1 = \int_{-\infty}^{\infty} \psi_1(w) R(w) \Omega(w) \, dw. \]  

(10)
Similarly, define a modulated line charge density $\lambda_1$ as

$$\lambda_1 = \int_{-\infty}^{\infty} \psi_1(w) \, dw.$$  \hspace{1cm} (11)

On insertion of the appropriate functions into equation (1) and integrating over all $w$, at constant $\theta$ and $t$, the following dispersion relation is obtained

$$1 = i \int_{-\infty}^{\infty} \frac{dw}{(w-n\omega)} \frac{e^{i\frac{1}{2}Z_1}}{2\pi \lambda_1} \left[ 1 + \frac{1}{2} \sum_{m=1}^{\infty} \cos(m\theta - \theta_1) \right] F(w, \theta, t)$$  \hspace{1cm} (12)

$$F = \left[ \frac{d\psi_0}{dw} e^{i(n\theta_1 - n\phi)} + \frac{d\psi_1}{dw} e^{i(n\theta_1 - wt)} \right].$$

Many of the terms on the right hand side of equation (12) are rapidly fluctuating in time. To find the important "stationary" terms in this equation, we change to a coordinate system that rotates with the particles at about the velocity of the E-M waves set up by the perturbation: i.e. we let

$$\theta = \theta^* + \frac{wt}{n} + \phi,$$

where $\phi$ is some arbitrary phase angle. $\theta^*$ is then a "stationary" or slowly varying quantity for a particular particle. On multiplying out the various trigonometric terms, one finds the only non-oscillating functions are those for which $m = n$, so that equation (12) becomes

$$1 = i \int_{-\infty}^{\infty} \frac{dw}{(w-n\omega)} \frac{e^{i\frac{1}{2}Z_1}}{2\pi \lambda_1} \left[ \frac{d\psi_0}{dw} + \frac{d\psi_1}{dw} \left( \cos \Theta + i \sin \Theta \right) \right].$$  \hspace{1cm} (13)
where, for convenience, \( n \theta^* + n \phi \) has been put equal to \( \mathbf{M} \); the latter being either stationary or slowly varying.

In the normal theory \( d \psi_1/d\omega \) is a small perturbation and is usually put equal to zero at this stage. Equation (13) would then yield the usual dispersion relations between \( \omega \) and the other parameters.

However, we wish to examine the case of a stack which is highly modulated in density where \( \psi_1 \) may be of the order of \( \psi_0 \); in this case we do not wish to neglect \( d \psi_1/d\omega \) but instead insert some reasonable form for the modulation and then re-examine the dispersion relation with this insertion.

3. DISCUSSION OF \( \psi_1 \) AND \( \psi \)

Assume that the situation in a typical stack is that shown by the diagram below

![Diagram](image)

In this case we may write

\[
\psi(w, \theta, t) = \psi(w, \mathbf{M}) = \frac{\psi_0(w)}{2\pi} \left[ 1 + \epsilon(\cos \mathbf{M} + i \sin \mathbf{M}) \right]
\]

(14)
\[ w_1 < w < w_2, \quad \text{and} \quad \epsilon \lesssim 1, \quad \text{with} \quad \psi_1(w) = \epsilon \psi_0(w). \]

Furthermore, for a sufficiently small enough region in the stack, \( \psi_0(w) \) may be considered constant, thus

\[ \psi_1(w) = \epsilon \psi_0 = \frac{\lambda_1}{\Delta w} \quad \text{if} \quad w_1 < w < w_2 \quad (15) \]

\[ \psi_1(w) = 0 \quad \text{everywhere else} \]

where

\[ \lambda_1 = \epsilon \int_{-\infty}^{\infty} \psi_0(w) \, dw \]

and

\[ \Delta w = w_2 - w_1 < < w_1. \]

Subsequently, putting

\[ \frac{d\psi_1}{dw} = \frac{\lambda_1}{\Delta w} \left[ \delta(w - w_1) - \delta(w - w_2) \right] \quad (16) \]

and

\[ \Omega(w_2) = \Omega(w_1) + \Delta w \left( \frac{d\Omega}{dw} \right)_{w_1} + \cdots \quad (17) \]

where

\[ \left( \frac{d\Omega}{dw} \right)_{w_1} = -|\Omega'| \quad \text{above transition}. \]

The dispersion relation, after some manipulation, becomes

\[ [\omega - n\Omega(w_1)]^2 = -i \mathcal{E} \left[ 1 + \epsilon(\cos \Theta + i \sin \Theta) \right] (2\pi)^{-1}, \quad (18) \]

where
\[ E = \left[ \frac{n |\Omega'| e^{i \frac{\Theta}{2}}}{2\pi} \right]. \quad (19) \]

For simplicity, we take \( \epsilon \approx 1 \), i.e. the largest possible modulation of density, then equation (18) becomes

\[ [\omega - n\Omega(w_1)]^2 = -i\epsilon E_2 \cos \frac{\Theta}{2} \left[ \cos \frac{\Theta}{2} + i \sin \frac{\Theta}{2} \right] (2\pi)^{-1} \]

or

\[ \omega = n\Omega(w_1) \pm (i - 1) \epsilon^{\frac{1}{2}} E_2 \cos^{\frac{1}{2}} \frac{\Theta}{2} \left[ \cos \frac{\Theta}{4} + i \sin \frac{\Theta}{4} \right] (2\pi)^{-\frac{1}{2}}, \]

i.e.

\[ \omega = n\Omega(w_1) \pm \epsilon^{\frac{1}{2}} E_2 \cos^{\frac{1}{2}} \frac{\Theta}{2} \times \]

\[ \left[ (\cos \frac{\Theta}{4} + \sin \frac{\Theta}{4}) - i(\cos \frac{\Theta}{4} - \sin \frac{\Theta}{4}) \right] (2\pi)^{-\frac{1}{2}} \]

and the distribution function becomes

\[ \psi(w, \theta, t) = \psi_0(w) \left[ 1 + e^{i(n\theta - n\Omega(w_1)) t + \epsilon^{\frac{1}{2}} E_2 \left( \alpha + \beta \right) (2\pi)^{-\frac{1}{2}}} \right], \quad (21) \]

where

\[ \omega = n\Omega(w_1) \pm \epsilon^{\frac{1}{2}} E_2 \left( \alpha + \beta \right) (2\pi)^{-\frac{1}{2}} \]

\[ \alpha = \left( 2 \cos^2 \frac{\Theta}{4} - 1 \right)^{\frac{1}{2}} \cos \left( \frac{\Theta}{4} + \frac{\pi}{4} \right) \quad (22) \]

\[ \beta = i \left( 2 \cos^2 \frac{\Theta}{4} - 1 \right)^{\frac{1}{2}} \cos \left( \frac{\Theta}{4} - \frac{\pi}{4} \right). \quad (23) \]

The following diagram of the trigonometric terms helps in understanding the overall effect!.
yields damping at $w_1 + \delta$, antidamping at $w_1 - \delta$

yields damping at $w_1 + \delta$, antidamping at $w_1 - \delta$

yields antidamping at $w_1 - \delta$, damping at $w_1 + \delta$

yields antidamping at $w_1 - \delta$, damping at $w_1 + \delta$

It is apparent from the diagram and equation (21) that the bunch amplitude may be damped or antidamped depending upon the value of $M$ and $w$.

One may imagine that the initial density modulation sets up a voltage across the lumped impedance which acts in the same way as an RF cavity. Namely, it produces buckets in the stack and makes the particles in them oscillate around in energy and phase. These particles moving along trajectories with energies $w_1 + \delta$ ($\delta > 0$) are "damped", in the sense that their average density is being reduced by their own induced fields, while those with energies $w_1 - \delta$ are "antidamped".

The antidamping "period" is therefore equal to the time taken by a particle to move through a phase angle of $M = \pi$. 
i.e. one half of a "self-induced" phase oscillation! One may then conclude that the bunch amplitude, or amplitude of the density modulation, will be enhanced by the factor

\[ R = \exp\left[|\vec{z}| \cdot \frac{e^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\tau}{2}\right], \tag{24} \]

where \(|\vec{z}|\) is the average value of the trigonometric terms (see the diagram) in the interval \(\pi/4 < \Omega/4 < 3\pi/4\); \(\tau\) is the period of the self-induced phase oscillations.

An estimate of this enhancement factor is given below.

4. **SELF-INDUCED PHASE OSCILLATIONS**

From the definitions given above,

\[ \frac{d}{dt} \Omega = n\Omega(w) - \omega \]

and for

\[ \Omega(w) = \Omega(w_1) - \omega^*|\Omega'| , \quad \omega^* = w - w_1 \]

where

\[ n\Omega(w_1) \approx \omega , \quad \text{for} \quad \varepsilon^{\frac{1}{2}} \ll n\Omega(w_1) \]

\[ \frac{d^2}{dt^2} \quad = -\frac{dw^*}{dt} \quad n|\Omega'| . \tag{25} \]

Now

\[ \frac{dw}{dt} = \frac{dw^*}{dt} = -\frac{e I Z}{2\pi} \left[ \cos \Omega + i \sin \Omega \right] , \quad m = n \tag{26} \]
\[ \frac{d^2 M}{dt^2} = \frac{e I_1 Z n|\Omega'|}{2\pi} \left[ \cos M + i \sin M \right] \]

\[ = \varepsilon \left[ \cos M + i \sin M \right]. \]

(27)

This indicates that the period of the phase oscillations is of the order of

\[ \tau = \frac{2\pi}{\varepsilon^{3/2}} = \frac{(2\pi)^{3/2}}{(n|\Omega'|e I_1 Z)^{3/2}} \]

(29)

and thus the enhancement factor is of the order of

\[ R = \exp \left| \bar{a} \right| e^{\frac{1}{2}} \left( \frac{n}{2} \right)^{3/2} \lesssim \exp \frac{1}{2} \left( \frac{\pi}{2} \right)^{3/2} = 1.9 \quad ; \quad \varepsilon \sim 1 \quad , \quad \left| \bar{a} \right| \sim \frac{1}{2}. \]

(29)

Since however, \( \varepsilon \) is already assumed to have its maximum value of unity, this result would appear to violate Liouville's theorem. However, in practice \( \varepsilon \) is to be interpreted as merely the amplitude of a Fourier series which properly describes the real density. In the above we have only considered the first harmonic of this series (or the \( n \)th harmonic of the revolution frequency). Thus \( \varepsilon \) may exceed unity at the expense of all the other amplitudes in the proper series.

Furthermore, from the equations (25) and (26) we may find a Hamiltonian

\[ H = \frac{\varepsilon}{n|\Omega'|} [\sin M - i \cos M] + n|\Omega'| \frac{\omega^2}{2} \]

(30)

and for \( \varepsilon \) real, dropping the \( i \cos M \) term yields the usual bucket theory Hamiltonian, from which it may be shown that
\[ w_{\text{max}}^* = \frac{4 \varepsilon}{n^2 |\Omega'|^2} = \frac{2}{\pi} \frac{e^{I_1/2}}{n|\Omega'|^2}, \tag{31} \]

where

\[ I_1 = e^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi_0(w) R(w) \Theta(w) \, dw. \]

Now since \( \varepsilon \) is enhanced by \( R \) then one might expect that the energy spread is, from equation (31), increased by \( R^{\frac{3}{2}} \), i.e. at most a factor of \( \sqrt{2} \).

All of the above, of course, is true above threshold, i.e. if instead of the rectangular energy spread, one with "tails" had been used, then the usual stability conditions will prevail. Above threshold, however, the analysis given here indicates that one might expect the "bunching factor" to increase and decrease at a frequency given by the self-induced phase oscillation frequency. This situation will be somewhat modified near threshold in that the increased energy spread might again tend to introduce stability. Even without this added "damping" effect, and even well above threshold, it would appear that the bunching factor will only be increased by a factor of two.

From this physical interpretation of the longitudinal bunching where, for half a phase oscillation, damping occurs and for the other half, antidamping occurs, one might expect to reduce the enhancement of the density modulation by speeding up the phase oscillations so that the density has less time in which to grow. This could be done by external means by imposing more rapid phase oscillations produced, say, by the scanning bucket system.

For example, in order to reduce the enhancement factor to about 1.05, one has to impose by external means a phase oscillation which has a frequency about twelve times that self-induced by the bunching. The scanning bucket system planned for the ISR.
is to operate with about 1.5 kV across its RF cavity at a harmonic number of 600. If a single stacked pulse has an RF amplitude at \( n = 30 \) of 150 mA (\( \equiv I_z \) above), then the scanning system should keep the enhancement factor down to 1.05 up to \( Z \) values of 100 \( \Omega \).

**N.B.** If the physics is similar in transverse phase space, then perhaps a "scanning Q-scheme" might help to control transverse instabilities.

**ACKNOWLEDGEMENTS**

If this paper is at all intelligible to anyone other than myself, it is due to the persistent efforts of my colleagues in the ISR Department and particularly in the theory group, to reduce the waving of my arms and increase the clarity of my perorations. I therefore make due acknowledgement to Erhard Fischer for the "imprimi potest", to Kurt Hübner and Bruno Zotter for the "nihil obstat" and to Hugh Hereward for the "imprimatur".