The Connes-Lott Program on the Sphere

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Abstract

We describe the classical Schwinger model as a study of the projective modules over the algebra of complex-valued functions on the sphere. On these modules, classified by \( \pi_2(S^2) \), we construct hermitian connections with values in the universal differential envelope which leads us to the Schwinger model on the sphere. The Connes-Lott program is then applied using the Hilbert space of complexified inhomogeneous forms with its Atiyah-Kähler structure. This Hilbert space splits in two minimal left ideals of the Clifford algebra preserved by the Dirac-Kähler operator \( D = i(d - \delta) \). The induced representation of the universal differential envelope, in order to recover its differential structure, is divided by the unwanted differential ideal and the obtained quotient is the usual complexified de Rham exterior algebra over the sphere with Clifford action on the "spinors" of the Hilbert space. The subsequent steps of the Connes-Lott program allow to define a matter action, and the field action is obtained using the Dixmier trace which reduces to the integral of the curvature squared.

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1 Introduction

Since Dirac’s seminal paper [10], the static magnetic monopole has been of considerable interest, not as much as a physically realised system but mainly as an example of a theory with nontrivial topological features. It is a curious coincidence that Dirac’s paper appeared in the same year as Hopf’s paper [13] on circle bundles over the sphere. The geometrical fibre bundle approach appeared in the seventies by Wu and Yang [19] in a language more accessible to physicists and by Greub and Petry [12] in a more formal way. Another approach using the differential characters of Cheeger and Simons [2] was made by Coquereaux [7]. The monopole is described by its magnetic field defined on $\mathbb{R}^3\setminus\{0\}$, homeomorphic to $S^2 \times \mathbb{R}_+$, of which the sphere $S^2$ is a deformation retract. The non-trivial topological features are thus common to both $\mathbb{R}^3\setminus\{0\}$ and $S^2$ and for simplicity reasons we restrict ourselves to the case of the sphere on which we include spinor fields. This actually means that we switched from the Dirac monopole in three dimensions to the Schwinger model on the sphere.

Here we examine this model in the light of Serre-Swan’s theorem on the correspondence between the sections of (complex) vector bundles over the sphere $S^2$ and the projective modules over the algebra of smooth complex-valued functions on the sphere $\mathcal{A} = \mathcal{C}^\infty(S^2, \mathbb{R}) \otimes \mathbb{C}$. We then apply the Connes-Lott program\footnote{We refer to the original Connes-Lott paper [6], to the review article [17] and to Connes’ book [3].} with a suitable Hilbert space and Dirac operator and obtain the results similar to those of the (classical) Schwinger model on the sphere, [14].

The main interest of our work consists in an explicitly worked out example of the Connes-Lott scheme applied to a relatively simple system. Since the approach is entirely algebraic, it is feasible to generalise to the tensor product $\mathcal{C}^\infty(M, \mathbb{R}) \otimes \mathcal{A}_F$, where $\mathcal{A}_F$ is a finite, involutive algebra, not necessarily commutative, e.g. the ”standard model” choice $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{M}_3(\mathbb{C})$.

In section 2 we define our notation for the relevant differential geometry on the sphere. The classification in topological sectors of the hermitian finite projective modules $\mathcal{M}_P$ over $\mathcal{A}$ is given in section 3. The hermitian connections $\nabla_P$ with values in the universal differential envelope $\Omega^\bullet(\mathcal{A})$ are also constructed. In section 4 we make our choice of a Hilbert space $\mathcal{H}$ and a Dirac
operator D. Instead of the usual square integrable spinors on the sphere and the usual Dirac operator associated to the Levi-Civita connection, we choose the space of sections of the Atiyah-Kähler bundle with its Clifford module structure and with $D = i(d - \delta)$ as Dirac operator. We do not discuss here the possible physical interpretation of these ”Kähler spinors” which remains controversial and for which we refer to [11],[1] and [9]. The representation of the algebra in $\mathcal{H}$, together with the Dirac operator $D$ induces a representation of the universal envelope $\Omega^\bullet(\mathcal{A})$ in $\mathcal{H}$. However, this representation is not a differential one and an unwanted ideal has to be quotiented out. The elimination of this so-called ”junk” is done in the standard manner [3] and yields a representation in $\mathcal{H}$ of the connection with values in the de Rham exterior algebra. Finally, in section 5 the projective modules $\mathcal{M}_{\mathcal{P}}$ are tensored over $\mathcal{A}$ with the Hilbert space $\mathcal{H}$ allowing to construct a covariant Dirac operator $D(\nabla_{\mathcal{P}})$ in $\mathcal{M}_{\mathcal{P}} \otimes_{\mathcal{A}} \mathcal{H}$, used in the matter action. The Yang-Mills functional, in each topological sector, is obtained by the Dixmier trace of the square of the corresponding curvature operator. Some conclusions are drawn and further prospects are presented in section 6.

The appendix contains a miscellanea of formulae useful in explicit calculations.

2 Differential Geometry on the Sphere $S^2$

The sphere of radius $r$ is defined as $S^2 = \{p = (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2\}$ and the stereographic projection on the equatorial plane is given in the austral chart $H_A = \{p \in S^2 | z < r\}$ by:

$$\phi_A : H_A \rightarrow \mathbb{R}^2 : (x, y, z) \rightarrow (\xi, \eta), \text{ where } \xi = \frac{x}{r - z}, \eta = \frac{y}{r - z},$$

(2.1)

with inverse

$$\phi_A^{-1} : \mathbb{R}^2 \rightarrow H_A : (\xi, \eta) \rightarrow (x, y, z), \text{ where }$$

$$\frac{x}{r} = \frac{2\xi}{1 + \xi^2 + \eta^2}, \frac{y}{r} = \frac{2\eta}{1 + \xi^2 + \eta^2}, \frac{z}{r} = \frac{\xi^2 + \eta^2 - 1}{1 + \xi^2 + \eta^2}.$$ 

In the boreal chart $H_B = \{p \in S^2 | z > r\}$ one has

$$\phi_B : H_B \rightarrow \mathbb{R}^2 : (x, y, z) \rightarrow (\xi', \eta'), \text{ where } \xi' = \frac{x}{r + z}, \eta' = \frac{y}{r + z},$$

(2.2)
with inverse
\[ \phi_B^{-1} : \mathbb{R}^2 \to H_B : (\xi', \eta') \to (x, y, z), \]
where
\[ \begin{align*}
   x &= \frac{2\xi'}{1 + \xi'^2 + \eta'^2}, \\
   y &= \frac{2\eta'}{1 + \xi'^2 + \eta'^2}, \\
   z &= \frac{1 - \xi'^2 - \eta'^2}{1 + \xi'^2 + \eta'^2}.
\end{align*} \]

It is useful to introduce the complex coordinates \( \zeta_A = \xi + i\eta \) with its complex conjugate \( \zeta_A^* = \xi - i\eta \) in \( H_A \) and \( \zeta_B = \xi' - i\eta' \), \( \zeta_B^* = \xi' + i\eta' \) in \( H_B \).

In the overlap \( H_A \cap H_B = \{(x, y, z) \in S^2 | -r < z < r\} \), both coordinates are related by \( \zeta_A \zeta_B = 1 \), displaying the complex structure of \( S^2 \cong \mathbb{C}P_1 \).

Spherical coordinates are defined on this overlap by:
\[
\begin{align*}
x &= r \sin \theta \cos \varphi, \\
y &= r \sin \theta \sin \varphi, \\
z &= r \cos \theta,
\end{align*}
\]
related to the coordinates above by
\[
\zeta_A = \cotg(\theta/2) \exp(+i\varphi), \quad \zeta_B = \tg(\theta/2) \exp(-i\varphi). \tag{2.3}
\]

Note that the signs are chosen such that the orientation defined by \((\xi, \eta)\) corresponds to a positive inward oriented normal on \( S^2 \) as imbedded in \( \mathbb{R}^3 \) and is opposed to that of \((\xi', \eta')\) and \((\theta, \varphi)\).

The metric on the sphere \( S^2 \) is obtained as the pull-back of the Euclidean metric on \( \mathbb{R}^3 \): \( g_E = dx \otimes dx + dy \otimes dy + dz \otimes dz \). In \( H_A \) it is written as:
\[
g|_A = \theta^\xi \otimes \theta^\xi + \theta^n \otimes \theta^n, \tag{2.4}
\]
with the Zweibein
\[
\theta^\xi = \frac{2}{f_A} d\xi \quad \text{and} \quad \theta^n = \frac{2}{f_A} d\eta, \quad \text{where} \quad f_A = 1 + |\zeta_A|^2.
\]

In terms of \( \theta_A = \theta^\xi + i \theta^n = \frac{2}{f_A} d\zeta_A \) and its conjugate \( \theta_A^* \), the metric reads:
\[
g|_A = \frac{1}{2} (\theta_A^* \otimes \theta_A + \theta_A \otimes \theta_A^*). \tag{2.5}
\]

Similarly in \( H_B \), the Zweibein is given by
\[
\theta^\xi' = \frac{2}{f_B} d\xi' \quad \text{and} \quad \theta^n' = \frac{2}{f_B} d\eta', \quad \text{where} \quad f_B = 1 + |\zeta_B|^2,
\]
or by $\theta_B = \theta^x - i \theta^y = \frac{2}{f_B} d\zeta_B$ and $f_B^2$, so that the metric reads:

$$g_{ij} = \partial^x \otimes \partial^x + \partial^y \otimes \partial^y = \frac{1}{2} (\partial^x \otimes \partial_B^x + \partial_B^x \otimes \partial^y).$$

(2.6)

In the overlap $H_A \cap H_B$, one has the relation

$$\theta_A = -\frac{\zeta_A^2}{|\zeta_A|^2} \theta_B,$$

and $\theta_A^c = -\frac{\zeta_A^2}{|\zeta_A|^2} \theta_B$.   

(2.7)

In this overlap, spherical coordinates can also be used and one has

$$f_A^{-1} = \sin^2(\theta/2), \quad f_B^{-1} = \cos^2(\theta/2);$$

$$\theta_A = -\exp(i\varphi)(d\theta - i \sin \theta d\varphi), \quad \theta_B = \exp(-i\varphi)(d\theta - i \sin \theta d\varphi).$$

In $H_A$ the structure functions of the Zweibein field are given by :

$$d\theta^x = \eta \theta^x \wedge \theta^y$$

and $d\theta^y = \xi \theta^y \wedge \theta^x$ or by

$$d\theta_A = 1/2 \zeta_A \theta_A \wedge \theta_A^c,$$

(2.8)

and the Levi-Civita connection reads :

$$\nabla \theta^x = -\left(\xi \theta^y - \eta \theta^x\right) \otimes \theta^y$$

and $\nabla \theta^y = -\left(\eta \theta^x - \xi \theta^y\right) \otimes \theta^x$ or by

$$\nabla \theta_A = -1/2 (\zeta_A \theta_A^c - \zeta_A^c \theta_A) \otimes \theta_A.$$

(2.9)

Similar expressions hold in $H_B$ and the relation with (2.8) and (2.9), in the intersection $H_A \cap H_B$, is easily worked out.

The oriented area element $\omega$ on the sphere is given, respectively in $H_A$ and $H_B$ by :

$$\omega_A = \tau := \frac{i}{2} \theta^x \wedge \theta^y = \frac{i}{2} \theta_A \wedge \theta_A^c,$$

$$\omega_B = -\tau' := -\frac{i}{2} \theta^x \wedge \theta^y = \frac{i}{2} \theta_B \wedge \theta_B^c.$$

(2.10)

and in the overlap $H_A \cap H_B$, one has

$$\omega_A = \omega_B = -d\theta \wedge \sin \theta d\varphi.$$

5
On the sphere, a local basis of the space of differential forms \( F^\bullet(M) \), is given in \( H_A \) by

\[
\{1, \theta^\xi, \theta^\eta, \tau\}.
\]

The only non zero \( C^\infty(S^2, \mathbb{R}) \)-valued scalar products of these basis vectors, as defined in (A.10), are:

\[
g^{-1}(1, 1) = g^{-1}(\theta^\xi, \theta^\xi) = g^{-1}(\theta^\eta, \theta^\eta) = g^{-1}(\tau, \tau) = 1.
\]

The multiplication table of the real Clifford algebra \( \mathcal{C}(S^2) \), defined in (A.26) of the appendix, is given in table (1).

Table 1: Clifford multiplication

<table>
<thead>
<tr>
<th>( \lor )</th>
<th>1</th>
<th>( \theta^\xi )</th>
<th>( \theta^\eta )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( \theta^\xi )</td>
<td>( \theta^\eta )</td>
<td>( \tau )</td>
</tr>
<tr>
<td>( \theta^\xi )</td>
<td>( \theta^\xi )</td>
<td>1</td>
<td>( \tau )</td>
<td>( \theta^\eta )</td>
</tr>
<tr>
<td>( \theta^\eta )</td>
<td>( \theta^\eta )</td>
<td>-( \tau )</td>
<td>1</td>
<td>-( \theta^\xi )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>( \tau )</td>
<td>-( \theta^\eta )</td>
<td>( \theta^\xi )</td>
<td>-1</td>
</tr>
</tbody>
</table>

Besides the trivial idempotent 1, the primitive idempotents are of the form

\[
P = 1/2(1 + b_\xi \theta^\xi + b_\eta \theta^\eta + c\tau),\quad \text{with } b_\xi^2 + b_\eta^2 = 1 + c^2, \ b_\xi, b_\eta, c \in C^\infty(S^2, \mathbb{R}) .
\]

The Dirac-Kähler operators, \( D = i(\delta - \delta) \) and \( D^\delta = i(\delta^D - \delta^D) \), defined in (A.32) and (A.33), conserve the left-, \( I^E_\pm = \mathcal{C}(S^2) \vee P_\pm \), and right-ideals, \( I^D_+ = P_+ \vee \mathcal{C}(S^2) \), if and only if \( \nabla_X P = 0 \) for any vector field \( X \) on the sphere. This happens if \( b_\xi = b_\eta = 0 \) and \( 1 + c^2 = 0 \), so that there is no real solution. Thus, although the real representation of \( \mathcal{C}(S^2) \) is reducible, the invariant subspaces are not conserved by the Dirac-Kähler operators. However, in the complexified Clifford algebra \( \mathcal{C}(S^2) \), there is a solution, namely \( P_\pm = 1/2(1 \pm i\omega) \), so that the minimal left-, \( I^E_\pm = \mathcal{C}(S^2) \vee P_\pm \), and right-ideals, \( I^D_\pm = P_\pm \vee \mathcal{C}(S^2) \), are conserved by \( D \), respectively \( D^\delta \).

The complexified Clifford algebra \( \mathcal{C}(S^2) \) can then be decomposed in a sum of minimal left ideals: \( \mathcal{C}(S^2) = I^E_+ + I^E_- \), providing two equivalent representations of \( \mathcal{C}(S^2) \) on these ideals \( I^E_\pm \). To write down explicit matrices for these representations, one chooses a local basis of \( I^E_\pm \) in each chart,
e.g. in $H_A$, one takes $(P^A_\pm Q^A_\pm)$, where $Q^A_\pm = \theta^\pm \lor P^A_\pm$.

\[
P^A_+ = \frac{1}{2}(1 + i\tau), \quad Q^A_+ = \frac{1}{2}(\theta^\epsilon + i\theta^\prime) = \frac{1}{f_A} d\zeta_A, \quad (2.11)
\]

\[
P^-_+ = \frac{1}{2}(1 - i\tau), \quad Q^-_+ = \frac{1}{2}(\theta^\epsilon - i\theta^\prime) = \frac{1}{f_A} d\zeta^\epsilon_A. \quad (2.12)
\]

In $H_B$, one has $(P^B_\pm Q^B_\pm)$, where $Q^B_\pm = \theta^\epsilon' \lor P^B_\pm$ and

\[
P^B_+ = \frac{1}{2}(1 - i\tau'), \quad Q^B_+ = \frac{1}{2}(\theta^\epsilon' + i\theta'^\prime) = \frac{1}{f_B} d\zeta_B, \quad (2.13)
\]

\[
P^-_+ = \frac{1}{2}(1 + i\tau'), \quad Q^-_+ = \frac{1}{2}(\theta^\epsilon' - i\theta'^\prime) = \frac{1}{f_B} d\zeta^\epsilon_B, \quad (2.14)
\]

with the relation in $H_A \cap H_B$ given by:

\[
(P^A_\pm Q^A_\pm) = (P^B_\pm Q^B_\pm) T^BA_{\pm},
\]

where

\[
T^BA_{\pm} = \begin{pmatrix}
1 & 0 \\
0 & -(\frac{\zeta_A}{|\zeta_A|})^2
\end{pmatrix}, \quad T_{\pm}BA = \begin{pmatrix}
1 & 0 \\
0 & -(\frac{\zeta_A}{|\zeta_A|})^2
\end{pmatrix}.
\]

A general inhomogeneous form is now written as $\Psi = \Psi^{(\pm)} + \Psi^{(\mp)}$, with

\[
\Psi^{(\pm)}|_A = (P^A_\pm Q^A_\pm) \begin{pmatrix}
\psi^A_{\pm,(0)} \\
\psi^A_{\pm,(1)}
\end{pmatrix},
\]

\[
\Psi^{(\pm)}|_B = (P^B_\pm Q^B_\pm) \begin{pmatrix}
\psi^B_{\pm,(0)} \\
\psi^B_{\pm,(1)}
\end{pmatrix} \quad (2.16)
\]

and, in $H_A \cap H_B$, one has the gauge transformation:

\[
\begin{pmatrix}
\psi^B_{\pm,(0)} \\
\psi^B_{\pm,(1)}
\end{pmatrix} = T_{\pm}BA \begin{pmatrix}
\psi^A_{\pm,(0)} \\
\psi^A_{\pm,(1)}
\end{pmatrix}. \quad (2.17)
\]

An element $\Phi$ of the Clifford algebra is then represented in each ideal $T^E_\pm$, locally by the matrices $R^A_\pm(\Phi), R^B_\pm(\Phi)$:

\[
\Phi \lor (P^A_\pm Q^A_\pm) = (P^A_\pm Q^A_\pm) \mathcal{R}^A_\pm(\Phi), \quad (2.18)
\]

\[
\Phi \lor (P^B_\pm Q^B_\pm) = (P^B_\pm Q^B_\pm) \mathcal{R}^B_\pm(\Phi). \quad (2.19)
\]

7
Explicit matrices are obtained from table (2).

Table 2: Representation of the Clifford algebra

<table>
<thead>
<tr>
<th>$\lor$</th>
<th>$P_+^A$</th>
<th>$Q_+^A$</th>
<th>$P_-^A$</th>
<th>$Q_-^A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$P_+^A$</td>
<td>$Q_+^A$</td>
<td>$P_-^A$</td>
<td>$Q_-^A$</td>
</tr>
<tr>
<td>$\theta^+$</td>
<td>$Q_+^A$</td>
<td>$P_+^A$</td>
<td>$Q_-^A$</td>
<td>$P_-^A$</td>
</tr>
<tr>
<td>$\theta^-$</td>
<td>$-iQ_+^A$</td>
<td>$iP_+^A$</td>
<td>$iQ_-^A$</td>
<td>$-iP_-^A$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$-iP_+^A$</td>
<td>$iQ_+^A$</td>
<td>$iP_-^A$</td>
<td>$-iQ_-^A$</td>
</tr>
</tbody>
</table>

\[
\mathcal{R}_\pm^A(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{R}_\pm^A(\tau) = \begin{pmatrix} \mp i & 0 \\ 0 & \pm i \end{pmatrix},
\]

\[
\mathcal{R}_\pm^A(\theta^+) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{R}_\pm^A(\theta^-) = \begin{pmatrix} 0 & \pm i \\ \mp i & 0 \end{pmatrix}. \tag{2.20}
\]

In the overlap region $H_A \cap H_B$, one has:

\[
\mathcal{R}_\pm^B(\Phi) = T_{\pm}^{BA)} \mathcal{R}_\pm^A(\Phi) \left( T_{\pm}^{BA)} \right)^{-1}. \tag{2.21}
\]

In $\mathcal{C}(S^2)$, the complex conjugation yields $(P_\pm^A)^c = P_\mp^A$, $(Q_\pm^A)^c = Q_\mp^A$ so that the basis $(P_\pm^A, Q_\pm^A)$ is orthonormal (see table (3)) with respect to the sesquilinear form $h^{-1}(\Psi, \Phi) = g^{-1}(\Psi^c, \Phi^c)$, given in (A.21).

Table 3: Orthonormality relations

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$P_+^A$</th>
<th>$Q_+^A$</th>
<th>$P_-^A$</th>
<th>$Q_-^A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_+^A$</td>
<td>$1/2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$Q_+^A$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$P_-^A$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$Q_-^A$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

This hermitian scalar product with values in $\mathcal{C}(S^2, \mathbb{R})^C$ is given by:

\[
h^{-1}(\Psi, \Phi) = h^{-1}(\Psi^{(+)}, \Phi^{(+)}) + h^{-1}(\Psi^{(-)}, \Phi^{(-)}), \tag{2.22}
\]
where, in chart $H_A$ say,

$$h^{-1}(\Psi^\pm_A, \Phi^\pm_A) = \frac{1}{2} \left( (\psi^{A,0}_\pm c^A_\pm + (\psi^{A,1}_\pm c^A_\pm) \right).$$

Integrating over the sphere yields, in each $\mathcal{I}^E_\pm$, a hermitian scalar product, as in (A.22), with complex values:

$$\langle \Psi^\pm | \Phi^\pm \rangle = \int_{S^2} h^{-1}(\Psi^\pm, \Phi^\pm) \omega.$$  \hspace{1cm} (2.24)

Each $\mathcal{I}^E_\pm$ is then completed with this product to form a Hilbert space $\mathcal{H}(P_\pm)$ and the total Hilbert space is the direct orthogonal sum $\mathcal{H}(P_+) \oplus \mathcal{H}(P_-)$. The Dirac-Kähler operator conserves this splitting and reads:

$$\mathcal{D}\Psi = \mathcal{D}_{(+)}\Psi^{(+)} + \mathcal{D}_{(-)}\Psi^{(-)}$$  \hspace{1cm} (2.25)

with, on each $\mathcal{H}(P_\pm)$:

$$\mathcal{D}_A^\pm \Psi^\pm_A = (P^A_\pm Q^A_\pm) \left( \begin{array}{cc} 0 & \mathcal{D}_A^{A,1}_\pm \Psi^\pm_A \\ \mathcal{D}_A^{A,0}_\pm & 0 \end{array} \right) \left( \begin{array}{c} (\psi^{A,0}_\pm) \\ (\psi^{A,1}_\pm) \end{array} \right).$$  \hspace{1cm} (2.26)

Each Hilbert space $\mathcal{H}(P_\pm)$ is $\mathbb{Z}_2$ graded by the parity of the corresponding differential form with grading operator $\pm \omega$ represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (2.27)

With respect to this grading, $\mathcal{H}(P_\pm)$ is further decomposed as $\mathcal{H}(P_\pm) = \mathcal{H}^{(0)}(P_\pm) \oplus \mathcal{H}^{(1)}(P_\pm)$ and the Dirac-Kähler operator is odd. The operators $\mathcal{D}^{(0)}_\pm : \mathcal{H}^{(0)}(P_\pm) \rightarrow \mathcal{H}^{(1)}(P_\pm)$ and $\mathcal{D}^{(1)}_\pm : \mathcal{H}^{(1)}(P_\pm) \rightarrow \mathcal{H}^{(0)}(P_\pm)$ are formally adjoint in the sense that

$$\langle \Psi^\pm, \mathcal{D}^{(0)}_\pm \Phi^\pm \rangle - \langle \mathcal{D}^{(1)}_\pm \Psi^\pm, \Phi^\pm \rangle = 0.$$  \hspace{1cm}

It is easier to use the complex coordinates $\zeta_A$ and $\zeta^A_\pm$ to write explicit expressions for $\mathcal{D}^A_\pm$:

$$\mathcal{D}^A_\pm = i \left( \begin{array}{cc} 0 & f_A \frac{\partial}{\partial \zeta_A^A} - \zeta_A \\ f_A \frac{\partial}{\partial \zeta^A_\pm} & 0 \end{array} \right).$$  \hspace{1cm} (2.28)

$$\mathcal{D}^A_\mp = i \left( \begin{array}{cc} 0 & f_A \frac{\partial}{\partial \zeta_A^A} - \zeta_A \\ f_A \frac{\partial}{\partial \zeta^A_\pm} & 0 \end{array} \right).$$  \hspace{1cm} (2.29)
The index of the Dirac-Kähler operator, restricted to the even forms, is:

\[
\text{Index} \left( \mathcal{D}_{\pm}^0 \right) = \dim(\ker \mathcal{D}_{\pm}^0) - \dim(\coker \mathcal{D}_{\pm}^0)
\]

\[
= \dim(\ker \mathcal{D}_{\pm}^0) - \dim(\ker \mathcal{D}_{\pm}^1).
\]

Now, \( \ker \mathcal{D}_{\pm}^0 \) is given by the functions \( \psi \) such that \( f \frac{\partial}{\partial \zeta_A} \psi = 0 \), i.e. the anti-holomorphic functions on the sphere, and these are the constants, so that \( \dim(\ker \mathcal{D}_{\pm}^0) = 1 \). On the other hand, \( \ker \mathcal{D}_{\pm}^1 \) is given by \( \phi \) such that \( f \frac{\partial}{\partial \zeta_A} \phi = 0 \). The substitution \( \phi = f \gamma \) yields \( f^2 \frac{\partial}{\partial \zeta_A} \gamma = 0 \), so that the solutions are \( \phi = f \times \text{constant} \). These, however, are not integrable over the sphere, so that \( \dim(\ker \mathcal{D}_{\pm}^1) = 0 \) and \( \text{Index} \mathcal{D}_{\pm}^0 = 1 - 0 \).

The same argument applies to \( \mathcal{D}_{\pm}^0 \) and the total index is

\[
\text{Index} \left( \mathcal{D}_{\pm}^0 + \mathcal{D}_{\pm}^0 \right) = 2 - 0,
\]

which equals the Euler number of the sphere. The index could also be obtained as the difference between the even and the odd zero modes of \( \mathcal{D} \).

The spectrum of the Dirac operator is obtained solving the eigenvalue equation:

\[
i \left( f_A \frac{\partial}{\partial \zeta_A} - \zeta_A \right) \psi_+^{A,(1)} = \lambda \psi_+^{A,(0)}
\]

\[
i \left( f_A \frac{\partial}{\partial \zeta_A} \right) \psi_+^{A,(0)} = \lambda \psi_+^{A,(1)},
\]

which yields

\[
-f_A^2 \frac{\partial^2}{\partial \zeta_A \partial \zeta_A} \psi_+^{A,(0)} = \lambda^2 \psi_+^{A,(0)}.
\]

To make a link with well-known facts, it is useful to introduce the Killing vector fields generating the \( \text{SO}(3) \) action on \( S^2 \):

\[
\hat{\ell}_+ = - \left( \frac{\partial}{\partial \zeta_A} + \zeta_A^2 \frac{\partial}{\partial \zeta_A} \right),
\]

\[
\hat{\ell}_- = \left( \frac{\partial}{\partial \zeta_A} + (\zeta_A^c)^2 \frac{\partial}{\partial \zeta_A} \right),
\]

\[
\hat{\ell}_z = \left( \zeta_A \frac{\partial}{\partial \zeta_A} - \zeta_A^c \frac{\partial}{\partial \zeta_A} \right).
\]
with the Casimir
\[
\hat{L}^2 = \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) + \hat{L}_z^2 = -f_A^2 \frac{\partial^2}{\partial \zeta_A \partial \zeta_A^*}.
\]  

(2.35)

It follows that the eigenvalues are \( \lambda = \pm \sqrt{\ell(\ell + 1)} \) with multiplicity \( 2\ell + 1 \) and eigenvectors given by
\[
\psi^A_+(\pm, \ell, m) = \begin{pmatrix} \psi^A_{+,0}(\pm, \ell, m) \\ \psi^A_{+,1}(\pm, \ell, m) \end{pmatrix},
\]  

(2.36)

with
\[
\psi^A_{+,0}(\pm, \ell, m) = Y_{t,m}, \\
\psi^A_{+,1}(\pm, \ell, m) = \frac{\pm i}{\sqrt{\ell(\ell + 1)}}(\hat{L}_- + \zeta_{A}^* \hat{L}_z)Y_{t,m} \\
= \frac{\pm i}{\sqrt{\ell(\ell + 1)}}(\sqrt{(\ell + m)(\ell - m + 1)}Y_{t,m-1} + m \zeta_{A}^* Y_{t,m})
\]

3 The Projective Modules over \( C^\infty(S^2, \mathbb{R}) \otimes \mathbb{C} \)

Let \( \mathcal{M} \) be a (right) module over the algebra \( \mathcal{A} = C^\infty(S^2, \mathbb{R}) \otimes \mathbb{C} \) with involution \( a \rightarrow a^+ \) given here by complex conjugation \( a^+ = a^c \).

Assume \( \mathcal{M} \) to be endowed with a sesquilinear form \( h \), i.e. a mapping
\[
h : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A} : X, Y \rightarrow h(X, Y),
\]  

(3.1)

which is bi-additive and obeys \( h(Xa, Yb) = a^+ h(X, Y) b \).

It is nondegenerate if \( h(X, Y) = 0, \forall X \Rightarrow Y = 0 \) and \( h(X, Y) = 0, \forall Y \Rightarrow X = 0 \). It is hermitian if \( h(X, Y) = (h(Y, X))^+ \) and positive definite if it is hermitian and if \( h(X, X) \) is a positive element of \( \mathcal{A} \) for all \( X \neq 0 \).

The adjoint, \( S^\dagger \), with respect to \( h \) of an endomorphism \( S \in \text{END}_\mathcal{A}(\mathcal{M}) \) is defined by \( h(X, S^\dagger Y) = h(SX, Y) \).

---

*Throughout this section, elements of the algebra \( \mathcal{A} \) are denoted by \( \{a, b, \cdots\} \), vectors of the module \( \mathcal{M} \) by \( \{X, Y, \cdots\} \) and complex numbers by \( \{\kappa, \lambda, \cdots\} \).*
The universal graded differential envelope of $\mathcal{A}$ (see e.g. [8],[17]) is the graded differential algebra

$$\left\{ \Omega^\bullet_u(\mathcal{A}) = \bigoplus_{k=0}^\infty \Omega^{(k)}(\mathcal{A}) , \; \mathcal{d}_u \right\} . \quad (3.2)$$

Its most pragmatic definition goes as follows. Let $\mathcal{d}_u \mathcal{A}$ be a copy of $\mathcal{A}$ as a set and consider the free algebra over $\mathbb{C}$ generated by the elements of $\mathcal{A}$ and $\mathcal{d}_u \mathcal{A}$, then $\Omega^\bullet_u(\mathcal{A})$ can be defined as this free algebra modulo the relations

$$\mathcal{d}_u a + \lambda \mathcal{d}_u b = \mathcal{d}_u (\kappa a + \lambda b) ,$$

$$\mathcal{d}_u (a b) = (\mathcal{d}_u a) b + a (\mathcal{d}_u b) ,$$

$$\mathcal{d}_u 1 = 0 .$$

The involution in $\mathcal{A}$ can be extended to $\Omega^\bullet_u(\mathcal{A})$ such that

$$\left( \mathcal{d}_u a \right)^+ = \mathcal{d}_u a^+ , \quad (3.3)$$

and, with $\mathcal{a}_u$ be the main automorphism (see (A.1)) of the graded algebra $\Omega^\bullet_u(\mathcal{A})$, one obtains

$$\forall \psi_u \in \Omega^\bullet_u(\mathcal{A}) , \left( \mathcal{d}_u \psi_u \right)^+ = - \mathcal{a}_u \mathcal{d}_u \left( \psi_u^+ \right) . \quad (3.4)$$

In fact, $\Omega^\bullet_u(\mathcal{A})$ is an $\mathcal{A}$-bimodule so that its tensor product over $\mathcal{M}$, $\Omega^\bullet_u(\mathcal{M}) = \mathcal{M} \otimes_\mathcal{A} \Omega^\bullet_u(\mathcal{A})$, is well defined. The hermitian structure $\mathcal{h}$ of (3.1) can be extended to $\Omega^\bullet_u(\mathcal{M})$ by :

$$\mathcal{h} : \Omega^\bullet_u(\mathcal{M}) \times \Omega^\bullet_u(\mathcal{M}) \rightarrow \Omega^\bullet_u(\mathcal{A}) :$$

$$(X \otimes_a \psi_u , Y \otimes_a \phi_u ) \rightarrow \mathcal{h}(X \otimes_a \psi_u , Y \otimes_a \phi_u ) = \psi_u^+ \mathcal{h}(X , Y ) \phi_u . \quad (3.5)$$

A connection in $\mathcal{M}$ is a mapping $\nabla : \mathcal{M} \rightarrow \Omega^{[1]}_u(\mathcal{M}) : X \rightarrow \nabla X$, such that :

$$\nabla(X + Y ) = \nabla X + \nabla Y ,$$

$$\nabla(X a) = (\nabla X) a + X \otimes_a \mathcal{d}_u a . \quad (3.6)$$

The connection also can be extended to $\Omega^\bullet_u(\mathcal{M})$ :

$$\nabla : \Omega^{(k)}(\mathcal{M}) \rightarrow \Omega^{(k+1)}(\mathcal{M}) :$$

$$X \otimes_a \psi_u \rightarrow \nabla(X \otimes_a \psi_u ) = (\nabla X) \psi_u + X \otimes_a \mathcal{d}_u \psi_u . \quad (3.7)$$

\footnote{Several authors, e.g. Connes in [3], use a different convention : $\left( \mathcal{d}_u a \right)^+ = - \mathcal{d}_u a^+$.}
The square of the connection is its curvature which is a right module homomorphism:
\[ \nabla^2 : \Omega^k_u(M) \to \Omega^{k+2}_u(M) \; , \; \text{i.e.} \; \nabla^2(X \otimes_a \psi_u a) = \left( \nabla^2(X \otimes_a \psi_u) \right) a . \] (3.8)

The connection is compatible with the hermitian structure if
\[ d_a h(X, Y) = h(\nabla X, Y) + h(X, \nabla Y) . \] (3.9)

This is easily extended to \( \Omega^\bullet_u(M) \):
\[ \forall \hat{\psi}_u, \hat{\phi}_u \in \Omega^\bullet_u(M) , \; d_a h(\alpha_a(\hat{\psi}_u), \hat{\phi}_u) = h(\nabla \hat{\psi}_u, \hat{\phi}_u) + h(\hat{\psi}_u, \nabla \hat{\phi}_u) . \]

The curvature is then always a hermitian operator:
\[ h(\nabla^2 X, Y) - h(X, \nabla^2 Y) = 0 . \] (3.10)

A free module of finite rank is a module isomorphic to \( A^N \) and has a basis \( \{ E_i \; ; \; i = 1, \cdots, N \} \) so that each element of \( M \) can be written as \( X = E_i x^i \).

The hermitian structure is then given by \( h(X, Y) = (x^i) + h_{\bar{i} j} y^j \).

By definition, in a standard unitary basis:
\[ h_{\bar{i} j} = \begin{cases} 1 & \text{if } i = j , \\ 0 & \text{otherwise} . \end{cases} \]

In the basis \( \{ E_i \} \), the connection \( \nabla \) is given by a \( N \times N \) matrix with entries in \( \Omega^1_u(A) \) :
\[ \nabla E_i = E_j \otimes_a \omega^j_i , \] (3.11)
where \( \omega^j_i \in \Omega^1_u(A) \), so that
\[ \nabla X = E_i \otimes_a \left( d_a x^j + \omega^j_i x^j \right) . \] (3.12)

The curvature of the connection is then given by:
\[ \nabla^2 E_i = E_j \otimes_a \rho^j_i ; \quad \rho^j_i = d_a \omega^j_i + \omega^j_k \omega^k_i \in \Omega^2_u(A) . \] (3.13)

The compatibility with the hermitian structure reads:
\[ d_a h_{\bar{i} j} = \left( \omega^k_i \right)^+ h_{\bar{k} j} + h_{\bar{k} t} \omega^t_j , \] (3.14)
\[ \left( \rho^k_i \right)^+ h_{\bar{k} j} - h_{\bar{k} t} \left( \rho^t_i \right) = 0 . \] (3.15)
An endomorphism $S \in \text{END}_A(M)$, given by $SE_i = E_j s^j_i$, has adjoint $S^i_j$ given by $S^i_j E_i = E_j (s^i_j)^+$ such that $h_{ik} (s^i_j)^k = (s^i_j)^+ h_{ij}^k$.

A hermitian projective module of finite rank $M_P$ over $A$ is obtained from a free module $M$ as the image of a hermitian projection operator $P \in \text{END}_A(M)$ such that $P^2 = P$ and $P^i_j = P$.

An element $X \in M$ belongs to $M$ if $h_{ik} X = h_{ik} P X = h_{ik} P Y$, such that $h_{ik} = h_{ik} P$. Such a projection operator can be expanded in terms of the identity and the Pauli–Gell-Mann hermitian traceless matrices $\{\lambda_i ; \alpha = 1, \ldots, N^2 - 1\}$ as

$$P = \frac{1}{2} (a \mathbf{1} + b^a \lambda_a),$$

where $a$ and the $b^a$ are real valued functions on the manifold (here $S^2$).

Using the multiplication of the $\lambda$ matrices in standard notation:

$$\lambda_\alpha \lambda_\beta = \frac{2}{N} \delta_\alpha_\beta + \lambda_\gamma (\gamma d_\alpha_\beta + \gamma f_\alpha_\beta),$$

leads to:

$$a^2 + \frac{2}{N} \delta_{\alpha_\beta} b^\alpha b^\beta = 2a$$
$$2a b^\alpha + \gamma d_{\alpha_\beta} b^\alpha b^\beta = 2b^\gamma$$

It can be shown that, in the case of the sphere, it is enough to consider only $N = 2$ so that, besides the trivial solutions $P = 1$ and $P = 0$, the general solution is obtained as:

$$P(\vec{n}) = \frac{1}{2} (1 + n^a \sigma_a),$$

where $n^a$ are real functions on the sphere such that $\delta_{\alpha_\beta} n^\alpha n^\beta = 1$.

The projection operators are thus given by mappings $S^2 \to S^2$ and it can be shown that homotopic mappings define isomorphic projective modules. These are thus classified by the second homotopy group $\pi_2(S^2) \simeq \mathbb{Z}$.

In each homotopy class $[\vec{n}]$, a representative can be chosen in various ways. Our choice is the following. First we choose coordinates in the domain sphere and in the target sphere such that the point $(1,0,0)$ of the domain is mapped
on the fixed base point in the target sphere and, in this target sphere, coordinates are chosen such that the fixed base point is also given by \((1,0,0)\).

Let \(H_A^{im}\) and \(H_B^{im}\) be the austral and boreal charts of the image sphere with complex coordinates \(\nu_A, \nu_B\), then in each homotopy class \([\vec{n}]\) we choose:

\[
\begin{align*}
\nu_A &= (\zeta_A)^n, & \nu_B &= (\zeta_B)^n, & \text{if } [\vec{n}] = +n; \\
\nu_A &= (\zeta_A)^n, & \nu_B &= (\zeta_B)^n, & \text{if } [\vec{n}] = -n.
\end{align*}
\] (3.19) (3.20)

In this way, \(H_A^{im}\) and \(H_B^{im}\) are the images of \(H_A\) and \(H_B\).

In a standard unitary basis \(\{E_i; i = 1, 2\}\) of \(A^2\), the projection operator \(\mathcal{P}E_i = E_j \rho^j_i\), with the usual representation of the Pauli matrices, is given by the matrix:

\[
\mathcal{P} = \frac{1}{2} \begin{pmatrix}
1 + n_z & n_x - in_y \\
(n_x + in_y) & 1 - n_z
\end{pmatrix}.
\] (3.21)

It is given in \(H_A^{im}\), respectively in \(H_B^{im}\), by:\footnote{Here we obtain, in a quite natural way, the so-called Bott projection, used in algebraic K-theory [18]. We thank J.M. Gracia-Bondía for pointing this out.}

\[
\begin{align*}
\mathcal{P}_A &= \frac{1}{1 + |\nu_A|^2} \begin{pmatrix} |\nu_A| & \nu_A^* \\ \nu_A & 1 \end{pmatrix}, & \mathcal{P}_B &= \frac{1}{1 + |\nu_B|^2} \begin{pmatrix} 1 & \nu_B \\ \nu_B^* & |\nu_B|^2 \end{pmatrix}.
\end{align*}
\] (3.22)

An element \(X = E_i x^i \in A^2\), given by the column matrix \(X = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\), belongs to \(M_P\) if \(\mathcal{P}X = X\). Here \(x^1, x^2\) are functions on \(S^2\), given in the chart \(H_A\) by functions \(x_A^1, x_A^2\) of \((\zeta_A, \zeta_A')\), related by \(x_A^1 = \nu_A^* x_A^2\). In \(H_B\) they are given by \(x_B^1, x_B^2\), with the relation \(x_B^2 = \nu_B^* x_B^1\).

A local, normalized basis is defined in \(H_A\), by:

\[
E_A = \begin{pmatrix} E_1 & E_2 \end{pmatrix} \mathcal{E}_A; \mathcal{E}_A = \begin{pmatrix} \nu_A^* \\ 1 \end{pmatrix} \frac{1}{\sqrt{1 + |\nu_A|^2}},
\] (3.23)

and, in \(H_B\) by:

\[
E_B = \begin{pmatrix} E_1 & E_2 \end{pmatrix} \mathcal{E}_B; \mathcal{E}_B = \begin{pmatrix} \nu_B^* \\ 1 \end{pmatrix} \frac{1}{\sqrt{1 + |\nu_B|^2}}.
\] (3.24)
In $H_A \cap H_B$, they are related by the (passive) gauge transformation:

$$E_A = E_B g_A^B , \quad \text{with} \quad g_A^B = \frac{\nu_A^B}{\nu_A} . \quad (3.25)$$

In these local bases an element $X \in \mathcal{M}_P$ is written as $X = E_A x^A$ in $H_A$ and $X = E_B x^B$ in $H_B$ with:

$$x^B = g_A^B x^A . \quad (3.26)$$

The hermitian product of two elements $X$ and $Y$ of $\mathcal{M}_P$ reads

$$h_P(X, Y) = (x^A)^+ y^A \text{ in } H_A$$

$$= (x^B)^+ y^B \text{ in } H_B . \quad (3.27)$$

A connection $\nabla$ in $\mathcal{M}$ induces a connection $\nabla_P$ in $\mathcal{M}_P$.

$$\nabla_P = P \circ \nabla : \mathcal{M}_P \to \mathcal{M}_P \otimes \Omega^{(1)}_u(A) , \quad (3.28)$$

such that

$$\nabla_P X = E_i \otimes_a \left( p^j_i \partial_u x^j + (p^j_i \omega^k_i \nu^l_j) x^j \right) , \quad (3.29)$$

In matrix notation, this reads:

$$\nabla_P X = (P \partial_u X + (P(\omega)P)X) . \quad (3.30)$$

With $(\kappa) = (P(\omega)P)$, the curvature $\nabla^2_P$ reads:

$$\nabla^2_P X = (P(\partial_u(\kappa))P + (\kappa)^2 + P(\partial_uP)(\partial_uP)P)X . \quad (3.31)$$

A general hermitian connection in $\mathcal{A}^2$ is given by:

$$(\omega) = \frac{1}{i} \begin{pmatrix} \omega_1 & \sigma \\ \omega^+ & \omega_2 \end{pmatrix} ,$$

with $\omega_1, \omega_2$ and $\sigma$ in $\Omega^{(1)}_u(A)$ and $\omega_1^+ = \omega_1 , \omega_2^+ = \omega_2$.

The matrix $(\kappa)$ is given in $H_A$ by:

$$(\kappa) = \frac{1}{\sqrt{1 + |\nu_A|^2}} \begin{pmatrix} \nu_A^B \kappa_A & \nu_A^A \\ \kappa_A \nu_A & \kappa_A \end{pmatrix} \frac{1}{\sqrt{1 + |\nu_A|^2}} . \quad (3.32)$$
where
\[
\kappa_A = \frac{1}{\sqrt{1 + |\nu_A|^2}} \frac{1}{i} \left( \nu_A \omega_1 v_A^\sigma + \nu_A \sigma + \sigma^+ v_A^\sigma + \omega_2 \right) \frac{1}{\sqrt{1 + |\nu_A|^2}};
\]
and in \( H_B \) by:
\[
(\kappa) = \frac{1}{\sqrt{1 + |\nu_B|^2}} \left( \begin{array}{ccc} \kappa_B & \kappa_B \nu_B \\ \nu_B \kappa_B & \nu_B \kappa_B \nu_B \end{array} \right) \frac{1}{\sqrt{1 + |\nu_B|^2}},
\]
where
\[
\kappa_B = \frac{1}{\sqrt{1 + |\nu_B|^2}} \frac{1}{i} \left( \omega_1 + \nu_B \sigma^+ + \sigma v_B^e + \nu_B \omega_2 v_B^e \right) \frac{1}{\sqrt{1 + |\nu_B|^2}}.
\]

In \( H_A \), on \( X = E_A x^A \), one obtains:
\[
\nabla_P X|_A = E_A \left( d_u x^A + \gamma_A x^A \right),
\]
with the (universal) total potential \( \gamma_A = \kappa_A + \mu_A \), where
\[
\mu_A = \frac{1}{\sqrt{1 + |\nu_A|^2}} \left( \nu_A d_u v_A^\sigma - \sqrt{1 + |\nu_A|^2} d_u \sqrt{1 + |\nu_A|^2} \right) \frac{1}{\sqrt{1 + |\nu_A|^2}}.
\]
will be called the (universal) monopole potential.

In the same way, in \( H_B \), on \( X = E_B x^B \), one obtains:
\[
\nabla_P X|_B = E_B \left( d_u x^B + \gamma_B x^B \right),
\]
with \( \gamma_B = \kappa_B + \mu_B \), where
\[
\mu_B = \frac{1}{\sqrt{1 + |\nu_B|^2}} \left( \nu_B d_u v_B^\sigma - \sqrt{1 + |\nu_B|^2} d_u \sqrt{1 + |\nu_B|^2} \right) \frac{1}{\sqrt{1 + |\nu_B|^2}}.
\]

The curvature operator reads
\[
\nabla_P^2 X|_A = E_A R_A x^A, \text{ with } R_A = d_u \gamma_A + \gamma_A \gamma_A,
\]
\[
\nabla_P^2 X|_B = E_B R_B x^B, \text{ with } R_B = d_u \gamma_B + \gamma_B \gamma_B.
\]
The gauge transformations in the overlap region $H_A \cap H_B$ are:

\[
\kappa_A = \left( g_A^B \right)^{-1} \kappa_B g_A^B, \\
\mu_A = \left( g_A^B \right)^{-1} \mu_B g_A^B + \left( g_A^B \right)^{-1} d_u g_A^B, \\
\gamma_A = \left( g_A^B \right)^{-1} \gamma_B g_A^B + \left( g_A^B \right)^{-1} d_u g_A^B, \\
\mathcal{R}_A = \left( g_A^B \right)^{-1} \mathcal{R}_B g_A^B.
\]

(3.39) (3.40)

4 The spectral triple $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$

The spectral triple $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$, as defined by Connes [4, 5], is given here by the algebra $\mathcal{A} = C^\infty(S^2, \mathbb{R}) \otimes \mathbb{C}$ with complex conjugation as involution and $\|f\|_\mathcal{A} = \sup_{x \in S^2} |f(x)|$ as $\mathbb{C}^*$-algebra norm. The Hilbert space $\mathcal{H}$ is given by the completion $\mathcal{H}(|P_+|$ of the left ideal $\mathcal{I}_+$ of sections in the Clifford algebra bundle with inner product:

\[
\langle \Psi, \Phi \rangle = \int_{S^2} \Psi^c \wedge \Phi
\]

(4.1)

On this Hilbert space, there is a faithful $*$-representation $\pi_0$ of $\mathcal{A}$ given here by pointwise multiplication:

\[
\pi_0 : \mathcal{A} \to \mathcal{L}(\mathcal{H}) : f \to \hat{f} = \pi_0(f) \\
(\hat{f}\Psi)(x) = f(x) \Psi(x)
\]

(4.2)

Faithfulness implies that operator norm and $\mathbb{C}^*$-algebra norm coincide: $\|\hat{f}\| = \|f\|_\mathcal{A}$.

The unbounded, essentially selfadjoint, Dirac operator is $\mathcal{D} = i(d - \delta)$, restricted to $\mathcal{H}(|P_+|$. Its spectrum is $\{ \pm \sqrt{\ell(\ell + 1)} ; \ell = 0, 1, 2, \ldots \}$, so that the resolvent $(\mathcal{D} - z)^{-1}$, $z \not\in \mathbb{R}$, is compact.

Furthermore, its commutator with $\hat{f}$ is calculated using formula (A.30):

\[
\left[ \frac{1}{i} \mathcal{D}, \hat{f} \right] \Psi = (d - \delta)f \Psi - f(d - \delta)\Psi
\]

\[
= \theta^\mu \vee (\nabla_{e^\mu} \hat{f} \Psi) - f \theta^\mu \vee (\nabla_{e^\mu} \Psi)
\]

\[
= \theta^\mu \vee (\hat{e}_\mu(f)) \Psi = d f \vee \Psi
\]

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It is bounded, with norm squared:

\[
\left\| \frac{1}{i} \mathcal{D}, \hat{f} \right\|^2 = \sup_{x \in S^2} h^{-1}(\text{d}f, \text{d}f).
\]

(4.3)

The eigenvalues of \(|\mathcal{D}|\) are \(\sqrt{\ell(\ell + 1)}\); \(\ell = 0, 1, 2, \ldots\), with multiplicities

\[
\mu_\ell = \begin{cases} 
1 & \text{if } \ell = 0 \\
2(2\ell + 1) & \text{if } \ell \neq 0
\end{cases}
\]

(4.4)

These eigenvalues are ordered in an increasing sequence so that the order number \(n_L\) of the eigenvalue \(\lambda_{n_L} = \sqrt{\tilde{L}(\tilde{L} + 1)}\), counting the multiplicity, is

\[
n_L = 1 + \sum_{\ell=1}^{L-1} 2(2\ell + 1) = 2L^2 - 1.
\]

It follows then that the order of \(\lambda_{n_L}\) as \(n_L \to \infty\) is \((n_L)^{1/2}\). In Connes’ terminology, the spectral triple is called \((d, \infty)\) summable with \(d = 2\). It can then be shown that the Dixmier trace\(^9\) \(\text{Tr}_\text{Dir}(\hat{f}|\mathcal{D}|^{-d})\) exists and, in our case, is given by:

\[
\text{Tr}_\text{Dir}(\hat{f}|\mathcal{D}|^{-2}) = \frac{1}{2\pi} \int_{S^2} f \omega
\]

(4.5)

The Dirac operator extends \(\pi_0\) to a \(*\)-representation \(\pi\) of \(\Omega^*_u(A)\) in \(\mathcal{H}\):

\[
\pi : \Omega^*_u(A) \to \mathcal{L}(\mathcal{H}) ; \psi_u \mapsto \pi(\psi_u), \\
\pi(f_0 \text{d}f_1 \cdots \text{d}f_k) = \tilde{f}_0 \left[ \frac{1}{i} \mathcal{D}, \tilde{f}_1 \right] \cdots \left[ \frac{1}{i} \mathcal{D}, \tilde{f}_k \right]
\]

(given here by)

\[
= f_0 \text{d}f_1 \vee \cdots \vee \text{d}f_k \vee . \quad (4.6)
\]

However, this representation is not differential since \(\psi_u \in \text{Ker}(\pi)\) does not imply \(\pi(\text{d}_u \psi_u) = 0\), which is easily seen by the standard example:

\[
\pi(2f \text{d}_u f - \text{d}_u(f^2)) = (2f \text{d} f - \text{d}(f^2)) = 0 , \text{ while}
\]

\[
\pi(\text{d}_u(2f \text{d}_u f - \text{d}_u(f^2))) = \pi(2\text{d}_u f \text{d}_u f) = 2\text{d} f \vee \text{d} f = 2g^{-1}(\text{d} f, \text{d} f) \text{Id}.
\]

\(^9\)We refer to [3, 17] for the definition and properties of the Dixmier trace.
To obtain a graded differential algebra of operators in $\mathcal{H}$ one has to take the quotient of $\Omega^\bullet_u(A)$ by the graded differential ideal, often called "junk",

$$\mathcal{J} = \mathcal{J}_0 + d_u(\mathcal{J}_0) = \bigoplus_{k=0}^{\infty} \left( \mathcal{J}_0^{(k)} + d_u \mathcal{J}_0^{(k-1)} \right),$$

(4.7)

where $\mathcal{J}_0 = \text{Ker}(\pi)$. In this way, one obtains the graded differential algebra:

$$\Omega^\bullet_D(A) = \frac{\Omega^\bullet_u(A)}{\mathcal{J}} = \bigoplus_{k=0}^{\infty} \Omega_D^{(k)}(A),$$

(4.8)

with canonical projection

$$\pi_D : \Omega^\bullet_u(A) \to \Omega^\bullet_D(A)$$

(4.9)

The classical homomorphism theorem, applied to the representation $\pi$, yields the isomorphism

$$\Omega_D^{(k)}(A) \cong \frac{\pi(\Omega_u^{(k)}(A))}{\pi(d_u \mathcal{J}_0^{(k-1)})}.$$  

(4.10)

In $\pi\left(\Omega_u^{(k)}(A)\right)$, the scalar product of $R$ and $S$, belonging to $\pi\left(\Omega_u^{(k)}(A)\right)$, is defined by

$$\langle R, S \rangle_{(k)} = \text{Tr}_{Dix}(R^\dagger S\text{Tr}[D]^{-d}).$$

(4.11)

In the corresponding Hilbert space completion $\mathcal{H}_u^{(k)}$ of $\pi\left(\Omega_u^{(k)}(A)\right)$, let $P^{(k)}$ be the projector on the orthogonal complement of $\pi(d_u \mathcal{J}_0^{(k-1)})$. Then, $\langle P^{(k)}R, P^{(k)}S \rangle_{(k)}$ depends only on the equivalence classes of $R$ and $S$ in $\Omega_D^{(k)}(A)$ and defines a scalar product in $\mathcal{H}_D^{(k)} = P^{(k)} \mathcal{H}_u^{(k)} = \left(\pi(d_u \mathcal{J}_0^{(k-1)})\right)^\perp$ which contains $\Omega_D^{(k)}(A)$ as a dense subspace. Indeed, let $R_D$ and $S_D$ belong to $\Omega_D^{(k)}(A)$, which means that $R_D$ and $S_D$ are equivalence classes of elements $R$ and $S$ in $\Omega_u^{(k)}(A)$ modulo $\pi(d_u \mathcal{J}_0^{(k-1)})$, then

$$\langle R_D, S_D \rangle_{(k),D} = \langle P^{(k)}R, P^{(k)}S \rangle_{(k)}.$$  

(4.12)

Furthermore, with this scalar product, each $\mathcal{H}_D^{(k)}$ is endowed with a left- and a right representation of the unitary group $u^+u = uu^+ = 1$ of the algebra $A$:

$$\langle \hat{u}R_D, \hat{u}S_D \rangle_{(k),D} = \langle R_D, S_D \rangle_{(k),D} = \langle R_D \hat{u}, S_D \hat{u} \rangle_{(k),D}.$$  

(4.13)
This follows from \( \langle \hat{u} R, \hat{u} S \rangle_{(k)} = \langle R, S \rangle_{(k)} = \langle R \hat{u}, S \hat{u} \rangle_{(k)} \), consequence of the assumed "tameness" ([17]) of the Dixmier trace:

\[
\text{Tr}_{\text{Dix}}(\hat{f} R^1 |D|^{-d}) = \text{Tr}_{\text{Dix}}(R^1 \hat{f} |D|^{-d}) \quad f \in \mathcal{A} ,
\]

and from the fact that \( \mathcal{P}^{(k)} \) is a bimodule homomorphism:

\[
\mathcal{P}^{(k)}(\hat{f}_1 R \hat{f}_2) = \hat{f}_1 (\mathcal{P}^{(k)}(R)) \hat{f}_2 .
\]

It was shown by Connes ([3, 17]) that, in the commutative Riemannian case, this quotient amounts to identify \( \Omega^n_{\text{cl}}(\mathcal{A}) \) with the usual de Rham algebra of differential forms. This can be seen from formulae (A.27),(A.28), which yield

\[
f_0 \, df_1 \wedge df_2 = f_0 \, df_1 \wedge df_2 + f_0 \, g^{-1}(df_1, df_2) - g^{-1}(df_1, df_1) - g^{-1}(df_2, df_2) .
\]

Now, \( f \, g^{-1}(dh, dh) = f \, dh \wedge dh = df(h) \wedge dh + d\left(\frac{x^2}{2}\right) \wedge dh \) and, since \( (f \, h) dh + (\frac{x^2}{2}) dh = 0 \), \( f \, g^{-1}(dh, dh) \) belongs to \( \pi(\partial \mathcal{J}_0[1]) \), so that

\[
\mathcal{P}^{(2)}(f_0 \, df_1 \wedge df_2) = f_0 \, df_1 \wedge df_2 .
\]

This is generalised to arbitrary \( k \):

\[
\mathcal{P}^{(k)}(f_0 \, df_1 \wedge df_2 \wedge \cdots \wedge df_k) = f_0 \, df_1 \wedge df_2 \wedge \cdots \wedge df_k .
\]

Besides, the "Wick theorem" relating Clifford products with exterior products (see e.g. [15]), resulting from (A.27) and (A.28), yields an explicit expression for the "junk":

\[
f_0 df_1 \wedge df_2 \wedge \cdots \wedge df_k = f_0 df_1 \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_k
\]

\[
+ f_0 g^{-1}(df_1, df_2) df_3 \wedge df_4 \wedge \cdots \wedge df_k
\]

\[
- f_0 g^{-1}(df_1, df_3) df_2 \wedge df_4 \wedge \cdots \wedge df_k
\]

\[
+ \cdots
\]

\[
+ f_0 g^{-1}(df_1, df_2) g^{-1}(df_3, df_4) df_5 \wedge \cdots \wedge df_k
\]

\[
- \cdots .
\]

The trace theorem ([3, 17]) yields:

\[
\langle R_D, S_D \rangle_{(k), D} = \text{Tr}_{\text{Dix}}(\mathcal{P}^{(k)} R^1 \mathcal{P}^{(k)} S |D|^{-2}) = \frac{1}{2\pi} \int_{S^2} \rho^c \wedge *\sigma ,
\]

where \( \rho, \sigma \) are the de Rham forms corresponding to \( R_D \) and \( S_D \).
5 The Action

To construct an action functional for the potential and the matter field, the projective modules \( \mathcal{M}_P = \mathcal{P} \mathcal{M} \) are tensored over \( \mathcal{A} \) with the Hilbert space \( \mathcal{H} \), yielding a new Hilbert space:

\[
\mathcal{H}_F = \mathcal{M}_P \otimes_a \mathcal{H},
\]  

(5.20)

whose generic elements, denoted by \(|F\rangle\), are linear combinations of the factorisable states \(|X \otimes_a \Psi\rangle\), with \(X \in \mathcal{M}_P\) and \(\Psi \in \mathcal{H}\).

The scalar product, on these states, is defined by:

\[
\langle X \otimes_a \Psi \mid Y \otimes_a \Phi \rangle = \langle \Psi, \pi_0 \left( h_\mathcal{P}(X, Y) \right) \Phi \rangle.
\]  

(5.21)

The representation \(\pi\) of \(\Omega_u(\mathcal{A})\) (4.6) induces a right \(\mathcal{A}\)-module homomorphism

\[
\mathcal{R}_\pi : \Omega_u^*(\mathcal{M}_P) = \mathcal{M}_P \otimes_a \Omega_u^*(\mathcal{A}) \to \mathcal{L}(\mathcal{H}, \mathcal{H}_F),
\]  

(5.22)

defined by:

\[
\mathcal{R}_\pi \left( X \otimes_a \phi_a \right) \Psi = |X \otimes_a \pi(\phi_a) \Psi\rangle.
\]

This homomorphism, in turn, induces a linear map

\[
\mathcal{O}_\pi : \text{HOM}_\mathcal{A} \left( \mathcal{M}_P, \Omega_u^*(\mathcal{M}_P) \right) \to \mathcal{L}(\mathcal{H}_F) : T \to \mathcal{O}_\pi(T),
\]  

(5.23)

defined by:

\[
\mathcal{O}_\pi(T) |X \otimes_a \Psi\rangle = |\mathcal{R}_\pi(TX) \Psi\rangle.
\]

The Dirac operator in \(\mathcal{H}\) and the connection \(\nabla_\mathcal{P}\) in \(\mathcal{M}_P\) allow for the construction of a covariant Dirac operator \(\mathcal{D}(\nabla_\mathcal{P})\) in \(\mathcal{H}_F\):

\[
\mathcal{D}(\nabla_\mathcal{P}) |X \otimes_a \Psi\rangle = \left[ X \otimes_a \mathcal{D} \Psi \right] + i \mathcal{R}_a(\nabla_\mathcal{P} X) \Psi,
\]  

(5.24)

which has a well defined action on \(|X f \otimes_a \Psi\rangle = |X \otimes_a f \Psi\rangle\).

Furthermore, \(\mathcal{D}(\nabla_\mathcal{P})\) is formally self-adjoint in \(\mathcal{H}_F\) and is covariant under the unitary automorphisms \(U\) of \(\mathcal{M}_P\):

\[
\mathcal{D}(U \nabla_\mathcal{P}) U |F\rangle = U \mathcal{D}(\nabla_\mathcal{P}) |F\rangle,
\]  

(5.25)

where \(U \nabla_\mathcal{P} = U \circ \nabla_\mathcal{P} \circ U^\dagger\).
The linear map (5.23) \( O_n \) associates a selfadjoint operator \( R_F \) in \( H_F \) to the curvature \( \nabla_P^2 \in \text{HOM}_{\mathcal{A}}(\mathcal{M}_\rho, \Omega^{(2)}_{\mathcal{A}}(\mathcal{M}_\rho)) \) so that the Dixmier trace in \( H_F \) defines a functional:

\[
\mathcal{I}[\nabla_P] = \text{Tr}_{Dix}(R_F^{-1} R_F \ | D(\nabla_P)|^{-d}) .
\] (5.26)

On the other hand, the projection of \( \nabla_P \) with \( \text{Id} \otimes \pi_D \) yields a connection \( \nabla_P^D \) on \( \mathcal{M}_\rho \) with values in \( \Omega^{(1)}_D(\mathcal{A}) \). In the basis \( \{ E_i \} \) of \( \mathcal{M} \), this connection is given by (3.28):

\[
\nabla_P^D(PE_i) = PE_j (\gamma)^i_j ,
\] (5.27)

with \( (\gamma)^i_j \in \Omega^{(1)}_D(\mathcal{A}) \), obeying \( (\gamma)^i_j = \mathcal{P}^i_k (\gamma)^k_\ell \mathcal{P}^\ell_j \).

The curvature \( (\nabla_P^D)^2 \) is a homomorphism of \( \text{HOM}_{\mathcal{A}}(\mathcal{M}_\rho, \Omega^{(2)}_{\mathcal{A}}(\mathcal{M}_\rho)) \), and is given by (3.31):

\[
(\nabla_P^D)^2 PE_i = PE_j (\mathcal{R}_D)^j_i ,
\] (5.28)

with \( (\mathcal{R}_D)^j_i = \mathcal{P}^i_k \mathcal{D} \mathcal{P}^\ell_j + (\gamma)^i_k (\gamma)^k_\ell + \mathcal{P}^i_k \mathcal{D} \mathcal{P}^\ell_j \mathcal{D} \mathcal{P}^m_\ell \mathcal{P}^m_j \).

Let \( (\mathcal{R})^j_i \) be one of the operators in \( \mathcal{H} \) belonging to \( \pi(\Omega^{(2)}_{\mathcal{A}}) \), whose equivalence class is \( (\mathcal{R}_D)^j_i \). The Dixmier trace in \( \mathcal{H} \):

\[
\text{SYM}[\nabla_P^D] = \langle (\mathcal{R})^j_i(\mathcal{R}_D)^j_i(\mathcal{R})_i^j)_D = \text{Tr}_{Dix}(P^{(2)}(\mathcal{R})^j_i P^{(2)}(\mathcal{R}_D)^j_i P^{(2)(D)^{-d}}) \] (5.29)

is well defined and, due to tameness (4.14), does not depend on the basis \( \{ E_i \} \) used in \( \mathcal{M} \). According to Connes [3]:

\[
\text{SYM}[\nabla_P^D] = \text{Inf} \{ \mathcal{I}[\nabla_P] ; (\text{Id} \otimes \pi_D) \nabla_P = \nabla_P^D \} \] (5.30)

The action for the matter field, living in \( H_F \), is given by:

\[
S_F[\mathcal{F}], \nabla_P^D = \langle \mathcal{F} | D(\nabla_P^D) \mathcal{F} \rangle ,
\] (5.31)

where \( D(\nabla_P^D) \) is the Dirac operator with connection \( \nabla_P^D \). Finally, the total action for the matter field \( [\mathcal{F}] \) and the connection \( \nabla_P^D \), is:

\[
S[\mathcal{F}], \nabla_P^D = S_F[\mathcal{F}], \nabla_P^D + \lambda \text{SYM}[\nabla_P^D] ,
\] (5.32)

\(^{10}\)\( d_D \) is the differential in \( \Omega^{\bullet}_D(\mathcal{A}) \).
where \( \lambda \) is a coupling constant.
This action and the resulting Euler-Lagrange equations will be written down explicitly in our case study.
The connection \( \nabla^D \) in \( \mathcal{M}_P \) is obtained from (3.34),(3.36) as
\[
\nabla^D_P \mathcal{E}_A = \mathcal{E}_A \gamma^D_A, \quad \text{in } H_A; \quad \nabla^D_P \mathcal{E}_B = \mathcal{E}_B \gamma^D_B, \quad \text{in } H_B,
\]
where the total potentials are locally given by:
\[
\gamma^D_A = \kappa + \mu^D_A; \quad \gamma^D_B = \kappa + \mu^D_B.
\]
Here,
\[
\kappa = \pi_D(\kappa_A) = \pi_D(\kappa_B)
\]
is a globally defined one-form on \( S^2 \), while the monopole potentials
\[
\mu^D_A = \frac{1}{1 + |\nu_A|^2} \left( \nu_A d\nu_A^c - \nu_A^c d\nu_A \right), \quad \mu^D_B = \frac{1}{1 + |\nu_B|^2} \left( \nu_B d\nu_B^c - \nu_B^c d\nu_B \right),
\]
are local one-forms, related by a gauge transformation in the overlap region:
\[
\mu^D_A = \mu^D_B + (g_B^{-1}) d g_A^B.
\]
The curvature two-form is also globally defined:
\[
\rho = \partial \kappa + \rho_m,
\]
with the monopole field \( \rho_m = \partial \mu^D_A = \partial \mu^D_B \) given by:
\[
\rho_m = \frac{1}{(1 + |\nu_A|^2)^2} d\nu_A \wedge d\nu_A^c = \frac{1}{(1 + |\nu_B|^2)^2} d\nu_B \wedge d\nu_B^c.
\]
The integral of the curvature yields the Chern character of the projective module:
\[
\text{ch}(\mathcal{M}_P) = \frac{1}{2i\pi} \int_{S^2} \rho = \frac{1}{2i\pi} \int_{S^2} \rho_m.
\]
For example, when the homotopy class of $\pi_2(S^2)$ is $[\vec{a}] = \pm n$ and, provided we choose the representatives in (3.19) and (3.20), then

$$\text{ch}(\mathcal{M}_p) = \pm \frac{1}{2\pi i} \int_{S^2} \frac{n^2 |\zeta_A|^{2n-2}}{(1 + |\zeta_A|^{2n})^2} d\zeta_A \wedge d\zeta_A^c = \mp n . \quad (5.41)$$

The monopole potential naturally depends on the choice of the representative in the relevant homotopy class. With our choice in (3.19), they are given in spherical coordinates in $H_A \cap H_B$ by:

$$\mu_A^D = \mp i n \frac{(\cotg \theta/2)^{2n}}{1 + (\cotg \theta/2)^{2n}} d\varphi ; \quad \mu_B^D = \pm i n \frac{\ln (\tan \theta/2)^{2n}}{1 + (\tan \theta/2)^{2n}} d\varphi . \quad (5.42)$$

Both are related by the gauge transformation

$$\mu_A^D - \mu_B^D = \mp i n d\varphi . \quad (5.43)$$

It should be noted that the usual potentials, which are given by

$$\mu_A^D = \mp i n \frac{(\cotg \theta/2)^{2n}}{1 + (\cotg \theta/2)^{2n}} d\varphi ; \quad \mu_B^D = \pm i n \frac{\ln (\tan \theta/2)^{2n}}{1 + (\tan \theta/2)^{2n}} d\varphi , \quad (5.44)$$

differ from ours by a global differential form.

The action for the Yang-Mills potential is, again, obtained from the trace theorem:

$$S_{\text{YM}}[\nabla_P^D] = \frac{1}{2\pi} \int_{S^2} \rho^c \wedge *\rho \quad (5.45)$$

The elements $|F\rangle$ of $\mathcal{H}_F$ are given, locally, by $|F\rangle|_A = E_A \otimes_a \Psi_A$ and $|F\rangle|_B = E_B \otimes_a \Psi_B$, with $E_A$ and $E_B$ (or $\Psi_A$ and $\Psi_B$) related by the passive gauge transformation (3.25). Since $h_P(E_A, E_A) = 1$, the scalar product of $|F\rangle|_A$ with $|G\rangle|_A = E_A \otimes_a \Phi_A$ is:

$$\langle F \parallel G \rangle \parallel F \rangle = \langle \Phi_A, \Psi_A \rangle = \int_{S^2} g^{-1}(\Phi_A^c, \Psi_A) \omega . \quad (5.46)$$

Here $g^{-1}(\Phi_A^c, \Psi_A) \omega$ may be replaced by $\Phi_A^c \wedge (\ast \Psi_A) = (\ast \Phi_A^c) \wedge \Psi_A$, since forms of degree less than the highest, vanish under the integration symbol.

The covariant Dirac operator acts on $|F\rangle$ as:

$$D(\nabla_P^D)|F\rangle = E_A \otimes_a D_A(\gamma_A^D) \Psi_A ,$$
with
\[ \mathcal{D}_A(\gamma^D_A)\Psi_A = i(d - \delta)\Psi_A + i\gamma^D_A \nabla \Psi_A. \] (5.47)

Using the selfadjointness of the Dirac operator, the action becomes:
\[ S_F = \frac{1}{2} \int_{S^2} \left( g^{-1}(\Psi^c_A, \mathcal{D}_A \Psi_A) + g^{-1}(\mathcal{D}_A \Psi_A)^c, \Psi_A \right) \omega. \] (5.48)

The Euler-Lagrange equations for the connection are obtained from the total action (5.32) by the variation \( \partial(\kappa) \) of \( \kappa \) in the connection \( \gamma^D_A = \kappa + \mu^D_A \), so that \( \partial(\rho) = d\partial(\kappa) \) and \( \partial(\mathcal{D}_A \Psi_A) = i\partial(\kappa) \nabla \Psi_A \).

Using the proper formulae of the appendix, it is easy to see that:
\[ \partial(S_{\text{YM}}) = \frac{1}{2\pi} \int_{S^2} \left( \partial(\kappa)^c \wedge \star \delta \rho + \star' \delta \rho^c \wedge \partial(\kappa) \right) \]
\[ = \frac{1}{2\pi} \int_{S^2} \left( g^{-1}(\partial(\kappa)^c, \delta \rho) + g^{-1}(\delta \rho^c, \partial(\kappa)) \right) \omega \] (5.49)

Under this variation of \( \kappa \), the matter action varies as
\[ \partial(S_F) = \frac{1}{2} \int_{S^2} \left( g^{-1}(\Psi^c_A, i\partial(\kappa) \nabla \Psi_A) + g^{-1}(-i\partial(\kappa)^c \nabla \Psi_A^c) \right) \omega \]

Introducing the current of eq.(A.38):
\[ j = g_{\mu\nu} g^{-1}(\psi^c_A, \theta^\mu \nabla \Psi_A) \theta^\nu = j^c, \]
(5.50)

the variation of the matter action becomes:
\[ \partial(S_F) = \frac{i}{2} \int_{S^2} \left( g^{-1}(j^c, \partial(\kappa)) - g^{-1}(\partial(\kappa)^c, j) \right) \omega. \] (5.51)

Combining equations (5.49) and (5.51) yield the Euler-Lagrange equations:
\[ \frac{\lambda}{2\pi} \delta \rho - \frac{i}{2} j = 0 \]
\[ \frac{\lambda}{2\pi} \delta \rho^c + \frac{i}{2} j^c = 0. \] (5.52)

The matter equation results from the variation \( \partial(\Psi_A) \) in (5.48), which yields the covariant Dirac equation of Benn and Tucker [1]:
\[ i(d - \delta)\Psi_A + i(\kappa + \mu^D_A) \nabla \Psi_A = 0. \] (5.53)
The system of coupled equations (5.52) and (5.53) is consistent provided \( \delta j = 0 \) is satisfied and this results from (A.39) which holds also for the covariant Dirac equation.

In the absence of matter, the absolute minimum of \( S_{\text{YM}} \) is reached when \( \rho = 0 \), i.e.
\[
d\kappa + \rho_m = 0 .
\]

Using the Hodge decomposition and \( H^1_{\text{deRham}}(S^2) = 0 \), we may write
\[
\kappa = d\chi_0 + \delta \phi_2 ,
\]
where \( \chi_0 \) is a 0-form and \( \phi_2 \) is a two-form. Substitution in (5.54) yields \( d\delta \phi_2 + \rho_m = 0 \). If we put \( f = \star \phi_2 \), this becomes \( d \star \alpha (d f) + \rho_m = 0 \) or
\[
\Delta f + \star^{-1} \rho_m = 0 ,
\]
where \( \Delta \) is the Laplacian on the sphere. This is essentially a result of Jayewardena [14] for the (classical) Schwinger model on \( S^2 \).

6 Conclusions

In this article, we made explicit the procedure of Connes and Lott [6] for the algebra of complex valued functions on the sphere \( S^2 \), describing the classical Schwinger model on the sphere.

Here, the non-trivial topological features of the theory show up in the projective modules over this algebra with their connections. For each projective module (i.e. each "sector" of the theory), the Lagrangian appears as that of a constrained theory in the sense that the monopole field is fixed and the Euler-Lagrange equations look accordingly. The Connes-Lott program, restricted here to a commutative algebra, provides a systematic and consistent way of dealing with topologically non-trivial aspects of gauge theories.

A similar treatment of the, still commutative, algebra \( \mathcal{C}^\infty(S^2, \mathbb{R}) \otimes (\mathbb{C} \oplus \mathbb{C}) \) will lead to a generalised Schwinger model on the sphere and should include "Higgs" type of phenomena. Finally, the genuine non-commutative algebra of quaternionic-valued functions on \( S^4 \) should describes instantons in the Connes-Lott framework.
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Appendix

Let $M$ be a Riemannian manifold of dimension $N$ with cotangent bundle $\tau^*(M) : T^*(M) \to M$ and let $\Lambda^*(\tau^*(M))$ be the exterior product bundle. Its space of sections consists of the differential forms $\mathcal{F}^\bullet(M) = \sum_{k=0}^N \mathcal{F}^k(M)$ which, with the exterior product $\wedge$ and the exterior differential $d$, becomes a graded differential algebra.\(^{11}\)

The main automorphism $\alpha$ of the graded algebra $\{\mathcal{F}^\bullet(M), \wedge\}$ is defined by:

\[
\alpha(\psi \wedge \phi) = \alpha(\psi) \wedge \alpha(\phi) , \\
\alpha(f) = f , f \in \mathcal{F}^0(M) , \text{ and} \\
\alpha(\xi) = -\xi , \xi \in \mathcal{F}^1(M).
\]  
(A.1)

and the main antiautomorphism $\bar{\alpha}$ by:

\[
\bar{\alpha}(\psi \wedge \phi) = \bar{\alpha}(\phi) \wedge \bar{\alpha}(\psi) , \\
\bar{\alpha}(f) = f , f \in \mathcal{F}^0(M) , \text{ and} \\
\bar{\alpha}(\xi) = \xi , \xi \in \mathcal{F}^1(M).
\]  
(A.2)

Besides the usual exterior differential $d$ and interior product $\iota(X)$ which are antiderivations on $\mathcal{F}^\bullet(M)$ acting from the left, i.e.

\[
d(\psi \wedge \phi) = (d\psi) \wedge \phi + \alpha(\psi) \wedge (d\phi) \text{ and}
\]  
(A.3)

\[
\iota(X)(\psi \wedge \phi) = (\iota(X)\psi) \wedge \phi + \alpha(\psi) \wedge (\iota(X)\phi) ,
\]  
(A.4)

it appears useful to define also a "right" exterior differential and interior product by:

\[
d^D(\psi \wedge \phi) = \psi \wedge (d^D\phi) + (d^D\psi) \wedge \alpha(\phi) \text{ and}
\]  
(A.5)

\(^{11}\)We follow the conventions of Kobayashi-Nomizu [16] for the Cartan exterior calculus.
\[ \iota^D(X)(\psi \wedge \phi) = \psi \wedge (\iota^D(X)\phi) + (\iota^D(X)\psi) \wedge \alpha(\phi) . \] (A.6)

They are related to the usual \( d \) and \( \iota(X) \) by 12:

\[ d^D = d \circ \alpha = -\alpha \circ d , \quad \text{and} \quad \iota^D(X) = -\iota(X) \circ \alpha = \alpha \circ \iota(X) . \] (A.7)

The sections of the tangent bundle \( \tau(M) : T(M) \to M \) are the vector fields \( X(M) \) on the manifold. A vector field \( X \in X(M) \) acts on the differential graded algebra \( \{ \mathcal{F}^*(M), \wedge, d \} \) through the interior product \( \iota(X) \) and the Lie derivative

\[ \mathcal{L}(X) = d \circ \iota(X) + \iota(X) \circ d = d^D \circ \iota^D(X) + \iota^D(X) \circ d^D . \] (A.8)

Let \( \{ \varepsilon^\mu \}, \{ \theta^\mu \} \) be a pair of dual local bases of the tangent, respectively cotangent, bundle with structure functions given by:

\[ [\varepsilon^\mu, \varepsilon^\nu] = \varepsilon^\kappa C_{\mu\nu} \quad \text{or} \quad d \theta^\kappa = -\frac{1}{2} C_{\mu\nu} \theta^\mu \wedge \theta^\nu . \] (A.9)

The metric on the manifold defines a scalar product on \( \mathcal{F}^*(M) \) with values in \( C^\infty(M) \):

\[ g^{-1} : \mathcal{F}^*(M) \times \mathcal{F}^*(M) \to C^\infty(M) : (\psi, \phi) \mapsto g^{-1}(\psi, \phi) , \]

which is \( C^\infty(M) \)-bilinear, symmetric and such that forms of different order are orthogonal. Let \( \psi = \psi_{\alpha_1 \cdots \alpha_k} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_k} , \quad \phi = \phi_{\beta_1 \cdots \beta_k} \theta^{\beta_1} \wedge \cdots \wedge \theta^{\beta_k} , \)

then

\[ g^{-1}(\psi, \phi) = k! \psi_{\alpha_1 \cdots \alpha_k} \phi_{\beta_1 \cdots \beta_k} g^{\alpha_1 \beta_1} \cdots g^{\alpha_k \beta_k} \] (A.10)

This product has the following properties

\[ g^{-1}(\xi \wedge \chi, \phi) = g^{-1}(\chi, \xi| \phi) , \quad g^{-1}(\chi \wedge \xi, \phi) = g^{-1}(\chi, \phi \xi| \xi) , \] (A.11)

where \( \xi \) is the vector field defined by

\[ \xi| \eta = \eta| \xi = g^{-1}(\eta, \xi) . \] (A.12)

The Hodge duals \( \star \) and \( \star' \) are the \( C^\infty(M) \)-linear mappings:

\[ \star_k : \mathcal{F}^k(M) \to \mathcal{F}^{N-k}(M) : \phi \mapsto \star_k \phi , \]
\[ \star'_k : \mathcal{F}^k(M) \to \mathcal{F}^{N-k}(M) : \psi \mapsto \star'_k \psi \]

12 We will also write \( \iota(X)\psi = X| \psi \) and \( \iota^D(X)\psi = \psi| X \)
given by:
\[ \psi \wedge (\ast_k \phi) = (\ast'_k \psi) \wedge \phi = g^{-1}(\psi, \phi) \omega. \] (A.13)

They are related by
\[ \ast_k = (-1)^{k(N-k)} \ast_k', \] (A.14)
and on inhomogeneous forms the action of \( \ast \) and \( \ast' \) is straightforward.

The above properties (A.11) of \( g^{-1} \) imply that
\[ \ast(\phi \wedge \xi) = \xi| \ast(\phi) \quad \ast'(\eta \wedge \psi) = (\ast' \psi) | \eta. \] (A.15)

Since \( \ast = \ast' = \omega \), the repeated application of (A.15) leads to:
\[ \ast(\xi^1 \wedge \cdots \xi^k) = \xi^k | \cdots | \xi^1 \omega, \]
\[ \ast'(\xi^k \wedge \cdots \xi^1) = \omega | \xi^k \cdots | \xi^1. \] (A.16)

The inverse of the Hodge star-operators are given by:
\[ \ast'_{N-k} \circ \ast_k = \ast_{N-k} \circ \ast_k' = \text{Sign}(\det g) \text{Id}_k. \] (A.17)

Other useful properties are:
\[ \ast \circ \circ = (-1)^N \circ \ast, \quad \ast' \circ \circ = (-1)^N \circ \ast', \] (A.18)
\[ g^{-1}(\ast \psi, \phi) = g^{-1}(\psi, \ast' \phi), \] (A.19)
\[ \xi \wedge (\ast \phi) = \ast(\phi | \xi) \quad (\ast' \psi) \wedge \eta = \ast'(\eta | \psi). \] (A.20)

The complexification of the exterior bundle yields complex-valued differential forms \( \mathcal{F}^\bullet(M)^C \) with a naturally defined operation of complex conjugation \( \psi \rightarrow \psi^c \) and an involution \( \psi \rightarrow \psi^\dagger \) related by \( \psi^c = \beta(\psi^\dagger) \).

On \( \mathcal{F}^\bullet(M)^C \) a \( C^\infty(S^2, R)^C \)-sesquilinear form is given by:
\[ h^{-1}(\psi, \phi) = g^{-1}(\psi^c, \phi). \] (A.21)

When \( M \) is compact\(^{13}\), it defines a Hermitian scalar product in \( \mathcal{F}^\bullet(M)^C \):
\[ \langle \psi \mid \phi \rangle = \int_M h^{-1}(\psi, \phi) \omega = \int_M \psi^c \wedge (\ast \phi) = \int_M (\ast' \psi^c) \wedge \phi. \] (A.22)

\(^{13}\) If \( M \) were not compact, the scalar product can be defined for differential forms of compact support or for square integrable forms with the Riemannian invariant measure.
The completion of \( \mathcal{F}^\bullet(M)^C \) with this product (A.22) yields a Hilbert space \( \mathcal{H}(M) \). The adjoint operators of \( d \) and \( d^D \) with respect to this scalar product are the codifferentials \( \delta \) and \( \delta^D \) obtained as follows:

\[
\langle d\psi|\phi \rangle = \int_M (d\psi^c) \wedge (\star \phi) = \int_M d(\psi^c \wedge (\star \phi)) - \int_M \alpha(\psi^c) \wedge d(\star \phi) = \int_M \psi^c \wedge (\star(\star^{-1}(d(\star \alpha(\phi)))))) = \langle \psi | \delta \phi \rangle,
\]

so that

\[
\delta = \star^{-1} \circ d \circ \star \circ \alpha = (-1)^{N+1} \star^{-1} \circ d \circ \star \circ \alpha. \quad (A.23)
\]

In the same way

\[
\langle \psi | d^D \phi \rangle = \int_M (\star' \psi^c) \wedge (d^D \phi) = \cdots = \langle \delta^D \psi | \phi \rangle,
\]

and

\[
\delta^D = \star'^{-1} \circ d^D \circ \star' \circ \alpha = (-1)^{N+1} \star'^{-1} \circ d^D \circ \star \circ \alpha. \quad (A.24)
\]

Both codifferentials are related by:

\[
\delta^D = \alpha \circ \delta = -\delta \circ \alpha. \quad (A.25)
\]

Just as the exterior algebra \( \Lambda^\bullet(\tau^\ast) \) is obtained as the quotient of the tensor algebra \( \otimes^\bullet(\tau^\ast) \) by the two-sided ideal \( I_{ext} \) generated by elements of the form \( \{\xi \otimes \eta + \eta \otimes \xi\} \):

\[
\{\Lambda(\tau^\ast) = \otimes(\tau^\ast)/I_{ext}, \wedge\},
\]

the Clifford algebra\(^{14}\) is obtained as the quotient by the ideal \( I_{Cliff} \) generated by elements of the form \( \{\xi \otimes \eta + \eta \otimes \xi - 2g^{-1}(\xi, \eta) b f 1\}, \)

\[
\{Cl(\pm)(\tau^\ast) = \otimes(\tau^\ast)/I_{Cliff}, \vee\}.
\]

Since \( I_{ext} \) is generated by homogeneous elements, the \( \mathbb{Z} \)-grading of \( \otimes^\bullet(\tau^\ast) \) persists in \( \Lambda^\bullet(\tau^\ast) \), while \( I_{Cliff} \) being generated by inhomogeneous but even elements of \( \otimes^\bullet(\tau^\ast) \), only a \( \mathbb{Z}_2 \) grading survives in \( Cl(\pm)(\tau^\ast) \).

\(^{14}\)A detailed account on Clifford algebra and the usual Dirac operator with special emphasis to applications in noncommutative geometry, can be found in the lecture notes "CLIFFORD GEOMETRY: A Seminar" by J.C. Várilly and J.M. Gracia-Bondía of the University of Costa Rica.
As vector spaces both $\Lambda^\bullet(\tau^*)$ and $C^{[\pm]}(\tau^*)$ are isomorphic so that they can be considered as a single vector space with two different products $\wedge$ and $\vee$, yielding the so-called Atiyah-Kähler algebra. This algebraic construction can be done in the cotangent space of each point of the manifold $M$, yielding an Atiyah-Kähler algebra-bundle with its space of sections $\{F^\bullet(M), \wedge, \vee\}$. The relation between the two products was given by Kähler (see [11]) and, in our notation, reads

$$\psi \vee \phi = \sum_{k=1}^N \frac{1}{k!} g^{\alpha_1 \beta_1} \cdots g^{\alpha_k \beta_k} ((\psi | e_{\alpha_1}^-- \cdots | e_{\alpha_k}^--) \vee (e_{\beta_1}^- \cdots (e_{\beta_k}^- | \phi) \cdots)) .$$

(A.26)

In particular

$$e_\xi \vee \phi = e_\xi \wedge \tilde{\phi} ,$$

(A.27)

and

$$\psi \vee \eta = \psi \wedge \eta + \psi \tilde{\eta} .$$

(A.28)

Further useful formulae are:

$$g^{-1}(\xi \vee \psi, \phi) = g^{-1}(\psi, \xi \vee \phi) , \quad g^{-1}(\psi, \phi \vee \eta) = g^{-1}(\psi \vee \eta, \phi) .$$

Also

$$\star(\psi \vee \eta) = \overline{\eta} \wedge (\star \psi) , \quad \star^0(\xi \vee \phi) = (\star^0 \phi) \vee \xi ,$$

which imply:

$$\star(\eta^1 \vee \cdots \vee \eta^k) = (\eta^k \wedge \cdots \wedge \eta^1) \wedge \omega ,$$

$$\star^0(\xi^1 \vee \cdots \vee \xi^k) = \omega \wedge (\xi^k \vee \cdots \vee \xi^1) ,$$

or, more generally:

$$\star \phi = \overline{\beta}(\phi) \vee \omega , \quad \star^0 \psi = \omega \wedge \beta(\psi) .$$

(A.29)

The exterior differentials and codifferentials can be written in terms of the Levi-Civita covariant derivative as follows:

$$d\psi = \theta^\mu \wedge (\nabla e_\mu \psi) \quad \text{and} \quad \delta \psi = -\tilde{\theta}^\mu (\nabla e_\mu \psi) ;$$

(A.30)

$$d^D \psi = (\nabla e_\mu \psi) \wedge \theta^\mu , \quad \text{and} \quad \delta^D \psi = -(\nabla e_\mu \psi) \tilde{\theta}^\mu .$$

(A.31)
The Hermitian Kähler-Dirac operators are defined by:

\[ \mathcal{D}\psi = i(d - \delta)\psi = i\bar{\theta}^\mu \vee (\nabla_{e_\mu} \psi), \quad (A.32) \]
\[ \mathcal{D}^D\psi = i(d^D - \delta^D)\psi = i(\nabla_{e_\mu} \psi) \vee \bar{\theta}^\mu. \quad (A.33) \]

A local current can be constructed as follows:

\[ g^{-1}(\tfrac{1}{i} \mathcal{D}\psi^c, \phi) = g^{-1}(\bar{\theta}^\mu \vee (\nabla_{e_\mu} \psi^c), \phi) = g^{-1}(\nabla_{e_\mu} \psi^c, \bar{\theta}^\mu \vee \phi) \]
\[ = \partial_\mu(g^{-1}(\psi^c, \bar{\theta}^\mu \vee \phi)) - g^{-1}(\psi^c, \nabla_{e_\mu}(\bar{\theta}^\mu \vee \phi)) \]
\[ = \partial_\mu(g^{-1}(\psi^c, \bar{\theta}^\mu \vee \phi)) - g^{-1}(\psi^c, (\nabla_{e_\mu} \bar{\theta}^\mu \vee \phi)) \]
\[ - g^{-1}(\psi^c, \bar{\theta}^\mu \vee (\nabla_{e_\mu} \phi)). \]

So that the current

\[ J^\mu = g^{-1}(\psi^c, \bar{\theta}^\mu \vee \phi) \quad (A.34) \]

obeys

\[ \partial_\mu(J^\mu) + \Gamma^\mu_{\nu\rho} J^\nu = g^{-1}(\tfrac{1}{i} \mathcal{D}\psi^c, \phi) + g^{-1}(\psi^c, \tfrac{1}{i} \mathcal{D}\phi) \quad (A.35) \]

and is covariantly conserved if \( \psi \) and \( \phi \) obey the Dirac-Kähler equation \( \mathcal{D}\psi = 0 = \mathcal{D}\phi \).

The current density

\[ J^\mu = g^{-1}(\psi^c, \bar{\theta}^\mu \vee \phi) \sqrt{\text{det } g} \quad (A.36) \]

is then divergence free:

\[ \text{div } J = \partial_\mu(J^\mu) + C^\mu_{\nu\rho} J^\nu = 0. \quad (A.37) \]

The current one-form

\[ j = g_{\mu\nu} g^{-1}(\psi^c, \bar{\theta}^\mu \vee \psi) \bar{\theta}^\nu, \quad (A.38) \]

obeys

\[ \delta j = 0 \quad (A.39) \]

and is dual to the (N-1)-form of Benn and Tucker [1] which is closed when the Dirac-Kähler equation is satisfied.
References