Reparameterization Invariance Revisited

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Abstract

Reparameterization invariance, a symmetry of heavy quark effective theory, appears in different forms in the literature. The most commonly cited forms of the reparameterization transformation are shown to induce the same constraints on operators that do not vanish under the equation of motion to order $1/m^2$, and to be related by a redefinition of the heavy quark field. We give a new, very straightforward proof that these constraints apply to all orders in $\alpha_s$ under matching to full QCD and renormalization-group running, at least up to and including $O(1/m^2)$.

1 Introduction

Heavy particle effective theories are useful in a variety of situations [1, 2, 3, 4, 5, 6, 7, 8]. In these effective theories, $S$ matrix elements are expanded around the limit $1/m \to 0$, in which limit the heavy particle becomes nearly static and the velocity $v$ of the heavy particle becomes a conserved quantum number. The momentum $p$ of the heavy particle is decomposed as

$$p = mv + k$$

where $m$ is the mass of the heavy particle, and $v$ is a four velocity ($v^2 = 1$), which must be chosen such that the residual momentum $k$ is small compared to $m$. Clearly, the decomposition $p = mv + k$ is not unique (see [9], for example). We can as well write $p = mv' + k'$ where $k' = k + (v - v')/m$, as long as $v'^2 = 1$. 
This leads to the requirement of reparameterization invariance for the effective Lagrangian [10, 11, 12]. In the case of a scalar field \( \phi(x) \) [8, 10], the issue is rather simple. Let us consider an infinitesimal reparameterization

\[
v \rightarrow v' = v + \delta v \quad \text{where} \quad v \cdot \delta v = 0
\]

The effective Lagrangian \( \mathcal{L}_v \) is written in terms of \( \phi_v(x) \), defined by

\[
\phi_v(x) = \sqrt{2m} \exp(imv \cdot x)\phi(x)
\]

and the reparameterization (2) leads to

\[
\phi_v \rightarrow \phi' = \exp(im\delta v \cdot x)\phi_v = [1 + im\delta v \cdot x]\phi_v
\]

Due to the appearance of \( m \) in the transformation law (4), the requirement of invariance of the effective Lagrangian under reparameterization leads to relations between couplings of different order in \( 1/m \).

In the case of spin 1/2, the situation is more complicated, because the reparameterization transformation of the field \( \Psi_{+v}(x) \) must involve a rotation in Dirac space, in order to ensure that the projection identity \( \bar{\psi}\Psi_{+v}(x) = \Psi_{+v}(x) \) is transformed into \( \bar{\psi}'\Psi_{+v'}(x) = \Psi_{+v'}(x) \).

Indeed, there is controversy in the literature on the correct form of the reparameterization transformation for heavy quark effective theory. In their paper on the issue, Luke and Manohar [10] propose a certain form for this transformation \( \Psi_{+v} \rightarrow \Psi_{+v'} \). This transformation law has been criticized by Yu-Qi Chen [11] as being incorrect at \( O(1/m^2) \), in a paper which got little attention (currently, the Spires database lists 2 citations of [11], as opposed to 70 citations of [10]).

Chen proposed a different transformation law, and in fact it is straightforward to calculate that the effective Lagrangian obtained from tree level matching to full QCD is invariant under Chen’s transformation, but not under the one proposed by Luke and Manohar. This alone, however, does not imply that Luke and Manohar’s transformation law is incorrect, because the form of the effective Lagrangian is not unique. Field redefinitions of the heavy quark field can change the Lagrangian without changing the physical predictions.

The purpose of this note is to shed some light on this question. In fact, we have not been able to follow the arguments in either [10] nor in [11] regarding the derivation of the reparameterization transformation step by step. To which extent this is due to our own inabilities, and to which extent the arguments are actually inconclusive or wrong, is not completely clear to us at each point, either. Therefore we decided to investigate the issue on our own along somewhat different lines.

Our main results are as follows. The difference between Chen’s transformation and Luke and Manohar’s, at least to order \( 1/m^2 \), has to do with the presence of the
“class II operators” that vanish under the leading-order equation of motion. They are therefore members of a family of reparameterization transformations which, interpreted as symmetries, impose the same constraints on the coefficients of the “class I operators” that do not vanish under the leading-order equation of motion. We have found a new, very straightforward proof that these constraints in fact hold to order $1/m^2$ in the heavy mass expansion, not merely at level, but to arbitrary order $\alpha_s^n$ in QCD perturbation theory. Although we might suspect that it will hold also at higher order $1/m^3$, we have not proven that. Furthermore, it is not really clear to us that reparameterization invariance constraints will hold unchanged for non-perturbative effects. We would like to encourage further study of this issue.

The present paper is organized as follows.

In Sec. 2, we review the structure of the heavy quark effective theory Lagrangian. In Sec. 3, we show how Chen’s transformation law can easily be derived on a classical level. We find that the tree level matching Lagrangian is invariant under this transformation law. Then, in Sec. 4, we compare this with Luke and Manohar’s transformation. We show that the two transformation laws differ by a redefinition of the fields.

In Sec. 5, we discuss reparameterization invariance constraints on the couplings of the effective Lagrangian, and a subtlety in applying the statements in [10] to the relations between the coupling coefficients at order $1/m^2$. There has been some confusion over the implications of Luke and Manohar’s version of reparameterization invariance. We show that Luke and Manohar’s transformation actually yields the same class I constraints as Chen’s.

In Sec. 6, we discuss loops and matching corrections. The Wilson coefficients $C_i(\mu)$ which multiply the various operators in the effective Lagrangian can be obtained in a two step process. In the first step, “matching”, one can use a renormalization scale $\mu = m$. The $C_i(m)$ are then determined by requiring Green’s functions in the full and the effective theory to be equal. In the second step, “running”, the renormalization group equations are used to evolve down from $m$ to scales $\mu \ll m$. We discuss why the invariance under Chen’s transformation is preserved to all orders in $\alpha_s$ and up to (including) order $1/m^2$ in these two steps. In Sec. 7 we draw our conclusions.

2 Operators in the heavy quark Lagrangian

The general form of the heavy quark effective Lagrangian is given by [13, 14]

$$\mathcal{L}^{\text{eff}} = \overline{\Psi}v + v\overline{\Psi}iD \cdot v + C_{\text{kin}}O_{\text{kin}} + C_{\text{mag}}O_{\text{mag}} + C_1O_1 + C_2O_2 + (\text{class II terms}) + O(1/m^3)$$

(5)
where

\[ O_{kin} = -\frac{1}{2m} \overline{\Psi}_{+v} D^2 \Psi_{+v} \]

\[ O_{mag} = \frac{g}{4m} \overline{\Psi}_{+v} \sigma^{\mu\nu} G_{\mu\nu} \Psi_{+v} \]

\[ O_1 = \frac{g}{8m^2} \overline{\Psi}_{+v} v^\mu [D^\nu, G_{\mu\nu}] \Psi_{+v} \]

\[ O_2 = \frac{ig}{8m^2} \overline{\Psi}_{+v} \sigma^{\alpha\mu} v^\nu \{ D_\alpha, G_{\mu\nu} \} \Psi_{+v} \]  

(6)

We have chosen to define the operators \( O_i \) such that tree level matching to full QCD yields \( C_1 = C_2 = C_{kin} = C_{mag} = 1 \).

Class II operators, which we have not explicitly written, have the general form

\[ O_i = \overline{\Psi}_{+v} (iD \cdot v A + \overline{A} D \cdot v) \Psi_{+v} \]  

(7)

and so vanish when applying the classical equations of motion. They can be removed from the effective Lagrangian by a field redefinition which does not change the coefficients of the class I operators \( C_1 \) through \( C_{mag} \) [14].

### 3 Derivation of Chen’s transformation law at the classical level

Let’s start from the heavy quark effective theory using \( v \). One defines

\[ \Psi_{\pm v} = e^{imv \cdot x} \frac{1 \mp i}{2} \Psi(x) \]  

(8)

where \( \Psi(x) \) is the quark field that appears in the QCD Lagrangian. This implies

\[ \Psi(x) = e^{-imv \cdot x} [\Psi_{+v}(x) + \Psi_{-v}(x)] \]  

(9)

The tree level matching Lagrangian is obtained by integration out the heavy field \( \Psi_{-v} \) using the classical equations of motion

\[ \Psi_{-v} = \frac{1}{2m + iv \cdot D} i(\partial - v \cdot D) \Psi_{+v} \]  

(10)

Now consider an effective theory using \( v' \) with

\[ v \rightarrow v' = v + \delta v \quad \text{where} \quad v \cdot \delta v = 0 \]  

(11)
We can express \( \Psi_{+v'} \) through \( \Psi_{+v} \) by using the classical equations of motion. We have

\[
\Psi_{+v'} = e^{imv' \cdot x} \, P_{+v} \, \Psi
\]

\[
= e^{im(v + \delta v) \cdot x} \frac{1 + \frac{\delta \phi}{2} + \frac{1}{2m + iv \cdot D} i(\mathcal{P} - v \cdot D)}{1 + \frac{1}{2m + iv \cdot D} i(\mathcal{P} - v \cdot D)} \Psi_{+v}
\]

\[
= \left[ 1 + im\delta v \cdot x + \frac{\delta \phi}{2} + \frac{1}{2m + iv \cdot D} i(\mathcal{P} - v \cdot D) \right] \Psi_{+v}
\]

(12)

which is Chen’s transformation law [11] (see also [12]).

In the above derivation of the transformation law, we have used the classical equations of motion for \( \Psi_{-v} \). It is therefore not obvious whether matching corrections to the effective Lagrangian will be invariant under the transformation law.

Furthermore, something funny has happened. In the effective theory based on \( v \), the two “heavy components” \( \Psi_{-v} \) are integrated out, and the two “light components” \( \Psi_{+v} \) are left as degrees of freedom. In the effective theory based on \( v' \), slightly different degrees of freedom, namely \( \Psi_{-v'} \) are integrated out. One might think that it should not be possible to recover \( \Psi_{+v'} \) from \( \Psi_{+v} \) (Note that in the case of a heavy particle effective theory for a scalar field, these problems do not appear because in that case there are no degrees of freedom which are integrated out.).

At the tree level, however, everything is certainly correct. We have checked explicitly that the tree level matching Lagrangian

\[
\mathcal{L}^{\text{tree}} = \overline{\Psi}_{+v} \left[ ivD + i\mathcal{P}_\perp \frac{1}{2m + ivD} i\mathcal{P}_\perp \right] \Psi_{+v}
\]

(13)

(expansion in \( 1/m \) is implied) is invariant under the transformation law (12). The calculation is somewhat lengthy, but straightforward. It is given here in Appendix A.

4 Comparing to Luke and Manohar’s transformation

4.1 The difference between the transformations

Luke and Manohar propose the following transformation law for the spinor \( \Psi_{+v} \) under reparameterization transformations [10]:

\[
\Psi_{+v}(x) \rightarrow \Psi_{+v}^{\text{LM}}(x) = e^{im\delta v \cdot x} A(v', \hat{u})A(v, \hat{u})^{-1} \Psi_{+v}(x)
\]

(14)
where
\[
\hat{u}^\mu = \frac{v^\mu + \frac{iD^\mu}{m}}{\sqrt{1 + \frac{2iv \cdot D}{m} - \frac{D^2}{m^2}}}
\]  
(15)

and
\[
\Lambda(w, v) = \frac{1 + \frac{\delta^\mu}{m}}{\sqrt{2(1 + v \cdot w)}}
\]  
(16)

Expanding this up to O(1/m), we find
\[
\Psi_{+v'}^{\text{LM}} = \left[1 + im\delta v \cdot x + \frac{\delta^\mu}{2} + \frac{i}{4m}\left(\delta^\mu(\not{D} - v \cdot D) - D \cdot \delta v\right) + O\left(\frac{1}{m^2}\right)\right] \Psi_{+v}
\]
\[
= \left[-\frac{i}{4m} D \cdot \delta v + O\left(\frac{1}{m^3}\right)\right] \Psi_{+v'}^{\text{Ch}}
\]  
(17)

4.2 Velocity-operator notation

Such a change in the reparameterization transformation may be induced in a simple way, because besides transforming the fields, a reparameterization transformation also changes the four-velocity \(v^\mu\).

At this point, it is mnemonically useful to adopt a notation in which the incorporation of different \(v^\mu\) into the Hilbert space of the theory is made explicit. This will make clear what happens when a transformation that changes \(v^\mu\) acts in the middle of a string of operators that depend on \(v^\mu\).

Define \(\Psi_+\) to be a column vector consisting of all of the heavy quark fields \(\Psi_{+v}\). (The + reminds us that the field is a HQET field that satisfies \(\not{D}\Psi_+ = \Psi_+\). All four-velocities are included in it, but not heavy antiquark fields, which would have to be dealt with separately, though analogously).

Then \(v^\mu\) may be treated as a four-vector operator \(\hat{v}^\mu\) that acts on \(\Psi_+\). Its eigenspaces consist of states of definite four-velocity with eigenvalue \(v^\mu\). \(\delta \hat{v}^\mu(\epsilon_i)\) is also an operator. It commutes with \(\hat{v}^\mu\), and is defined in terms of the \(\hat{v}^\mu\) operator via the formula for the change in four-velocity under an infinitesimal Lorentz transformation specified by the six infinitesimal parameters \(\epsilon_i\). (These could be boost rapidities and Euler angles, or any other convenient parameterization. What matters is that, unlike \(\delta v^\mu\), they do not depend on the value of \(v^\mu\).) It is the boost parameters which actually specify the reparameterization transformation.

The shift in velocity is now accomplished explicitly by a shifting operator \(\hat{S}(\epsilon_i) = \delta_{v',v} + \delta v(\epsilon_i)\), which obeys the commutation relations
\[
[\hat{v}^\mu, \hat{S}(\epsilon_i)] = \hat{S}(\epsilon_i) \delta \hat{v}^\mu(\epsilon_i)
\]
\[
[\delta \hat{v}^\mu(\epsilon_i), \hat{S}(\epsilon_i)] = 0
\]  
(18)
for infinitesimal $\epsilon_i$. Now everything about a reparameterization transformation, including the shift in four-velocity, is included in the action of the transformation operator on the field $\Psi_+$. We can handle both field and velocity transformations by manipulating operators in the usual way.

### 4.3 A field redefinition

Rewritten in velocity-operator notation (with the hats on the various operators omitted), Chen’s reparameterization transformation to order $\frac{1}{m}$ is

$$M^{\text{Ch}}(\epsilon_i)\Psi_+ = S(\epsilon_i) \left[ 1 + im\delta v(\epsilon_i) \cdot x + \frac{\delta \phi(\epsilon_i)}{2} \right.$$  
$$+ \frac{\delta \phi(\epsilon_i)}{4m} i(\slashed{p} - v \cdot D) + O \left( \frac{1}{m^2} \right) \right] \Psi_+ \quad (19)$$

and Luke and Manohar’s is

$$M^{\text{LM}}(\epsilon_i)\Psi_+ = S(\epsilon_i) \left[ 1 + im\delta v(\epsilon_i) \cdot x + \frac{\delta \phi(\epsilon_i)}{2} \right.$$  
$$+ \frac{\delta \phi(\epsilon_i)}{4m} i(\slashed{p} - v \cdot D) - \frac{i}{4m} D \cdot \delta v(\epsilon_i) + O \left( \frac{1}{m^2} \right) \right] \Psi_+ \quad (20)$$

where $S(\epsilon_i)$, $v^\mu$, and $\delta v^\mu(\epsilon_i)$ are now understood to be operators.

Consider the field-redefinition operator

$$R \Psi_+ = \left[ 1 - \frac{i}{4m} D \cdot v \right] \Psi_+ \quad (21)$$

The important thing about $R$ is that it is completely independent of $\epsilon_i$, so it may be applied to $\Psi_+$ even in situations that have nothing to do with reparameterization transformations. It is a valid means of redefining fields so as to obtain one formulation of HQET from another.

Then applying $R$ to Chen’s transformation reveals that

$$RM^{\text{Ch}}(\epsilon_i)\Psi_+ = RM^{\text{Ch}}(\epsilon_i)R^{-1}R \Psi_+$$
$$= M^{\text{Ch}}(\epsilon_i)R \Psi_+ + S(\epsilon_i) \left[ -\frac{i}{4m} D \cdot \delta v(\epsilon_i) \right] R \Psi_+$$
$$= M^{\text{LM}}(\epsilon_i)R \Psi_+ + O \left( \frac{1}{m^2} \right) \quad (22)$$

Even though the difference between Chen’s transformation and Luke and Manohar’s appears to depend on $\delta v(\epsilon_i)$, the shift-independent field redefinition $R$ turns Chen’s
transformation into Luke and Manohar’s, to order $1/m$. The redefined field $R\Psi$ transforms under Luke and Manohar’s reparameterization transformation.

The field redefinition $R$ is not a symmetry of the Lagrangian. However, since it is proportional to $D \cdot v$, the changes that it induces in the Lagrangian are manifestly class II operators. In fact, it is precisely the field redefinition necessary to absorb the class II operator $-\frac{1}{2m} (D \cdot v)^2$ in the Lagrangian obtained from tree-level matching, when the Lagrangian is written in terms of the field $R\Psi_+$. The field redefinition necessary to absorb order $1/m$ and order $1/m^2$ class II operators in the Lagrangian obtained from tree-level matching is

$$R'\Psi_+ = \left[1 - \frac{iD \cdot v}{4m} + \frac{3(iD \cdot v)^2}{32m^2}\right]\Psi_+ \quad (23)$$

In general, a derivative term at order $1/m^j$ might affect the form of the reparameterization transformation at order $1/m^{j-1}$, because of the order $m$ term in the reparameterization transformation. In this case, however, this does not happen, and the extra term in $R'$ has no effect on the reparameterization transformation to order $1/m$.

(In [14] the field redefinition shown is the inverse of (23), because of notational conventions. Here we define the new Lagrangian to be the original expression written in terms of the transformed fields.)

Luke and Manohar’s transformation, at least when expanded to first order in $1/m$, is a symmetry, not of the tree-level matching Lagrangian, but of the tree-level Lagrangian with the class II operators removed. Chen’s transformation, on the other hand, is a symmetry of the tree-level Lagrangian with class II operators included. In Appendix B, it is demonstrated that Chen’s transformation is not unique in this regard. There are other reparameterization transformations that preserve the entire tree-level matching Lagrangian to all orders in $1/m$.

5 Reparameterization invariance constraints on the effective Lagrangian

Reparameterization invariance leads to important constraints for the coupling constants in the effective Lagrangian. Due to the possibility of field redefinitions, neither the form of the Lagrangian nor the form of the reparameterization transformation $\Psi_{+v} \rightarrow \Psi_{+v'}$ is unique. However, a field redefinition such as $R$ above will induce only class II terms in the Lagrangian, and cannot change the constraints on the coefficients of class I terms in the Lagrangian.
Invariance under Chen’s transformation law sets the following constraints on the coefficients of the class I operators [14]

\[
\begin{align*}
C_{\text{kin}} &= 1 \\
2C_{\text{mag}} &= C_2 + 1
\end{align*}
\]  
(24)

Luke and Manohar derived the same constraint for \(C_{\text{kin}}\). When discussing the relationship between \(O_{\text{mag}}\) and \(O_2\), they noted that the combination

\[
O_{\text{mag}} + 2O_2 + O\left(\frac{1}{m^3}\right)
\]

is reparameterization invariant, and that \(O_{\text{mag}}\) is not related to the leading-order Lagrangian by reparameterization invariance.

The second of these statements needs qualification. \(C_{\text{mag}}\) may be varied independently of the leading-order Lagrangian. However, the presence of the leading-order Lagrangian does modify the relationship between \(C_{\text{mag}}\) and \(C_2\), because the reparameterization transformation acting on the leading-order Lagrangian yields a term at \(O(1/m)\)

\[
\delta L_0 = -\frac{g}{4m} \bar{\Psi}_+ \delta v_{\mu} \sigma^{\mu\nu} G_{\nu\rho} v^{\rho} \Psi_+
\]

which may only be cancelled by including a difference between \(C_{\text{mag}}\) and \(2C_2\). This is why the constraint resulting from either Chen’s transformation or Luke and Manohar’s is actually \(2C_{\text{mag}} = C_2 + 1\). The reparameterization invariance of (25) gives us the freedom to change \(C_{\text{mag}}\) and \(C_2\) subject to this constraint without violating reparameterization invariance. It is easy to jump from the statements in [10] to the incorrect conclusion that \(C_2 = 2C_{\text{mag}}\), but a close reading of [10] reveals that Luke and Manohar never actually state this, and it is not actually implied by what they do state. (Indeed, in [14], two of us jumped to exactly that erroneous conclusion, and then erroneously followed that one-loop running did not agree with the constraints from Luke and Manohar’s transformation).

A general field redefinition \(\Psi'_v = R \Psi_{+v}\) which preserves the projection property \(P_v \Psi_{+v} = \Psi_{+v}\) and which transforms the class I part of the general effective Lagrangian into itself has the form

\[
R \Psi_{+v} = \left[ 1 + \frac{a}{2m} i v \cdot D + \frac{b}{4m^2} D^2 + \frac{c}{4m^2} \sigma_{\mu\nu} D^\mu D^\nu + \frac{d}{4m^2} (i v \cdot D)^2 + O\left(\frac{1}{m^3}\right) \right] \Psi_{+v}
\]

where \(a, b, c, d\) are complex numbers. The field redefinitions (21) and (23) are redefinitions of this type. It is straightforward to check that this transformation applied to the general effective Lagrangian in (5) does not change the class I terms. Applying such redefinitions to a reparameterization symmetry \(M(\epsilon_i)\) yields a family
of reparameterization transformations $RM(\epsilon_i)R^{-1}$ which preserve the class I constraints.

This does not generalize to higher orders; at $1/m^3$, the coefficients of the class I terms may change under field redefinitions unless the form of the field redefinitions is restricted further (however, the transformation (21) induces only class II terms to all orders).

It has been checked explicitly that renormalization of the effective Lagrangian does fulfill the constraints in (24) [14, 15].

6 Loops, matching, and running

The effective theory does not have the same short distance behavior as full QCD. This must be taken into account by introducing suitable matching corrections. In this section, we show that Chen’s RPI symmetry still holds when this matching is performed to order $1/m^2$, but to all orders in $\alpha_s$.

6.1 Spinors and 1PI Green’s functions

The general prescription for matching one theory to another at some momentum scale is to ensure that the 1PI Green’s functions of the two theories describe the same physics at that scale, in an expansion in inverse powers of the effective theory cutoff. The same transitions must possess the same amplitudes when expanded in this way.

Spinors that appear on external legs of Feynman diagrams are always solutions in momentum space of the unperturbed equation of motion. The free field equations for the quark fields are different in QCD and HQET, since parts of the quark-quark Green’s function that arise from the leading equation of motion in full QCD are attributed to higher-order “interaction” terms in HQET.

Therefore, the spinors one puts on external legs in QCD are not the same as the ones used in HQET for the same physical situation. To find the Dirac spinor in terms of the corresponding HQET spinor, one substitutes $p^\mu = mv^\mu + k^\mu$ into the solutions of the momentum-space free-field Dirac equation, and writes the resulting expression in terms of a HQET spinor $u_{+v}$ for which $\not{p} u_{+v} = u_{+v}$. For a HQET spinor $u_{+v}$, the corresponding QCD spinor is

$$u_{QCD} = \left[ 1 + \frac{1}{2m + k \cdot v} (k - k \cdot v) \right] u_{+v} \quad (28)$$

The calculation may be simplified by putting external quark momenta on shell. This does not allow us to find, unambiguously, the coefficients of “class II operators”
which vanish according to the free-field equation of motion. However, since these operators may be removed by a field redefinition which (using our choice of operator basis) does not affect the class I operators to order $\frac{1}{m^2}$, we do not need to find their coefficients. This transforms factors such as $k \cdot v$ into higher-order quantities in $\frac{1}{m}$, simplifying the series expansion to finite order. Taking $(mv + k)^2 = m^2$ for external quarks gives the modified matching relation

$$u_{QCD} = \left[ 1 + \frac{1}{2m} \left( \frac{k^2}{2m} \right) \right] u_{+v}$$

and allows factors elsewhere in the 1PI Green’s functions to be similarly moved to higher orders in $1/m$.

### 6.2 Gauge invariance

When matching at tree level, it was possible to maintain gauge invariance explicitly at all steps of the calculation. This is because, at tree level, the generating functional of 1PI Green’s functions is identical to the Lagrangian. Therefore, one can match Lagrangians, deal with fields instead of spinors, and use covariant derivatives instead of momenta.

When calculating loop diagrams, on the other hand, it is necessary to choose a gauge. Gauge invariance can be made somewhat explicit by using background field gauge, but diagrams will still treat interactions with different numbers of gluons as separate vertices, and in the spinor-matching procedure above we treat momenta separately from gluon couplings. The consequences of gauge invariance then reappear later in the form of Ward identities relating different Green’s functions to one another. We will make use of one such identity when proving that Chen’s RPI symmetry holds under loop matching corrections to order $\frac{1}{m^2}$.

### 6.3 Regularization scheme

Since a matching prescription does not involve the infrared divergent terms in a theory, the regularization scheme used for infrared divergences does not matter, as long as it is used consistently in the two theories. Thus we can use dimensional regularization to regularize both ultraviolet and infrared divergences [4, 16]. When used with $\overline{\text{MS}}$, this eliminates all loop diagrams that do not possess a mass scale other than the renormalization scale $\mu$. This includes all loop diagrams in HQET, since there the quark mass becomes a factor in coupling constants rather than a contribution to the propagator.

Therefore, using this regularization scheme eliminates the need to calculate HQET loop diagrams when matching to any order in perturbation theory. We
calculate 1PI loop diagrams to any desired order in full QCD, with external quarks on shell and all divergences dimensionally regularized; apply the spinor substitution (29); and adjust the couplings in the HQET Lagrangian so that the derived 1PI Green’s function arises at tree level.

Using this regularization scheme affords us an opportunity to prove relations to all orders in $\alpha_s$. Lorentz and parity invariance of full QCD allow us to write its 1PI Green’s functions, with all loop corrections included, in terms of invariant form factors. If the Green’s functions in HQET may be computed at tree level, then the structure of the full QCD Green’s functions directly implies constraints upon the coupling constants of the HQET Lagrangian. To order $1/m^2$, it is sufficient to consider the 1PI quark-quark and quark-quark-gluon Green’s functions in QCD.

6.4 The quark two-point function

The matching of the quark two-point function just corresponds to what we already know about tree-level matching of free fields. Loops can only yield mass and field renormalizations in the full theory, so after these divergences have been subtracted off with counterterms, the amputated 1PI Green’s function is

$$i\bar{u}_{QCD}(\not{q} - m)u_{QCD}$$  \hspace{1cm} (30)

where $q^\mu$ is the full momentum of the quark. Making the substitution (29), and simplifying the result using the projection identity $\not{\phi}u_{+v} = u_{+v}$, yields the two-point function for HQET:

$$\Gamma_{\bar{q}q} = i\bar{u}_{+v} \left[ 1 + \frac{1}{2m - k^2/(2m)} \left( \frac{k + k^2/2m}{2} \right) \right] \left( m\not{\phi} + \not{k} - m \right)$$

$$\left[ 1 + \frac{1}{2m - k^2/(2m)} \left( \not{k} + \frac{k^2}{2m} \right) \right] u_{+v}$$  \hspace{1cm} (31)

This determines the coupling of every operator in HQET which contains a two-quark Feynman vertex with no gluons. To order $1/m^2$, applying the usual projection identities for heavy quark spinors, it is simply

$$i\bar{u}_{+v} \left( k \cdot v + \frac{k^2}{2m} \right) u_{+v} + O\left( \frac{1}{m^3} \right)$$  \hspace{1cm} (32)

and it ends up enforcing the RPI constraint $C_{\text{kin}} = 1$. It will also constrain the coefficients of many high-order operators such as $\frac{1}{m^{2n-1}} \not{\Psi}_{+v} (D^2)^n \not{\Psi}_{+v}$.  

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6.5 The quark-quark-gluon three-point function

The quark-quark-gluon vertex function is more interesting, because there can be quantum corrections to the structure in $p^2$, where $p^\mu$ is the transferred momentum. However, considerations of Lorentz invariance and parity limit the 1PI vertex function in a manner familiar from QED. There is a Dirac form factor $F_1$ and a Pauli form factor $F_2$, which can depend on the momenta only via $p^2$:

$$\Gamma_{aqq}^{\mu} = ig \bar{u}' \gamma^\mu T^a u F_1(p^2, g, m, \mu) - \frac{g}{2m} \bar{u}' \sigma^\mu T^a u p_\nu F_2(p^2, g, m, \mu)$$

(33)

Furthermore, $F_1(p^2 = 0) = 1$, because of gauge invariance. $F_2(p^2 = 0)$, giving the "anomalous chromomagnetic moment," is not constrained by symmetry and can be affected by loop corrections.

Regularizing all loop divergences with dimensional regularization, making the substitution (29), and applying $\vec{p} u_{+v} = u_{+v}$ as above gives the tree level vertex function in HQET. To order $1/m^2$, where $p^\mu$ is the transferred momentum and $k'^\mu$ is the final residual momentum of the heavy quark, it is

$$ig \bar{u}'_{+v} T^a u_{+v} v^\mu F_1 + \frac{ig}{2m} \bar{u}'_{+v} T^a u_{+v} (-2k'^\mu + p'^\mu) F_1$$

$$-\frac{g}{2m} \bar{u}'_{+v} \sigma^{\mu\nu} T^a u_{+v} p_\nu (F_1 + F_2)$$

$$+\frac{ig}{8m^2} \bar{u}'_{+v} T^a u_{+v} v^\mu p^2 (F_1 + 2F_2) + \frac{ig}{8m^2} \bar{u}'_{+v} T^a u_{+v} v^\mu [k'^2 + (k' - p)^2] F_1$$

$$+\frac{g}{4m^2} \bar{u}'_{+v} \sigma^{\alpha\beta} T^a u_{+v} k'_\alpha p_\beta v^\mu (F_1 + 2F_2)$$

(34)

Notice that putting the external quarks on shell has not eliminated all effects of class II operators. There are missing terms like $\vec{p} \cdot v$ which ought to arise from the class I part of our operator basis, were it not for class II operators canceling them out. Furthermore, the term that goes like $k'^2 + (k' - p)^2$ at order $1/m^2$ is a contribution from a class II operator, namely $\Psi_{+v} \{D^2, D \cdot v\} \Psi_{+v}$.

The remaining terms all come from the class I operators. Expanding the form factors as $F_1(p^2) = F_{10} + p^2/m^2 F_{12} + O(1/m^4)$ makes it possible to read off their coefficients directly:

$$C_{kin} = F_{10}$$
$$C_{mag} = F_{10} + F_{20}$$
$$C_1 = F_{10} + 8F_{12} + 2F_{20}$$
$$C_2 = F_{10} + 2F_{20}$$

(35)

This procedure yields no constraints on $C_1$, and, to this order, Chen’s RPI does not constrain it either. Applying the Ward identity $F_{10} = 1$ yields Chen’s RPI
constraints

\begin{align}
C_{\text{kin}} &= 1 \\
2C_{\text{mag}} &= C_2 + 1
\end{align}

(36)

Of course, the first relation already followed from the two-point function. That it shows up here as well is a consequence of gauge symmetry.

6.6 Running under the renormalization group

In heavy quark effective field theory, we typically want to know the values of coefficients at some momentum scale which is far below the scale where matching to the full theory is done. After matching to the full theory to some order in the number of loops, one uses the renormalization group equation to determine how the coefficients in the effective field theory Lagrangian evolve under large changes in scale. The anomalous dimensions to use are typically calculated using diagrams with one more loop than was used in matching.

It is known [15, 14] that running at one loop preserves the reparameterization invariance constraints on class I operators to order $1/m^2$. The result derived above implies that the constraints should apply for arbitrary numbers of loops. This is because renormalization group running can be seen as a special case of the matching procedure, which includes the terms from arbitrarily large orders in loops which dominate when the scale is far below the matching scale. If the RPI constraints apply at all orders in loops upon matching, they must therefore also apply to the coefficients found by running under the renormalization group. Therefore, we have shown not only that class I reparameterization invariance constraints apply to order $1/m^2$ upon matching to full QCD, but that they apply under renormalization group running as well, to all orders in $\alpha_s$.

Which transformation is actually a symmetry of the Lagrangian depends on the class II terms, and therefore on how the quark fields are defined. It is useful, as in [14], to eliminate class II terms at all stages of matching and running. One starts with the Lagrangian with class II terms absorbed by a field redefinition. Then the renormalization group running incorporates a field redefinition that continuously absorbs class II terms induced by the running. Under these conditions (if the class I operators are defined according to our operator definitions), the symmetry of the Lagrangian to order $1/m^2$ is Luke and Manohar’s, since it is a symmetry of a Lagrangian that satisfies the class I constraints and has no class II terms.
7 Conclusions

The form of a reparameterization transformation may be modified by conjugating it with other symmetry transformations, or with field redefinitions that affect the coefficients of class II operators. We have demonstrated that the forms of reparameterization invariance advocated by Chen and by Luke and Manohar are both members of the resulting family of viable reparameterization transformations. Of the two, only Chen’s is a member of the more restricted family of symmetries of the entire Lagrangian derived from tree-level matching.

Both transformations induce the same constraints on class I operator coefficients to order $1/m^2$. We have proven that these constraints hold not only at tree level, but to all orders in $\alpha_s$, upon matching between HQET and QCD and under renormalization-group running. The transformations, in this sense, are symmetries of the quantum theory as well as the classical theory.

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A Invariance of $\mathcal{L}_\text{tree}$

The tree level matching Lagrangian is given by

$$\mathcal{L}_\text{tree} = \overline{\Psi} A(v) \Psi$$

where

$$A(v) = ivD + iD/\sqrt{1 + 4m^2}$$

We want to prove invariance under the transformation

$$v \rightarrow v + \delta v$$
\[ \Psi_{+v} \rightarrow \left[ 1 + im\delta v_x + \frac{\delta \phi}{2} + \frac{\delta \phi}{2} \frac{1}{2m + ivD} iD_\perp \right] \Psi_{+v} \]

\[ \overline{\Psi}_{+v} \rightarrow \overline{\Psi}_{+v} \left[ 1 - im\delta v_x + \frac{\delta \phi}{2} + iD_\perp \frac{1}{2m + ivD} \frac{\delta \phi}{2} \right] \]

Now

\[ \delta \mathcal{L} = \delta \overline{\Psi}_{+v} A \Psi_{+v} + \overline{\Psi}_{+v} A \delta \Psi_{+v} + \overline{\Psi}_{+v} \delta A \Psi_{+v} \]

Here

\[ P_{+v} \delta A P_{+v} = P_{+v} \delta v^\mu \left[ iD_\mu - iD_\mu \frac{1}{2m + ivD} iD_\perp - iD_\perp \frac{1}{2m + ivD} iD_\mu \frac{1}{2m + ivD} iD_\perp \right] \]

\[ - iD_\perp \frac{1}{2m + ivD} iD_\mu ] P_{+v} \]

\[ = P_{+v} \left\{ i\delta vD - iD_\perp \frac{1}{2m + ivD} iD_\perp \frac{1}{2m + ivD} iD_\perp \right\} P_{+v} \]

\[ \text{where we have used} \]

\[ \frac{\partial}{\partial v^\mu} \frac{1}{2m + ivD} = \frac{1}{2m + ivD} \left( -iD^\mu \right) \frac{1}{2m + ivD} \]

Now consider

\[ \delta \overline{\Psi}_{+v} A \Psi_{+v} + \overline{\Psi}_{+v} A \delta \Psi_{+v} = \overline{\Psi}_{+v} \left\{ [A, im\delta v_x] + A \delta M + \overline{\delta M} A \right\} \Psi_{+v} \]

where

\[ \delta M = \frac{\delta \phi}{2} + \frac{\delta \phi}{2} \frac{1}{2m + ivD} iD_\perp \]

Now

\[ [A, im\delta v_x] = [iv D + iD_\perp \frac{1}{2m + ivD} iD_\perp, im\delta v_x] \]

Firstly

\[ [iv D, im\delta v_x] = [iv \partial, im\delta v_x] - g[v A, im\delta v_x] = 0 \]

because of \( v \delta v = 0 \). Secondly we have

\[ [iD_\perp, im\delta v_x] = [iD, im\delta v_x] \]

\[ = [i\partial, im\delta v_x] = -m \delta \phi \]

\[ 16 \]

Thirdly

\[ \left[ \frac{1}{2m + ivD}, im\delta vx \right] = 0 \]  \hspace{1cm} (48)

And so we obtain

\[ [A, im\delta vx] = [i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp}, im\delta vx] \]

\[ = -mi\mathcal{D}_{\perp} \frac{1}{2m + ivD} \delta \phi - m\delta \phi \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \]  \hspace{1cm} (49)

Furthermore

\[ A\delta M = ivD \frac{\delta \phi}{2} + \frac{\delta \phi}{2m + ivD} i\mathcal{D}_{\perp} + i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} \]

\[ +i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \]  \hspace{1cm} (50)

Now

\[ P_{+} ivD \frac{\delta \phi}{2} P_{+} = 0 \]

\[ P_{+} i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} P_{+} = P_{+} i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} P_{-} \delta \phi \frac{1}{2} P_{+} \]

\[ = P_{+} i\mathcal{D}_{\perp} \frac{-ivD}{2m + ivD} \delta \phi P_{+} \]  \hspace{1cm} (51)

and so

\[ P_{+} A\delta MP_{+} = P_{+} \left[ \frac{\delta \phi}{2} \frac{ivD}{2m + ivD} i\mathcal{D}_{\perp} - i\mathcal{D}_{\perp} \frac{ivD}{2m + ivD} \delta \phi \right] \]

\[ +i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \]  \hspace{1cm} (52)

Similarly

\[ P_{+} \bar{\delta} M A P_{+} = P_{+} \left[ -\frac{\delta \phi}{2} \frac{ivD}{2m + ivD} i\mathcal{D}_{\perp} + i\mathcal{D}_{\perp} \frac{ivD}{2m + ivD} \frac{\delta \phi}{2} \right] \]

\[ +i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \]  \hspace{1cm} (53)

and finally (sandwiching between a pair of \( P_{+} \)'s is implied)

\[ A\delta M + \bar{\delta} M A = \frac{-\delta \phi}{2} \frac{ivD}{2m + ivD} i\mathcal{D}_{\perp} - i\mathcal{D}_{\perp} \frac{ivD}{2m + ivD} \frac{\delta \phi}{2} \]

\[ +i\mathcal{D}_{\perp} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \frac{\delta \phi}{2} \frac{1}{2m + ivD} i\mathcal{D}_{\perp} \]  \hspace{1cm} (54)
Plugging everything together, we find the variation of the Lagrangian:

\[
\delta L = \Psi + v \left\{ -m \frac{1}{2m + ivD} \delta \phi - m \delta \phi \frac{1}{2m + ivD} iD_\perp - \frac{1}{2} iD_\perp - \frac{1}{2m + ivD} iD_\perp \right\}
\]

\[
-iD_\perp \frac{ivD}{2m + ivD} \frac{1}{2} + iD_\perp \frac{1}{2m + ivD} i\delta vD - \frac{1}{2m + ivD} iD_\perp + i\delta vD
\]

\[
- \frac{1}{iD_\perp} \frac{1}{2m + ivD} i\delta vD - \frac{1}{2m + ivD} iD_\perp \Psi_+v
\]

\[
= \Psi_+v \left\{ i\delta vD - \frac{\delta \phi ivD + 2m}{2m + ivD} iD_\perp - iD_\perp \frac{ivD + 2m \delta \phi}{2m + ivD} \right\} \Psi_+v
\]

\[
= \Psi_+v \left\{ i\delta vD - i\delta vD \right\} \Psi_+v = 0
\]

which concludes the proof.

**B  Is Chen’s transformation unique?**

In this appendix we will show that Chen’s transformation law is not unique even in the restricted family of symmetries of the full Lagrangian derived from tree-level matching.

So let’s try to find the class of all reparameterization transformation which leave the Lagrangian

\[
\mathcal{L}^{\text{tree}}(v, \Psi_+v) = \Psi_+v A(v) \Psi_+v
\]

invariant (\(A(v)\) has been given in the previous section). So consider an infinitesimal transformation

\[
v \rightarrow v' = v + \delta v
\]

with

\[
\delta v \cdot v = 0
\]

What is the most general ansatz for the transformation \(\Psi_+v \rightarrow \Psi_{+v'}\)? The field \(\Psi_{+v'}\) must have two properties: (i) it must have the correct projection property \(P_{+v'} \Psi_{+v'} = \Psi_{+v'}\) and (ii) the derivative acting on it must produce the correct residual momentum \(k'\) instead of \(k\). The most general ansatz compatible with these two requirements is

\[
\Psi_{+v'} = \left[1 + im \delta v \cdot x\right] P_{+v'} B \Psi_+v
\]

where \(B\) must not depend explicitly on \(x\), but is otherwise arbitrary. \(B\) is a matrix in Dirac space and will contain covariant derivatives. Define

\[
B_\pm := P_{\pm v} B
\]
Then

\[ \Psi' = \left[ \left( 1 + i m \delta v \cdot x + \frac{\delta \phi}{2} \right) B_+ + \frac{\delta \phi}{2} B_- \right] \Psi \]  

(61)

For \( \delta v \to 0 \), we must have \( \Psi' = \Psi \). Therefore we can assume \( B_- \) to be of lowest order in \( \delta v \), i.e. \( B_- = O(\delta v)^0 \) and \( B_+ = 1 + \delta B_+ \), where \( \delta B_+ = O(\delta v) \). And so

\[ \Psi' = \left[ 1 + \delta B_+ + i m \delta v \cdot x + \frac{\delta \phi}{2} + \frac{\delta \phi}{2} B_- \right] \Psi \]  

(62)

At this point one can notice that the only combination of \( B_+ \) and \( B_- \) which enters the transformation law is

\[ \delta B_+ + \frac{\delta \phi}{2} B_- \]

but not \( B_+ \) or \( B_- \) themselves.

Using the various tricks and techniques of the previous section, one can now inserte this general transformation into the Lagrangian, and require its variation to vanish. Defining

\[ C_- := B_- - \frac{1}{2m + i \nu \cdot \vec{D}} i \bar{\psi} \]  

(63)

this finally leads to

\[ 0 = \delta \mathcal{L} = \bar{\psi} \left\{ A(v) \left[ \delta B_+ + \frac{\delta \phi}{2} C_- \right] + \left[ \frac{\delta B_+}{2} + \frac{\delta \phi}{2} \right] A(v) \right\} \psi \]  

(64)

A solution to this equation is \( \delta B_+ = 0 \) and \( C_- = 0 \). This is Chen’s transformation. Are there other solutions?

Firstly note again, that only \( \delta B_+ + \frac{\delta v}{2} B_- \) enters in the transformation law, i.e. only the sum \( \delta B_+ + \frac{\delta v}{2} C_- \) matters and solutions with \( \delta B_+ + \frac{\delta v}{2} C_- = 0 \) do not lead to different reparameterization transformations.

Secondly, however, there are solutions with \( \delta B_+ + \frac{\delta v}{2} C_- \neq 0 \). A non-trivial example is

\[ \delta B_+ + \frac{\delta \phi}{2} C_- = i \frac{i \delta v \cdot D A(v)}{m} \]  

(65)

Note that this transformation is 'class II' in a generalized sense, i.e. it vanishes for classical solutions of the full tree level effective Lagrangian with \( A(v) \psi = 0 \).
References


[13] C. L. Y. Lee, CALT-68-1663 (1991) (unpublished). Note that there exists a revised version of this preprint, in which a variety of errors have been corrected. The results in this revised version agree with those in [12, 14, 15].

