1992 – 1993 ACADEMIC TRAINING PROGRAMME

LECTURE SERIES FOR POSTGRADUATE STUDENTS

SPEAKER : A. MASIERO / CERN–TH
TITLE : Gauge theories for pedestrians
TIME : 8, 9, 12, 13 & 14 October, 11.00 to 12.00 hrs
PLACE : Auditorium

ABSTRACT

The series of lectures intends to provide a basic introduction to the major theoretical tools which enter the construction of the standard model of electroweak interactions and QCD. After reviewing the Fermi and intermediate vector boson IVB theories, I shall discuss global and local gauge symmetries and their spontaneous breaking. This will enable us to study spontaneously broken gauge theories where Higgs mechanism is operative. The final step of this course will be the presentation of the Lagrangian of the standard model and a discussion of its main features.
GAUGE THEORIES FOR PEDESTRIANS

A. Masiero
Cern and Univ. of Padua

WHY: to provide the basic ingredients of the framework (gauge theories) on which the standard model of electroweak interactions and QCD are based

HOW: at an elementary level (avoiding the theoretical intricacies of gauge theories)

WHAT:
1. Attempts to describe weak interactions outside the gauge theory framework
2. Global Symmetries
3. Local (gauge) symmetries
4. Yang-Mills Theories
5. Spontaneous Breaking of global symm.
6. The Higgs mechanism
7. General structure of the standard model Lagrangian
1. HOW TO DESCRIBE WEAK INTERACTIONS

ELM $\Rightarrow$ QED

WEAK $\Rightarrow$ QWD

in analogy with QED

perturbation theory

$\mathcal{H} \Psi_n = \mathcal{E}_n \Psi_n$

$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_I$

$\lambda \ll 1 \rightarrow$ interaction

renormalizability

QWD : first attempt FERMI THEORY (1933)

expression of the current-current interaction:

Feynman - Gell-Mann 1958

Hartshorn - Sudarshan 1958

Cabibbo 1963

$\mathcal{L}_F = \frac{\alpha}{\sqrt{2}} \, \bar{\Psi}(x) \, J^\mu(x) \, \Psi(x)$

basic structure of charged current hamiltonian at low energy

universal $V$-$A$ form

at the quark and lepton level
\[ \int \mu(x) = \ell_{\mu}(x) + h_{\mu}(x) \]

\[ \ell_{\mu}(x) = \bar{\mu}(x) \gamma_{\mu}(1 - \gamma_5) \gamma_\nu(x) + \bar{e}(x) \gamma_{\mu} (1 - \gamma_5) \gamma_\nu(x) \]

\[ V - A \]

\[ h_{\mu}(x) \rightarrow \text{can be identified with the currents of the chiral symmetry group } SU(3) \times SU(3) \text{ of the strong interactions} \]

\[ h_{\mu}(x) = \bar{F}(x) \gamma_{\mu} (1-\gamma_5) \left[ \cos \theta_c \eta(x) + \sin \theta_c \lambda(x) \right] \]

Good agreement of the Fermi theory with the experimental data (of that time) \textbf{BUT}

**PROBLEMS for the \textit{FERMI} THEORY**

**Unitarity limit**

\[ \frac{\sqrt{\sigma}}{\sqrt{\ell_{\mu}}} < \frac{\ell_{\mu}}{e} \]

\[ \sigma = \frac{G^2 s}{\pi} \]

Scattering theory \[ \sigma = \frac{\hbar^2 \pi}{s} \sum \frac{(2p + 1)(2q + 1)}{\ell} \]

\[ \ell = 0 \Rightarrow \frac{G^2 s}{\pi} > \frac{2 \pi}{3} \]

\[ s = \sqrt{\frac{\pi}{6}} (600 \text{ GeV})^2 \]

At very large energy at that time

\[ \int \frac{d^4 k}{k^2} \sim \Lambda^2 \]

\[ n^{th} \text{order ampl} \sim \mathcal{O}^n (\Lambda^2)^{n-1} \]
QUID: second attempt

$I^V_B$ (intermediate vector boson theory)

\[ \bar{u}_e \gamma_\mu (1 - \gamma_5) v_e - \frac{q^\mu q^\nu}{q^2 - M^2} \bar{u}_e \gamma_\nu (1 - \gamma_5) v_e \]

\[ g \int \frac{d^4x}{(2\pi)^4} \frac{g^\mu\nu}{k^2} \]

limit $M \gg m_e$, $M \gg q^2$ W propagator: $\sim \frac{g^\mu\nu}{M^2}$

\[ \frac{G}{12} \sim \frac{g^2}{M^2} \]

relation between $G_{\text{Fermi}}$ and g weak, $H_w$

* $q^\mu \sim m_e$ $q^\nu \sim m_e$ $\rightarrow \frac{g^\mu\nu}{q^2 - M^2}$ prevents infinite growth of $\sigma$ with $s$

**But unitarity and renormalizability problems still present**
$$e^+e^- \rightarrow W^+W^-$$

$$\sigma \sim \frac{G_F}{3\pi}$$

but one must add also

$$\gamma$$

the bad behaviour $\sigma \sim s$ can be eliminated in the sum of these two diagrams provided that the electromagnetic coupling $e$ and the weak coupling $g$ are RELATED

⚠️ AED + QWD $\Rightarrow$ electroweak theory

but even if one succeeds to realize the above cancellation, how do you cancel

the bad behaviour of $\nu\nu \rightarrow W^+W^-$

$$\sigma \sim s$$

this coupling does not exist!
if you believe in this theory and you are brave enough ⇒ invent a new intermediate vector boson which is NEUTRAL

Then:

\[ e^+ \gamma \rightarrow W \]

the mass and couplings of \( Z \) should be chosen to realize the planned cancellation

⚠ hint at the existence of NEUTRAL CURRENTS

however, even if we are ready to pay the above prices to save the unitarity (at tree level) of the IVB theory, still we are lacking the essential property of renormalizability

\[ \int \frac{d^4 k}{k^2} \]

QED is more convergent
2. **GLOBAL SYMMETRIES**

Lagrangian field theory $\mathcal{L}(\psi, \partial \psi)$

$$S = \int d^4x \, \mathcal{L} \quad 0 = \delta S = \int dx \left[ \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial \psi)} \delta (\partial \psi) \right]$$

eqs. of motion: $\frac{\partial \mathcal{L}}{\partial \psi} - \partial \mu \frac{\partial \mathcal{L}}{\partial \partial \mu \psi} = 0$

**global symmetry if:** $\psi \rightarrow \psi' = \psi + \delta \psi$ such that:

i) it leaves $\mathcal{L}$ invariant: $\mathcal{L}(\psi') = \mathcal{L}(\psi)$

ii) it is the same at all space-time points

**internal if it corresponds to transformations which commute with the space-time transformations**

**Ex.:** isospin $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ or $\psi = \begin{pmatrix} k^+ \\ k^0 \end{pmatrix}$

$$\psi' = \psi + \delta \psi \quad \delta \psi = i \, \epsilon^{a} \frac{\tau^{a}}{2} \, \psi(x)$$

$$\left[ \tau_{a}, \frac{\tau_{b}}{2} \right] = i \, \epsilon_{abc} \, \frac{\tau_{c}}{2}$$
\[ L \text{ is isospin invariant if } \delta L(\psi) = \mathcal{L}(\psi) \rightarrow \delta \mathcal{L} = 0 \]

\[ 0 = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi_x} \delta \psi_x + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_x} \delta \partial_\mu \psi_x = \frac{\partial \mathcal{L}}{\partial \psi_x} \left( i \varepsilon^a \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_x} \right) \right) \left( \frac{\tau^a}{2} \right)_{\alpha \beta} \psi_\beta + + i \varepsilon^a \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_x} \left( \frac{\tau^a}{2} \right)_{\alpha \beta} \psi_\beta \right] \]

\[ J^a_\mu = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_x} \left( \frac{\tau^a}{2} \right)_{\alpha \beta} \psi_\beta \Rightarrow \partial^\mu J^a_\mu = 0 \]

Noether's theorem: conserved charges

\[ I^a = \int d^3x J^a_0(\vec{x}, t) \]

\[ \frac{d}{dt} I^a = \int d^3x \partial_\mu J^a_0 = \int d^3x \left[ \partial^\nu J^a_\nu + \vec{\nabla} \cdot \vec{J}^a \right] = \int d^3x \vec{\nabla} \cdot \vec{J}^a = 0 \]

Upon quantization of the theory: \( I^a \) become operators, equal time commutation relations

\[ [I^a(t), I^b(t)] = i \varepsilon_{abc} I^c(t) \]
I give a realization (or representation) of the commutation rules that correspond to the algebra of the infinitesimal rotations in a three-dimensional isospin-space.

\[ I^a = i [H, I^a] = 0 \rightarrow I^a \text{ commute with } H \]

\[ \rightarrow \text{they connect states with the same energy} \]

(States appear in isospin multiplets)

Concerned currents for spin-\(\frac{1}{2}\) and spin-0 fields

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{interaction}} \]

\[ \mathcal{L}_0 = \overline{\psi} \left( i \gamma^\mu - m \right) \psi \quad (\text{spin-} \frac{1}{2} \text{ field}) \]

\[ \mathcal{L}_0 = (\partial_\mu \psi^\dagger) (\partial^\mu \psi) - m^2 \overline{\psi} \psi \quad (\text{spin-0 complex field}) \]

if direct does not contain derivative couplings

\[ \rightarrow \frac{\partial \mathcal{L}}{\partial \overline{\psi} \psi} = \frac{\partial \mathcal{L}_0}{\partial \overline{\psi} \psi} \]

\[ J_\mu = \overline{\psi} \gamma_\mu \frac{1}{2} \gamma^\tau \psi \quad \text{spin } \frac{1}{2} \]

\[ J^a_\mu = -i \left[ \partial_\mu \psi^\dagger \frac{1}{2} \gamma^a \psi - \psi^\dagger \frac{1}{2} \gamma^a \partial_\mu \psi \right] = -i \psi^\dagger \frac{1}{2} \gamma^a \partial_\mu \psi \quad \text{spin } 0 \]
For Fermi fields also $\gamma_5$ can be involved

$$\delta_5 \psi = i \epsilon^{\alpha} \frac{\gamma^\alpha}{2} \gamma_5 \psi$$

**CHIRAL TRANSF.**

$$\left[ \frac{\gamma^a}{2}, \frac{\gamma^b}{2} \right] = i \epsilon_{abc} \frac{\gamma^c}{2}$$

$$\left[ \frac{\gamma^a}{2}, \frac{\gamma_5}{2} \gamma_5 \right] = i \epsilon_{abc} \frac{\gamma^c}{2} \gamma_5$$

$$\left[ \frac{\gamma^a}{2} \gamma_5, \frac{\gamma^b}{2} \gamma_5 \right] = i \epsilon_{abc} \frac{\gamma^c}{2}$$

$$L^a = \frac{1 + \gamma_5}{2} \gamma^a; \quad R^a = \frac{1 - \gamma_5}{2} \gamma^a$$

$$[L^a, L^b] = i \epsilon_{abc} L^c; \quad [R^a, R^b] = i \epsilon_{abc} R^c$$

$$[L^a, R^b] = 0$$

**SU(2)_L \otimes SU(2)_R**

For manifold fermions $\gamma_5 = 2\chi \ (helicity)$

$$\delta_5 u(p) = \left( \frac{\vec{\gamma} \cdot \vec{p}}{E} + \frac{m}{E} \gamma_5 \gamma_0 \right) u(p)$$

Limit $m \to 0 \ (or \ E \to \infty) \to \gamma_5 \ u(p) = \frac{\vec{\gamma} \cdot \vec{p}}{E} u(p) = 2 \chi u(p)$

$L, R \ (L = \frac{1 + \gamma_5}{2}, \ R = \frac{1 - \gamma_5}{2})$ project out lefthanded or righthanded states

Chiral trans. are independent isospin rotations

performed over the two helicity states of the fermion field

$$\delta_5 \psi_{\alpha\beta} = -2 \epsilon^{\alpha} \gamma_5 \gamma_{\frac{3}{2}} \gamma_5 \psi_{\alpha\beta} \quad \text{if} \quad m = 0 \ \delta_5 \psi_{\alpha\beta} = 0$$
Generalization:

\[ \Phi^i(x) \quad i = 1, \ldots, n \quad \mathcal{L} \left[ \Phi^i(x), \partial_\mu \Phi^j(x) \right] \]

Internal symmetry is an invariance of \( \mathcal{L} \) under a group \( G \) of transformations acting on \( \Phi^i(x) \):

\[ \Phi^i(x) \rightarrow \Phi^i(x) + \Theta^a (T^a)^j_i \Phi^j(x) \quad i = 1, \ldots, n \quad a = 1, \ldots, N \]

\( N \rightarrow \) number of generators of \( G \)

\( T^a \) matrices of the representation to which the \( \Phi^i(x) \) belong.

ex. \( G = SU(2) \) \( \Phi^i \) isodoublet \( \rightarrow \) \( T^a = i \sigma^a \)

\( \Phi^i \) isovector \( \rightarrow \) \( (T^a)^i_j = \epsilon_{aij} \)

\( \Theta^a \rightarrow N \) c-numbers, infinitesimal, \( x \)-independent function.

\( \checkmark \) since \( \Theta^a \) are \( x \)-independent

\[ \Rightarrow \partial_\mu \Phi^i(x) \rightarrow \partial_\mu \Phi^i(x) + \Theta^a (T^a)^i_j \partial_\mu \Phi^j(x) \]

i.e. \( \Phi^i \) and \( \partial_\mu \Phi^i \) transform in the same way.
3. FROM **GLOBAL** TO **LOCAL** SYMMETRY

symmetry $\rightarrow$ certain choices are purely conventional and have no effect on the dynamics

- in **LOCAL FIELD THEORIES** it is more desirable that such choices can be performed locally, i.e. the choice I make at $x$ should be independent from what I chose at $x'$

$\Rightarrow$ the parameter of the transformation becomes $x$-dependent

$$\delta \psi = i \frac{\bar{\psi}(x)}{2} \varepsilon^a(x) A^a \psi(x)$$

local or gauge transformation
the QED path:

\[ \mathcal{L} = \overline{\psi}(x) \left( i \gamma^\mu D_\mu - m \right) \psi(x) \]

invariant under:

\[ \psi(x) \rightarrow e^{i \Theta(x)} \psi(x) \]

\[ \text{global symm.} \]

\[ \partial_\mu \psi(x) \rightarrow e^{i \Theta(x)} \partial_\mu \psi(x) \]

\[ \rightarrow j_\mu(x) = \overline{\psi}(x) \gamma_\mu \psi(x) \quad \partial_\mu j_\mu(x) = 0 \]

try now to promote \( U(1) \) to a local symmetry:

\[ \psi(x) \rightarrow e^{i \Theta(x)} \psi(x) \]

\[ \partial_\mu \psi(x) \rightarrow e^{i \Theta(x)} \partial_\mu \psi(x) + i e^{i \Theta(x)} \psi(x) \partial_\mu \Theta(x) \]

⇒ invent a new derivative \( D_\mu \) such that

\[ D_\mu \psi(x) \rightarrow e^{i \Theta(x)} D_\mu \psi(x) \]

i.e. such that \( \psi(x) \) and \( D_\mu \psi(x) \) transform in the same way under the \( U(1) \) local symmetry

answer:

\[ D_\mu \equiv \partial_\mu + i e A_\mu \]

with \( A_\mu \) transforming as:

\[ A_\mu(x) \rightarrow A_\mu(x) - \frac{i}{e} \partial_\mu \Theta(x) \]

under the local \( U(1) \) symm. (prove per credere)
\[ \mathcal{L}_0 = \bar{\psi}(x) \left( i \slashed{D} - m \right) \psi(x) \]

invariant under global \( U(1) \)

\[ \mathcal{L}_1 = \bar{\psi}(x) \left( i \slashed{D} - m \right) \psi(x) = \]

\[ = \bar{\psi}(x) \left( i \slashed{D} - m \right) \psi(x) - e \bar{\psi}(x) \gamma_\mu \dot{\psi}(x) A_\mu(x) \]

invariant under local \( U(1) \)

to interpret \( A^\mu(x) \) as the photon \( \Rightarrow \) add kinetic energy terms for \( A^\mu \) :

\[ \mathcal{L}_2 = \mathcal{L}_1 - \frac{e}{4} \mathcal{F}_{\mu\nu}(x) \mathcal{F}^{\mu\nu}(x) \]

\[ \mathcal{F}_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \]

⚠️ \( \mathcal{L}_2 \) does not contain a term proportional to \( A_\mu A^\mu \) since such a term is not gauge invariant (i.e. it is not left invariant under a local \( U(1) \) transformation).
Summary of the 1st Lecture

1. Attempts for a description of weak interaction, based on the QED example
   a) Fermi theory
      \[ \frac{\alpha}{4\pi} \text{ four fermion contact interaction} \]
      \[ \text{current-current V-A} \]
      problems: violations of unitarity and renormalizability.
      \[ \alpha \sim \frac{\pi}{137} \sim 1 \]

   b) IVB theory
      \( \gamma \rightarrow W \)
      unitarity at tree level recovered if
      e.g. connected + introduction of \( Z \)
      problem: renormalizability still lacking
      hint: in QED \( U(1) \) local invariance responsible for better convergence properties

2. Global Symmetries
   Noether theorem
   \( \xi, D \xi \xi \) transform in the same way

3. Local (gauge) symmetries
   \[ \delta \Phi = i \epsilon^a(x) \frac{\tau^a}{2} \Phi(x) \]
   \( D \mu \Rightarrow D^{\mu} \) covariant derivative
Non-Abelian Gauge Symmetries

ex.: $SU(2)$ local (gauge) $SU(2)$ symmetry

$$
\delta \psi = i \varepsilon^a(x) \frac{\tau^a}{2} \psi(x)
$$

$$
\delta \phi = i \varepsilon^a(x) \frac{\tau^a}{2} \partial_\mu \phi + i \left( \partial_\mu \varepsilon^a(x) \right) \frac{\tau^a}{2} \phi(x)
$$

$$
L_{\text{free}} = i \bar{\psi} \gamma^\mu \partial_\mu - m \bar{\psi} \psi
$$

$$
\delta (\bar{\psi} \gamma_\mu \psi) = i \bar{\psi} \varepsilon^a(x) \gamma_\mu \tau^a \gamma_\nu \frac{\tau^a}{2} \psi \neq 0
$$

$$
A^a_\mu : D_\mu \psi = \left( \partial_\mu + i g A^a_\mu \frac{\tau^a}{2} \right) \psi
$$

$$
\delta (D_\mu \psi) = i \varepsilon^a(x) \frac{\tau^a}{2} D_\mu \psi
$$

$$
\delta A^a_\mu = - \frac{i}{g} \partial_\mu \varepsilon^a(x) - \varepsilon^{abc} E^b(x) A^c_\mu (x)
$$

NEW peculiar of non-abelian gauge symm.

A function of $\psi$ and $D_\mu \psi$ invariant under the $SU(2)$ global transformations is also invariant under $SU(2)$ local.
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g \varepsilon^{abc} A_\mu A_\nu^b A_\nu^c \]

\[ \delta F_{\mu\nu} = - \varepsilon_{abc} \varepsilon^b(\alpha) F_{\mu\nu}^c \]

\[ \Rightarrow L = - \frac{i}{4} F_{\mu\nu}^a (F^2)^{\mu\nu} \text{ is gauge invariant} \]

From this SU(2) ex. \( \Rightarrow \) extension to general simple groups \( G \) (i.e. groups such that we cannot divide the generators into two or more sets of mutually commuting generators)
RECIPE:
\[ L(\psi, \partial \mu \psi) \text{ invariant under } G \]  
\[ \Downarrow \]
\[ L' \text{ invariant under } G \text{ LOCAL is obtained by the minimal prescription} \]
\[ \partial \mu \psi \rightarrow \partial \mu \psi \]
\[ L' = L(\psi, \partial \mu \psi) \]

⚠️ the interaction of \( \psi \) with \( A_{\mu} \) is COMPLETELY prescribed only unknown parameter \( g \) coupling \( \overline{\psi} \gamma_{\mu} \psi A_{\mu} x g \) (geometrical principle to constrain the \( \psi - A \) interactions)
4. **YANG-MILLS INTERACTIONS**

In the case where $\psi$ has no other interaction but the Y-M ones, $L_{\text{total}}$ is **COMPLETELY DETERMINED** by kinematics and by the requirement of **GAUGE Symmetry**.

Ex.: $\psi \rightarrow$ doublet under $SU(2)$ gauge symmetry

\[ \psi \rightarrow \text{choice of the representation of } \psi \]
\[ \text{choice of the gauge group} \]

$L_{\text{total}}$ is completely specified

\[ L = \bar{\psi} (i\gamma_\mu - m) \gamma^\mu \frac{\partial}{\partial \gamma^\mu} - \frac{1}{2} \partial_\mu A^a_{\nu} (\partial^\nu A^{\mu a} - \frac{1}{2} \delta^{\mu \nu} A_{\mu}^{aa'}) \]

$- g A^a_{\nu} \bar{\psi} \gamma^\nu \gamma^a \psi \rightarrow$ free propagator of $A, \psi$

$+ g \epsilon_{abc} A^a_{\mu} A^b_\nu \gamma^\nu A^{\mu c} - \frac{g^2}{4} \epsilon_{abc} \epsilon_{adp} A^a_{\mu} A^b_\nu A^c_{\rho} A^d_{\tau}$

self-interaction of $A^a_{\mu}$ with themselves
eq. of motion in the free limit \( g = 0 \)

\[
\frac{\partial L}{\partial \psi} - \partial_{\mu} \frac{\partial L}{\partial \partial_{\mu} \psi} = 0 \quad \frac{\partial L}{\partial A_{\mu}^a} = 0
\]

\[
\frac{\partial L}{\partial A_{\mu}^a} = \partial_{\nu} A_\nu^a - \partial_{\mu} A_\mu^a
\]

\[
(g^{\mu \lambda} \Box - \partial^\mu \partial^\lambda) A_\lambda^a (x) = 0
\]

Fourier transform \((\partial \mu \to k_\mu)\)

\[
(-g^{\rho \lambda} k_\rho + k_\mu k_\lambda) A_{\lambda}^a = G_{\rho \lambda}^{\mu \lambda} A_\lambda^a (k) = 0
\]

The inverse of \( G^{\rho \lambda} \) is the propagator for the A-field.

**BUT** \( G^{\rho \lambda} \) has \( \Box \) inverse!

\[
[G^{\rho \lambda}]^{-1} = A k_\rho g^{\lambda \sigma} + B k_\rho k_\sigma \quad \text{form dictated by}
\]

\( (A \text{ and } B \text{ functions of } k^2) \)

\( \text{Lorentz covariance} \)

\[
(-g^{\rho \lambda} k_\rho + k_\mu k_\lambda) (A k_\sigma g^{\lambda \sigma} + B k_\lambda k_\sigma) = A (-\delta_\rho^\sigma k_\sigma + k_\rho k_\sigma)
\]

\( \text{A function } A \text{ and } B \)
Why is it so? Redundancy of the degrees of freedom

\[ A^\mu \rightarrow 4\text{-}components \ of \ A^\mu \ are \ NOT \ independent \ dynamical \ degrees \ of \ freedom \ (remember \ a \ similar \ situation \ for \ the \ photon \ in \ QED) \]

we may require \[ \partial^\mu A^\mu(x) = 0 \] (gauge condition).

then the eq. of motion becomes: \[ \Box A^\mu(x) = 0 \]

\[ \Rightarrow \ A^\mu(x) = 0 \]

\[ g_{\mu\lambda} \frac{\partial^\lambda}{\partial^2} A^\mu(x) = 0 \]

\[ \Rightarrow \ Feynman \ propagator: \ D_{\mu\nu}(k) = G^{-1}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2} \]

\[ \text{or} \]

method of Lagrangian multipliers

\[ \text{add to } \mathcal{L}_{\text{free}} - \frac{1}{2\alpha} \left( \partial^\mu A^\mu(x) \right) \left( \partial^\lambda A^\lambda(x) \right) \text{ that vanishes when } \partial^\mu A^\mu = 0. \]

Then, take the variation of the Lagrangian with fixed \( \alpha \)

\[ \left[ -g_{\mu\lambda} \Box + \left( 1 - \frac{1}{\alpha} \right) \partial^\mu \partial^\lambda \right] A^\mu(x) = 0 \]

\[ \Rightarrow \text{propagator} \ D_{\mu\nu} = \frac{1}{k^2} \left[ g_{\mu\nu} - \left( 1 - \frac{1}{\alpha} \right) \frac{k_{\mu} k_{\nu}}{k^2} \right] \]
\[ \alpha = 1 \rightarrow g_{\mu \nu}/k^2 \quad \text{Feynman propagator} \]
\[ \alpha = 0 \rightarrow \left[ g_{\mu \nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right] \frac{1}{k^2} \quad \text{Landau gauge propagator} \]

Pole at \( k^2 = m^2 \) of the propagating particle
in our case \( k^2 = 0 \Rightarrow A_{\mu} \) are MASSLESS PARTICLES !

Interaction \( A - \psi \) terms:

\[-g A^a_{\mu} \bar{\psi} \gamma^\mu \frac{\tau^a}{2} \psi = -g A^a_{\mu} (J^a)^{\mu}_{\nu} \]

\((J^a)^{\mu}_{\nu}\) Noether current associated to the global \( I \)-spin symmetry before the introduction of the \( A_{\mu} \).

Valid also in the presence of further matter fields \( \psi \Rightarrow \text{UNIVERSALITY of the coupling of gauge fields to } I \)-spin carrying matter fields

\[ \triangle J^a_{\mu} \text{ is NOT conserved in the presence of gauge fields} \rightarrow \text{conserved } J^a_{\mu \text{ total}} = i \varepsilon^{abc} f^{bc}_{\nu} A^c_{\nu} + J^a_{\mu} \]
gauge fields themselves carry \( I \)-spin.
\[ A \cdot \psi \cdot \bar{\psi} \Rightarrow \text{matrix element of } \mathcal{L} \text{ for emission or absorption of a gauge particle} \]

\[ A \mu \text{ by an electron or a neutrino if } \psi = (\gamma \text{ e}) \]

\[ \psi_{\text{out}} \]
\[ x = (\vec{x}, t) \]
\[ \psi_{\text{in}} \]
\[ \phi \]
\[ \phi' \]
\[ x' = (\vec{x'}, t') \]

in this case \( h \mu J^{\mu} = h' \mu J^{\mu'} = 0 \) (\( J^{\mu} \) and \( J^{\mu'} \) are conserved \( \Rightarrow \) no \( A \mu \) interactions among \( A \mu \) fields themselves)

\[ A = g^2 J^{\mu} D^{\nu}(k) J^{\nu} = g^2 \frac{1}{h^2} \frac{1}{i} J^{\mu} \]

\[ = \frac{g^2}{h^2} \left[ \vec{J}_0 \cdot \vec{J}' - \vec{J} \cdot \vec{J}' \right] \]
\( k = (\omega, \mathbf{q}) \quad k^2 = \omega^2 - \mathbf{q}^2 \)

choose \( \mathbf{q} \parallel z \) axis

\( k^\mu J_\mu = 0 \rightarrow \omega J_0 - q J_3 = 0 \quad \Rightarrow J_3 = \omega J_0 \)

\[
A = q^2 \left[ \frac{J_0 J_0'}{k^2} - \left( \frac{J_1 J_1' + J_2 J_2'}{k^2} \right) - \frac{\omega^2}{k^2} J_0 J_0' \right]
\]

\[
= -q^2 \left[ \frac{1}{q^2} J_0 J_0' + \frac{1}{k^2} \left( J_1 J_1' + J_2 J_2' \right) \right]
\]

\( \text{instantaneous coulombic} \)

\( \text{interactions among} \)

\( \text{fermions} \quad \frac{1}{q^2} \rightarrow \delta(t-t') \frac{1}{|x-x'|} \)

\( J_0, J_0' \) give the charge densities of the two external particles

\( * \ J_1 J_1' + J_2 J_2' = \left( \frac{J_1 + i J_2}{\sqrt{2}} \right) \left( \frac{J_1' - i J_2'}{\sqrt{2}} \right) + \left( \frac{J_1 - i J_2}{\sqrt{2}} \right) \left( \frac{J_1' + i J_2'}{\sqrt{2}} \right) \)

\( \text{two polarizations} \quad J_2 = \pm \mathbf{z} \quad \Rightarrow \text{massless} \ \tau \)
But we don't see manifest vector bosons around in the electroweak world apart from the photon.

How can we provide a mass to $A_\mu^\alpha$?

1st method → by brute force (as it often occurs, it doesn't work)

Add an explicit mass term

$$\frac{1}{2} M^2 A_\mu^\alpha A^{\alpha \mu} \leftrightarrow$$

this term violates the $SU(2)$ gauge invariance

then the eq. of motion reads:

$$(g^{\beta \lambda} \Box - \partial^\beta \partial^\lambda + M^2 g^{\beta \lambda}) A_\lambda^\alpha (x) = 0$$

$\Rightarrow$ (F.T.)

$$(-g^{\beta \lambda} \partial^2 + \partial^\beta \partial^\lambda + M^2 g^{\beta \lambda}) A_\lambda^\alpha (k) = 0$$

this $G^{\beta \lambda}$ admits an inverse (indeed, gauge invariance has been broken!)
\[ D_{\mu\nu} (k, M^2) = \frac{1}{\not{u}^2 - M^2} \left(-g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{M^2} \right) \]

\[ A = g^2 \frac{1}{\not{u}^2 - M^2} \mathcal{J}_\mu \mathcal{J}^\mu \]

for \( M^2 \gg k^2 \Rightarrow A = -\frac{g^2}{M^2} \mathcal{J}_\mu \mathcal{J}^\mu \)

But for large \( k \)

\[ D_{\mu\nu} \sim 1 \quad \text{(massive case)} \]

\[ D_{\mu\nu} \sim k^{-2} \rightarrow 0 \quad \text{(previous massless case)} \]

\[ \Rightarrow \text{massive theory is much less convergent in the ultraviolet region} \]

\[ A = -g^2 \left\{ \frac{\mathcal{J}_0 \mathcal{J}^0}{k^2 + M^2} + \frac{1}{\not{u}^2 - M^2} \left( \frac{M^2}{k^2 + M^2} \mathcal{J}_3 \mathcal{J}^3 + \mathcal{J}_1 \mathcal{J}^1 + \mathcal{J}_2 \mathcal{J}^2 \right) \right\} \]

Instantaneous interaction

is no more coulombic

\[ \frac{1}{\not{k}^2 + M^2} \Rightarrow \frac{e}{k^2 - \not{k}^2} \]

Yukawa type interaction

Pole at \( \not{u}^2 = M^2 \)

Propagation of massive waves

3 types of waves

\( J_\sigma = 0 \) new degree of freedom

Longitudinal polarized waves
THE Y.-M. DILEMMA

unbroken gauge invariance

\[ \text{RENAME} \]

\[ \text{NO PHENOM. RELEVANCE} \]

all the intermediate vector bosons are massless

\[ \text{broken gauge invariance (through explicit mass terms for } A_\mu) \]

\[ \text{phenomenologically satisfactory} \]

\[ \text{lacking Renormalizability} \]

look for some way of breaking gauge symmetry (hence allowing for massive vector bosons)

without putting EXPLICIT breaking terms
5. **SPONTANEOUS SYMMETRY BREAKING OF GLOBAL SYMMETRIES**

1) symmetric problems possess symmetric solutions

2) symmetric solutions are **stable** $\Rightarrow$ small deviations from perfect symmetry will induce only small departures from the symmetric solutions

1) an symm. solution 2) its stability

2) is not always true

Ex.:

A rod on which acts the force $F$ along $\xi$

$X(\xi)$ and $Y(\xi)$ $\Rightarrow$ deviations along the $x$ and $y$ directions of the axis of the rod at $\xi$ from the symmetric solution (i.e. the solution in which the rod is aligned) $\xi(\xi)$

Eqs. of elasticity

\[
\begin{align*}
IE\frac{d^4X}{d\xi^4} + F\frac{d^2X}{d\xi^2} &= 0 \\
IE\frac{d^4Y}{d\xi^4} + F\frac{d^2Y}{d\xi^2} &= 0
\end{align*}
\]
\[ I = \frac{\pi R^4}{4} \quad \text{moment of inertia of the rod} \]

\[ E \quad \text{Young modulus} \]

\[ \text{symmetric solution} \quad X = Y = 0 \]

Is there any other possible solution?

Yes when \( F = F_{\text{crit}} = \frac{\pi^2 EI}{L^2} \)

\[ \text{non-symmetric solution} \quad X = C \sin \theta \quad \text{and} \quad Y = n \pi ; \quad n = 1, \ldots \]

The study of the stability problem proves that the non-symmetric solutions correspond to lower energy. (At \( F > F_{\text{crit}} \) the previous eqs. of elasticity — valid for small deflections — are no longer valid and one can show that the new non-symmetric solutions hold for any \( F > F_{\text{crit}} \)).

⚠️ Symmetry still present: we cannot predict in which direction in the x-y plane the rod is going to bend — applying a symm. transformation — rotation around z — to an asymmetric solution, we obtain another asymmetric solution degenerate with the first one.
There exist various physical systems (in classical and quantum physics) which exhibit analogous behaviour concerning their symmetries; other ex. the Heisenberg ferromagnet.

In these cases the symmetry is said to be *spontaneously broken*

features: *a) E critical point
(critical value of some quantity which determines the appearance of non-symmetric solutions)

*b) beyond the critical point
the symmetric solution becomes **UNSTABLE**

*c) beyond the critical point
the ground state becomes **degenerate**
Ex. of spontaneous symmetry breaking in field theory

\( \phi(x) \) complex scalar field

\[ L = (\partial \phi)(\partial \phi^*) - \mu^2 \phi \phi^* - \lambda (\phi \phi^*)^2 \]

\( L \) invariant under: \( \phi(x) \rightarrow e^{i\sigma} \phi(x) \)

\[ H = \pi^\phi + \pi^+ \phi^+ - L \]

\[ \pi = \frac{\partial L}{\partial (\partial \phi)} = \partial \phi^* \], \[ \pi^+ = \partial \phi \]

\[ H = (\partial \phi)(\partial \phi^*) + (\vec{\partial} \phi)(\vec{\partial} \phi^*) + V \]

\[ V = \mu^2 \phi \phi^* + \lambda (\phi \phi^*)^2 \]

\[ V \) vanishes for \( \phi = \text{const.} \)

\( V \) must be bounded from below \( \rightarrow \lambda > 0 \)

GROUND STATE \( \rightarrow \) configuration which corresponds to absolute minimum of the energy (HAMILTON) density

\[ \phi(x) = \text{const} = \phi_0 \] \( V(\phi_0^* \phi_0) \) minimum
The position of the minimum of $V$ depends on the sign of $\mu^2$

a) $\mu^2 \geq 0$  minimum at $\phi = 0$ (symmetric solution)

\[ V \]
\[ \phi \]

b) $\mu^2 < 0$  there exists a whole circle of minima at the complex $\phi$-plane with radius $\sqrt{-\mu^2/2\lambda} \equiv \nu$

\[ V \]
\[ \phi \]

→ any point on the circle corresponds to a spontaneous breaking of the symmetry $\phi \rightarrow e^{i\theta}\phi$

*critical point $\mu^2 = 0$
* $\mu^2 \geq 0$  symmetric solution is stable
* $\mu^2 < 0$  spontaneous symmetry breaking
1) \( \mu^2 \geq 0 \) \( (\Box^2 + \mu^2) \phi(x) = 0 \)

propagation of a spin zero complex field with mass \( \mu \) particle, associated to oscillations of \( \phi \) around \( \phi_0 = 0 \) \[ L'' = \phi'' + \phi^4, \]

\[ \phi'' = (\partial^2 x \phi) (\partial \phi) - \mu^2 \phi^3 + \phi \]

2) \( \mu^2 < 0 \) lowest energy configuration (vacuum) corresponds to \( \phi = \phi_0 \neq 0 \)
in this case it doesn't make sense to conjugate oscillations of \( \phi \) around \( \phi = 0 \), i.e. to consider \[ L'' = \phi'' + \phi^4, \]
since \( \phi = 0 \) is not a minimum, rather one should "expand" around \( \phi = \phi_0 \) \[ \phi(x) = \phi_0 + \eta(x) ; \quad L'' = \phi'' + \phi^3'' + \phi^4, \]
the true particle spectrum is determined by the eq. of motion obtained from the "\( \eta'' \) terms"
Choice: there is no loss of generality by choosing a particular point on the circle

ex.: choose the point on the real axis in the $\Psi$-plane

$$\Psi(x) = \frac{1}{\sqrt{2}} \left[ \psi(x) + i \chi(x) \right]$$

this is the $\gamma(x)$ of the previous way of writing

$$\Psi(x) = \psi(x) + i \chi(x)$$

$$L(\Psi) = (\partial_\mu \psi)(\partial^\mu \psi^*) - \mu^2 \psi^* \psi - \frac{i}{2} (\psi^* \psi)^2$$

$$L(\Psi, \chi) = \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 \rightarrow \text{kin. terms}$$

$$- \frac{1}{2} (2 \lambda \psi^2) \psi \rightarrow \text{mass term for } \psi$$

$$- \lambda \psi (\psi^2 + \chi^2) - \frac{1}{4} (\psi^2 + \chi^2)^2 \rightarrow \text{interaction terms}$$
Important! \( L(\phi) \) and \( L(\phi, \chi) \) are completely equivalent: both describe the dynamics of the same physical system and a change of variable \( \phi \rightarrow \chi \) cannot change the physics.

But this equivalence is only true if we can solve the problem exactly. In applying perturbation theory the equivalence is no longer guaranteed.

If we use \( L(\phi) \) with \( \mu^2 < 0 \), the unperturbed Hamiltonian would consist of particles with negative square mass and no perturbative corrections, at any finite order, could change that.

In \( \mu^2 < 0 \), \( L(\phi, \chi) \) gives a reasonable spectrum:

\( \phi \) and \( \chi \) two interacting real scalar particles with \( M_\phi^2 = 2 \lambda \sigma^2 \), \( M_\chi^2 = 0 \).
$m \chi = 0 \rightarrow$ example of a general situation that arises when one breaks a global symmetry spontaneously:

**Goldstone's Theorem**: To every generator of a spontaneously broken symmetry there corresponds a massless particle ($\rightarrow$ Goldstone particle)

(this theorem is just the translation into the quantum field theory language of the statement about the degeneracy of the ground state)
EXAMPLE OF SPONTANEOUS BREAKING OF A GLOBAL NON-ABELIAN SYMMETRY

Consider $SU(2)$ I-spin $\varphi = \left(\begin{array}{c} \varphi^+ \\ \varphi^0 \end{array}\right) = \left(\begin{array}{c} \frac{\varphi_1 + i \varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i \varphi_4}{\sqrt{2}} \end{array}\right)$

usual potential $V = \mu^2 \varphi \varphi^* + \lambda (\varphi \varphi^*)^2$

$\varphi \varphi^* = \frac{1}{2} \sum_{i=1}^{4} \varphi_i^2$

choice of $\mu^2 > 0$ that spontaneous breaking occurs $\varphi_0 \neq 0$ $\varphi_0$ is a doublet since we have the $SU(2)$ I-spin invariance we can rotate the I-spin frame so that $\varphi_0$ is a "down" isospinor $\varphi_0 = (\eta) \eta = \sqrt{-\frac{\mu^2}{\lambda}}$

\[ \begin{array}{c}
\text{I-spin frame} \\
\text{\varphi_0 is "down" everywhere} \\
\rightarrow \text{global symm.}
\end{array} \]

perturbation of the ground state $\varphi_0$ at the point $x \Rightarrow$ small deviation from $\varphi_0$, i.e.

\[ \varphi_0(x,t) = \left(\begin{array}{c} \eta \\ \eta_+ \eta(x) \end{array}\right) \]

in a new I-spin frame slightly tilted from the previous one.
The two I-spin frames are related by an I-spin rotation:

$$U = e^{i \tau_a \Theta^a(x)}$$

(don't be confused: the SU(2) symmetry is global, but the "angle" of rotation \( \Theta^a \) depends on \( x \) since the perturbation I perform from the ground state can be different at every space-time point)

Then \( \varphi(x) \) is not a "down" isospinor in the original I-spin frame where \( \varphi_0 \) is "down" and the amount of deviation from being "down" is the rotation \( U(x) \). Hence in the original I-spin frame \( \varphi(x) \) reads:

$$\varphi(x) = U(x) \left( \begin{array}{c} 0 \\ \eta + \frac{\delta(x)}{\sqrt{2}} \end{array} \right)$$

$$\delta(x) = \varphi(x) - \varphi_0 = U(x) \left( \begin{array}{c} 0 \\ \eta + \frac{\delta(x)}{\sqrt{2}} \end{array} \right) - \left( \begin{array}{c} 0 \\ \eta \end{array} \right)$$

for small perturbations we obtain

$$\delta(x) \simeq \left( \begin{array}{c} \eta (\theta^2 + i \theta^3) \\ \frac{\delta(x)}{\sqrt{2}} - i \eta \theta^3 \end{array} \right) \rightarrow 4 \text{ real functions (fields)} \ \theta^1, \theta^2, \theta^3 \text{ and } \sigma$$
We rewrite $L(q, \phi^+)$ in terms of the new fields $\theta_1(x), \theta_2(x), \theta_3(x)$ and $\sigma(x)$:

$$\partial_{\mu} \phi^+ \partial_{\mu} \phi = \partial_{\mu} (\phi^+_0 + s^+(x)) \partial_{\mu} (\phi^+_0 + s(x)) =$$

$$= \partial_{\mu} s^+ \partial_{\mu} s = \frac{1}{2} (\partial_{\mu} \sigma)(\partial_{\mu} \sigma) + \eta^2 (\partial_{\mu} \Theta^a)(\partial_{\mu} \bar{\Theta}^a)$$

+ higher order terms

$$-V(\phi^+ \phi) = -V[(\eta + \frac{\sigma}{\sqrt{2}})] = \text{const} - \frac{1}{2} (4 \lambda \eta^2) \sigma^2 + \text{h.o.t.}$$

$$\Rightarrow UV^+ = 1 \Rightarrow \theta_1, \theta_2, \theta_3 \text{ are not present!}$$

Hence the quadratic terms are:

$$\tilde{\sigma} = \frac{1}{2} (\partial_{\mu} \sigma)(\partial_{\mu} \sigma) - \frac{1}{2} m^2 \sigma^2 + \frac{1}{2} (\partial_{\mu} \bar{\Theta}^a)(\partial_{\mu} \Theta^a)$$

$$\bar{\Theta}^a = \sqrt{2} \eta \Theta^a; \quad m^2 = 4 \lambda \eta^2 = -2 \mu^2 > 0$$

particle spectrum

$$\begin{cases} 
3 \text{ massless bosons } \bar{\Theta}^a(x) \\
1 \text{ massive scalar boson with } m = \sqrt{-2\mu^2} \text{ associated to } \sigma
\end{cases}$$

3 Goldstone bosons!

(However this does not mean that $SU(2)$ is broken to "nothing".... (see next page)
\[ \mathcal{L} = (\partial \mu \varphi^+)(\partial^\mu \varphi) - \mu^2 \varphi^+ \varphi - \lambda (\varphi^+ \varphi)^2 \]

the invariance group of \( \mathcal{L} \) is **larger than** \( SU(2) \)

**SU(2) rotations**: \( \delta \varphi = i \varepsilon^a \frac{\sigma^a}{2} \varphi \)

but also if I read \( \varphi \) into \( e^{i \varepsilon} \varphi \), \( \mathcal{L} \) is left invariant

**U(1) rotation**: \( \delta \varphi = i \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varphi \)

the unit matrix commutes with the Pauli matrices \( \tau^a \). Hence the correct symmetry group is:

\[ SU(2) \otimes U(1) \]

Which symmetry remains unbroken when \( \varphi \) gets the **vacuum expectation value** \( \varphi_0 \) \( (\langle \varphi \rangle = \varphi_0) \)?

symmetry of the vacuum, i.e. What are the transfs. which leave \( \varphi_0 \) invariant?

\[ \downarrow \text{translation} \]

which generators of \( SU(2) \times U(1) \) give zero when applied to \( \varphi_0 \)?
$T\Phi_0 = 0$ implies that

$$e^{i\mathbf{T}\Phi_0} = (1 + \frac{i}{2}\mathbf{T})\Phi_0 = \Phi_0$$

terms which contain $T$ acting on $\Phi_0$.

If $T\Phi_0 = 0$ then the symmetry transformation associated to the generator $T$ leaves $\Phi_0$ (i.e. the vacuum) invariant. In other words, the symmetry associated to $T$ remains unbroken when $\Phi_0$ gets the VEV $\bar{\Phi}_0$

$$\mathbf{T}\Phi_0 = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) (\begin{pmatrix} \eta \\ 0 \end{pmatrix}) = (\begin{pmatrix} \eta \\ 0 \end{pmatrix}) \neq 0$$

$\mathbf{T}_2\Phi_0 \neq 0$, $\mathbf{T}_3\Phi_0 \neq 0$, $\mathbf{T}_3\Phi_0 \neq 0$

However, there exist a combination of $\mathbf{T}_3$ and $\mathbf{1}$ which leaves $\Phi_0$ invariant:

$$\frac{1 + \mathbf{T}_3}{2} = \frac{1}{2}(\begin{pmatrix} 1 + 1 & 0 \\ 0 & 1 - 1 \end{pmatrix}) = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \rightarrow \frac{1 + \mathbf{T}_3}{2}\Phi_0 = 0$$

$$SU(2) \otimes U(1) \rightarrow U(1)'$$

$\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{1} \rightarrow \frac{1 + \mathbf{T}_3}{2}$
6. **Spontaneous Breaking of Local Symmetries: The Higgs Phenomenon**

We consider the previous Lagrangian

\[ \mathcal{L} = (\partial_{\mu} \phi^*) (\partial^\mu \phi) - V(\phi^* \phi) \]

invariant under the global \( SU(2) \times U(1) \) symmetry.

We promote the global \( SU(2) \times U(1) \) to a local (gauge) \( SU(2) \times U(1) \) symmetry by using the minimal substitution \( \partial_{\mu} \rightarrow D_{\mu} \)

\[ \mathcal{L} = (D_{\mu} \phi)^* (D^\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{g} G_{\mu \nu} G^{\mu \nu} \]

\( G_{\mu \nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \)  
\( B_{\mu} \) gauge field of \( U(1) \)

\( F_{\mu \nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a - g \epsilon^{abc} A_{\mu}^b A_{\nu}^c \)  
\( A_{\mu}^a \) gauge fields of \( SU(2) \)

\[ D_{\mu} \phi = (\partial_{\mu} + i g A_{\mu}^a \frac{\epsilon^a}{2} + i g' B_{\mu} \frac{1}{2}) \phi \]

\( g \rightarrow \text{gauge coupling of } SU(2) \)

\( g' \rightarrow \text{gauge coupling of } U(1) \)
vacuum: \( \varphi_0 (A^\mu)_0 (B^\mu)_0 \)

ground state must be Lorentz invariant

\[ (A^\mu)_0 = (B^\mu)_0 = 0 \]

Since at the minimum \( A^\mu \) and \( B^\mu \) are zero, we are back to the same situation as in the global case, i.e.

\( \mu^2 < 0 \) symmetric solution \( \varphi_0 = 0 \)

\( \mu^2 > 0 \) spontaneous symmetry breaking \( \varphi_0 \neq 0 \)

Let's discuss this case

Important difference between global and local cases:

GLOBAL

\( \varphi(x) \) is a "down spinor," i.e. I choose the \( \text{I}-\text{spin} \) frame in such a way that \( \varphi(x) \) is "down" then at \( x' \) I cannot choose the \( \text{I}-\text{spin} \) frame so that \( \varphi(x') \) is again "down" because the previous choice is made at all space-time points

\( \varphi(x) \rightarrow \text{point } x' \rightarrow \) I need a rotation \( U(x') \) to bring \( \varphi(x) \) to the "down spinor position"
**LOCAL CASE**: at \( x' \), I can choose a DIFFERENT orientation of the \( I \)-spin frame so that \( \psi(x') \) is also a "down spinor"

\[
\Rightarrow \forall x \quad \psi(x) = \begin{pmatrix} 0 \\ \eta + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix}
\]

without any need of introducing the rotation \( U(x) \), i.e. the functions (fields) \( \Theta^1(x), \Theta^2(x), \Theta^3(x) \) do not show up in this case.

in other words the gauge \( SU(2) \) symmetry allow us to absorb the matrix \( U(x) \) into the redefinition of the isospin axes.

the paradox of the degrees of freedom:

\[
\begin{align*}
I-\text{doublet} & \quad \rightarrow \quad \psi = \begin{pmatrix} 0 \\ \eta + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix} \\
4 \text{ real fields} & \quad \rightarrow \quad \text{only ONE real field } \sigma(x)
\end{align*}
\]
Solution of the paradox:

particle mass spectrum $\Rightarrow$ plug $\varphi(x) = \left( \begin{array}{c} 0 \\ \eta + \sigma(x) \end{array} \right)$ in

the initial lagrangian $\mathcal{L} = (D_\mu \varphi)(D^\mu \varphi) - V(\varphi^+ \varphi) - \frac{F_{\mu \nu} F^{\mu \nu}}{4} - G_{\mu \nu} G^{\mu \nu}$ and find all the terms which are bilinear in the fields $\sigma$, $A_\mu^a$ and $B_\mu^a$

$$(D_\mu \varphi^+)(D^\mu \varphi) \rightarrow \left( \frac{1}{2} \left( \partial_\mu \sigma \right) \left( \partial^\mu \sigma \right) +$$

$$+ \frac{1}{2} \left( \hat{g}^2 \hat{y}^2 \right) \left[ A_\mu^a A_\mu^a + A_\mu^a A_\mu^a \right] +$$

$$+ \frac{1}{4} \eta^2 \left( g A^3_{\mu} - g' B^3_{\mu} \right) \left( g A^3_{\mu} - g' B^3_{\mu} \right)$$

$$- V(\varphi^+ \varphi) \rightarrow \left( \frac{1}{2} \left( -2 \mu^2 \right) \sigma^2$$

$\left( \hat{g}^2 \hat{y}^2 \right) \left[ A_\mu^a A_\mu^a + A_\mu^a A_\mu^a \right] +$}

$$- \frac{F_{\mu \nu} F^{\mu \nu}}{4} - \frac{G_{\mu \nu} G^{\mu \nu}}{4} \rightarrow - \frac{1}{4} \left( \partial_\nu A^a_{\tau} - \partial_\tau A^a_{\nu} \right)^2 - \frac{1}{4} \left( \partial_\nu B^a_{\tau} - \partial_\tau B^a_{\nu} \right)^2$$

$*$ $A_\mu^a$ and $B_\mu^a$ are not mass eigenstates

$\Rightarrow$ need to diagonalize the $A_\mu^3 - B_\mu^3$ mass matrix
\[
\begin{pmatrix}
A_\mu^3 \\
A_\mu^3 \\
B_\mu
\end{pmatrix}
\begin{pmatrix}
g^2 \\
-g^2 \\
g^2
\end{pmatrix}
\times \frac{\eta^2}{4}

\rightarrow \text{ one massless eigenstate (indeed only}
\text{ the combination } g A_\mu^3 - g' B_\mu \text{ has a}
\text{ mass term, while the orthogonal combination}
\text{ remains massless)}

we diagonalize the above mass matrix making
use of the orthogonal matrix \((\cos \theta \ -\sin \theta)
\text{ Relation between the initial (current) eigenstates}
\text{ and the mass eigenstates (}Z_\mu \text{ and } A_\mu\text{)}:
\[
\begin{pmatrix}
Z_\mu \\
A_\mu
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
A_\mu^3 \\
B_\mu
\end{pmatrix}
\]
\[ Z_\mu = \cos \theta \ A_\mu^3 - \sin \theta \ B_\mu \]
\[ A_\mu = \sin \theta \ A_\mu^3 + \cos \theta \ B_\mu \]
\[ \tan \theta = g'/g \]

We rewrite the parts containing the terms bilinear in the fields in the new basis \( \sigma, A_\mu, A_\mu, A_\mu \) and \( Z_\mu \):

\[ \mathcal{L}_{\text{free}} = \frac{1}{2} (\partial_\mu \sigma)(\partial_\nu \sigma) - \frac{1}{2} (-\xi^2) \sigma^2 - \]
\[ - \frac{1}{4} A_{\mu \nu}^1 A_{\mu \nu}^1 + \frac{1}{2} \left( \frac{g^2 \eta^2}{2} \right) A_{\mu}^1 A_{\mu}^1 \]
\[ - \xi \rightarrow 2 \]
\[ - \frac{1}{4} Z_{\mu \nu} Z_{\mu \nu} + \frac{1}{2} \left( \frac{g^2 \eta^2}{2 \cos^2 \theta} \right) Z_{\mu} Z_{\mu} \]
\[ - \frac{1}{4} A_{\mu \nu} A_{\mu \nu} \]
Masses of the gauge bosons:

\[ \frac{1}{2} A_\mu^2 A_\mu^2 \frac{g^2 \eta^2}{2} \quad \frac{1}{2} A_\mu^2 A_\mu^2 \frac{g^2 \eta^2}{2} \quad M^2 = \frac{g^2 \eta^2}{2} \]

\[ \frac{1}{2} \bar{Z}_\mu Z_\mu \frac{g^2 \eta^2}{2 \cos^2 \theta} \quad A_\mu A_\mu = 0 \]

\[ M_Z^2 = M^2 / \cos^2 \theta \]

\[ \tan \theta = g' / g \]

**Identification of** \( A_\mu \) **with the photon**

in \( D_\mu \Rightarrow g A_\mu^3 \frac{\tau_3}{2} + g' \frac{1}{2} \gamma^\mu B_\mu = \)

= \( g \frac{\tau_3}{2} (\cos \theta Z_\mu + \sin \theta A_\mu) + g' \frac{1}{2} \gamma^\mu (\sin \theta Z_\mu + \cos \theta A_\mu) \)

= \( \bar{Z}_\mu \left( g \frac{\tau_3}{2} \cos \theta - g' \frac{1}{2} \sin \theta \right) + A_\mu \left( g \frac{\tau_3}{2} \sin \theta + g' \frac{1}{2} \cos \theta \right) \)

= \( \bar{Z}_\mu \frac{\tau_3}{2 \cos \theta} - \sin^2 \theta \left( \frac{\tau_3 + 1}{2} \right) + A_\mu \frac{g \sin \theta}{2} \left( \frac{\tau_3 + 1}{2} \right) \)

\( \frac{\tau_3 + 1}{2} \Rightarrow \) generator corresponding to the remaining \( U(1)' \) unbroken symmetry.

\( \Rightarrow \frac{\tau_3 + 1}{2} = Q \) generator of \( U(1)_{\text{em}} \) \( A_\mu \) coupled to \( Q \Rightarrow g \sin \theta \).
Summary of the spectrum:

**unbroken phase:** when $SU(2) \times U(1)$ is the local symmetry, i.e. before spont. symm. breaking

- $4$ I-doublet degrees of freedom
- $A^\mu$ I-particle massless
- $B^\mu$ massless

\[
\frac{2}{12}
\]

**broken phase:** after the spontaneous breaking

$SU(2) \times U(1) \rightarrow U(1)_{\text{em}}$

- $\sigma$ massive (real scalar field) #freedom 1
- $A^\mu, A^\rho$ massive mass $H^2 = \frac{g^2}{2} \# \rightarrow 3 \times 2$
- $Z^\mu$ mass $H^2 = \frac{M^2}{\cos^2 \theta} \# \rightarrow 3$
- $A^\mu$ massless (photon) \# \rightarrow $\frac{2}{12}$

$Q = \frac{1 + q_2}{2}$; relation between $\mu$ and $q$:

$e = g \sin \theta$ (since $g = g'/g \Rightarrow e = g' \cos \theta$)
7. Comments on the Renormalizability of
Spontaneously Broken Gauge Theories

\[ L_{1} = \left| (\partial_{\mu} - i e A_{\mu}) \phi \right|^{2} - \mu^{2} \phi \phi^{*} - \lambda (\phi \phi^{*})^{2} - \frac{1}{4} F_{\mu \nu}^{2} \]

\( \phi \) complex scalar \( L_{1} \) invariant under the local \( U(1) \) symmetry: \( \phi \to e^{i \theta(x)} \phi; A_{\mu} \to A_{\mu} - \frac{1}{e} \partial_{\mu} \theta(x) \)

degrees of freedom: \( 2 + 2 = 4 \)

After spontaneous breaking: \( \phi(x) = \frac{1}{\Delta} \left[ \sigma + g(x) + i \chi(x) \right] \)

\[ L_{2} = \frac{1}{2} (\partial_{\mu} \phi)^{2} + \frac{1}{2} (\partial_{\mu} \chi)^{2} - \frac{1}{4} F_{\mu \nu}^{2} + \]

\[ + \frac{e^{2} \sigma^{2}}{2} A_{\mu}^{2} - \frac{1}{2} (2 \lambda \sigma^{2}) \phi^{2} - e \sigma A_{\mu} \partial_{\mu} \chi + h.s.t. \]

\( L_{2} \) is still \( U(1) \) gauge invariant:

\[ \sigma(x) \to \cos \theta(x) \left[ \sigma(x) + \nu \right] - \sin \theta(x) \chi(x) - \nu \]

\[ \chi(x) \to \cos \theta(x) \chi(x) + \sin \theta(x) \left[ \sigma(x) + \nu \right] \]

\( A_{\mu}(x) \to A_{\mu}(x) - \frac{1}{e} \partial_{\mu} \theta(x) \)

\( \Rightarrow \) notice that it is possible to have a mass term \( A_{\mu}^{2} \) and \( U(1) \) still present!!
degree of freedom: $5 \times A_\mu \\

1 + 1 + 3 (A_\mu \text{ vanirve}) \\

in $L_1$ only 4. In this spectrum 1 degree of freedom is unphysical

hint: $A_\mu \not\partial_\mu \not\nabla$ \\

we must redefine $A_\mu$ and $X$ to eliminate this "off-diagonal" derivative coupling.

$\Phi(x) = \frac{1}{\sqrt{2}} \left[ \phi + \sigma(x) \right] e^{i\frac{\Phi(x)}{\sqrt{2}}}$ \\

$A_\mu (x) = B_\mu + \frac{1}{i\nu} \not\partial \Phi(x)$ \\

$\not\nabla = -\frac{i}{4} B_\mu \nu + \epsilon^{\mu \nu \rho \sigma} B_\mu B_\rho + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} (2\lambda \nu^2) j^\mu \\

- \frac{i}{4} j_\mu + \frac{i}{\nu} \not\partial X_\nu (2\nu \phi + \xi^2)$ \\

$\Phi(x)$ disappears in $L_3$!

degree of freedom: $B_\mu \text{ vanirve 3 + } \Phi \text{ real scalar field and 1} \\
3+1 = 4$!

($L_3$ is not invariant under gauge transformation)
for his proof of renormalizability 't Hooft starts with $L_2$ (which is still gauge invariant) and adds
\[ -\frac{1}{\varepsilon} \left[ \partial_\mu A_\mu(x) - H X(x) \right]^2 \]
with the gauge condition $\partial_\mu A_\mu(x) - H X(x) = 0$ \( (H^2 = e^2 \nu^2) \)

Then $A_\mu(x) \partial_\mu X(x)$ disappears – the quadratic terms in $A_\mu(x)$ are:
\[ \frac{1}{\varepsilon} H^2 A^2 - \frac{1}{\varepsilon} (\partial_\mu A_\mu)^2 \]

\[ X(x) = \frac{1}{\varepsilon} (\partial_\mu X)^2 - \frac{1}{\varepsilon} H^2 X^2 \]

\[ (\not{D} + H^2) A_\mu(x) = 0 \Rightarrow D^{\mu \nu} = \frac{g^{\mu \nu}}{H^2 - H^2} \]

\[ \frac{\not{X}}{H^2} \]

disappears in the $A_\mu$ propagator

$D^{\mu \nu} \Rightarrow$ good convergence properties essential for proof of renorm.

$X$ unphysical $\Rightarrow$ eliminated in the unitary gauge ($L_3$) similarly to the elimination of the longitudinal and scalar photons in QED

($X$ is gauge dependent but all the observable quantities are gauge invariant like in QED)
8. A REALISTIC EXAMPLE OF SPONT. BROKEN
GAUGE THEORY: THE STANDARD MODEL (of EW inter.)

from the IUVB theory:

\[ \Sigma(x) \gamma_\mu \left( \frac{1-\gamma_5}{2} \right) \psi \]

\[ = \bar{e}_L \gamma_\mu e_L \Rightarrow \text{this suggests to put } \chi_L \text{ and } e_L \]
in an SU(2) doublet

\[ \psi = (v_e) \quad \text{SU(2) infinitesimal transformation on } \psi : \]

\[ \delta \psi = i \varepsilon^a \frac{\gamma^a}{2} \left( \frac{1-\gamma_5}{2} \right) \psi = i \varepsilon^a \left[ L^a \right] \psi \]

\[ L^a = \frac{i}{2} \left( 1-\gamma_5 \right) \Rightarrow \text{chiral isospin generator} \quad \text{(see 1st lecture)} \]

introduce \[ \psi_L \equiv \left( \frac{1-\gamma_5}{2} \right) (v_e) \equiv (\chi_L) \text{ or } (e_L) \]

then:

\[ \delta \psi_L = i \varepsilon^a \frac{\gamma^a}{2} \psi_L \]

no V+A current \Rightarrow \[ \delta e_R = \delta \left( \frac{1+\gamma_5}{2} e \right) = 0 \]

right-handed fields behave like SU(2) weak iso-singlets
According to our previous example in which \( SU(3) \times U(1) \) could be spontaneously broken to \( U(1)_{em} \), we choose our gauge symmetry to be:

\[
\begin{cases}
SU(2) \times U(1) \\
\text{Gauge Group}
\end{cases}
\]

of the ELW interactions

since \( SU(2) \) acts only on the left-handed fermionic fields it is sometimes denoted as \( SU(2)_L \) — it is called gauged (weak) isospin)

\( U(1) \); this charge is called hypercharge \( U(1)_Y \)

what are the values of hypercharge of \( \nu \) and \( e^- \)?

\[
\delta \nu = 1 \times e^{-} \]

Knowing the electric charge and the isospin number of the fermions, it is possible to find the values of the hypercharge that they carry.

The \( Y \) eigenvalues are determined by requiring the photon \( (A_{\mu} = \pi \nu \sigma A^3 + \nu \nu \sigma B_{\mu}) \) to couple to the electric charge. Let’s see how this works determining \( Y_{L}, Y_{R}, Y_{L}, Y_{R} \)
\[ \psi = \begin{pmatrix} \nu_L \\ e_L \\ e_R \\ \nu_R \end{pmatrix} \quad \delta_Y \chi = i e \chi \chi \rightarrow 4 \times 4 \text{ matrix} \]

\[ \Rightarrow \bar{\psi} \gamma^\mu \left[ g A^3_{\mu} L^3 + g^1 B^\mu Y \right] \psi = \]

\[ = \bar{\psi} \gamma^\mu \left[ A_{\mu} (g \sin \theta L^3 + g^1 \cos \theta Y) + \right. \]

\[ \left. Z_{\mu} (g \cos \theta L^3 - g^1 \sin \theta Y) \right] \psi \]

\[ = \bar{\psi} \gamma^\mu \left\{ A_{\mu} \begin{pmatrix} \nu \nu \nu \nu \end{pmatrix} \right. \]

\[ \left. + \underbrace{Z_{\mu}}_{\Omega = L^3 + Y} \begin{pmatrix} \nu \nu \nu \nu \end{pmatrix} \right\} \psi \]

\[ L_3 = \begin{pmatrix} \tau_{3/2} & 0 \\ 0 & \vdots \end{pmatrix} \Rightarrow \begin{array}{c|cccc} L_3 & \nu_L & e_L & \nu_R & e_R \\ \hline \hline Y & +1/2 & -1/2 & 0 & -1 \\ \hline \end{array} \]

\[ D_\mu \psi_L = D_\mu (e_L) = (\partial_\mu + ig \frac{\gamma^a}{2} A^a_\mu - ig^1 \frac{\gamma^1}{2} B_\mu) \psi_L \]

\[ D_\mu e_R = (\partial_\mu - ig^1 B_\mu) e_R \]

\[ D_\mu \nu_R = \partial_\mu \nu_R \]
the part of the $\mathcal{L}$ describing the weak and
elem. interactions of the leptons is:

\[ \mathcal{L}_{\text{eff}} = i \left[ \bar{\psi}_e \gamma_\mu D^\mu \psi_e + \bar{e}_\mu \gamma_\mu D^\mu e_e \right] \]

where $\psi_e = (\nu_i, e_i)^T$, $i = e, \mu, \tau$ ($\nu_\alpha$ does not carry $\text{SU}(2) \times \text{U}(1)$
charges)

charged leptonic currents:

\[ \sum_{\nu_e} \bar{\nu}(x) \gamma^\mu (1 - \gamma_5) \nu(x) \]

\[ J^\mu_a(x) = \bar{\psi}_e \gamma^\mu \frac{\tau_a}{2} \psi_e \]

\[ J^\mu_1(x) = i \left[ \frac{1}{2} \bar{\psi}(x) i \gamma_5 \gamma^\mu \psi(x) \right] = \bar{\nu}(x) \gamma^\mu (1 - \gamma_5) \nu(x) \]

\[ J^\mu_2(x) = \bar{\nu}(x) \gamma^\mu (1 - \gamma_5) e(x) \]

**Definition:** \[ W_\mu(x) = \frac{i}{\sqrt{2}} \left[ A^\mu_\tau(x) - i A^\mu_e(x) \right] \]

\[ \Rightarrow \int J^\mu_a(x) A^\mu_a(x) + \int J^\mu_1(x) A^\mu_2(x) = \frac{1}{2\sqrt{2}} \left[ \int J^\mu_1(x) W_\mu(x) + \int J^\mu_2(x) W_\mu^+(x) \right] \]

$W_\mu$ and $W_\mu^+$ (its conjugate) have well-defined el. charge:

$W_\mu \rightarrow \text{el. charge} + 1$ \hspace{1cm} ($J^\mu_+ W_\mu$ must be neutral)

$W_\mu^+ \rightarrow \text{el. charge} - 1$
Re-expressing $\psi^\mu$ in the new basis of the gauge bosons which makes use of $A_\mu, Z_\mu, W^\mu$

$$\psi_{\text{left gauge bosons}} = e \bar{e}_\gamma \gamma_\mu e_i A_\mu$$

$$- \frac{g}{4\cos\theta} \left[ \bar{e}_\gamma \gamma_\mu (1 - \gamma_5) e_i - \bar{e}_\gamma \gamma_\mu (1 - \gamma_5 \sin^2\theta - \gamma_5) e_i \right] Z_\mu$$

$$- \frac{g}{2\sqrt{2}} \bar{e}_\gamma \gamma_\mu (1 - \gamma_5) e_i W_\mu + \frac{\hbar c}{\sqrt{2}} \bar{e}_\gamma \gamma_\mu (1 - \gamma_5) e_i W^\mu$$

Relation between $M_W$ and $G_F$

$\nu_e + e \rightarrow \mu + \bar{\nu}_e$

at energies $\ll M_W$

charged current interactions

$$A = \frac{g^2}{8 M_W^2} \left[ \bar{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e \right] \left[ \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e \right]$$

Fermi current current theory

$$A = \frac{G}{\sqrt{2}} \left[ \bar{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e \right] \left[ \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e \right]$$

$$M_W^2 = \frac{g^2}{4\sqrt{2} G} = \frac{e^2}{4\sqrt{2} G} \frac{1}{\sin^2\theta} = \left( \frac{\alpha}{\sqrt{2} G} \right) \frac{1}{\sin^2\theta} = \frac{(37.5 \text{ GeV})^2}{\sin^2\theta}$$
Neutral current interactions

\[ \nu_e + e \rightarrow \nu_e + e \]

\[ A = \frac{g^2}{16 \cos^2 \theta W^2} \left[ \bar{\nu}_e \gamma^\mu \left( 1 - \gamma^5 \right) \nu_e \right] \left[ \bar{e} \gamma_\mu \left( g_V - g_A \gamma^5 \right) e \right] \]

\[ g_V = 1 - 4 \cos^2 \theta \quad ; \quad g_A = 1 \]

The electron neutral current is not pure V-A

\[ \frac{g^2}{16 \cos^2 \theta W^2} = \frac{g^2}{16 M_W^2} = \frac{G_F}{2 \sqrt{2}} \rightarrow \text{the scale of the above amplitude is determined by } G \text{ only!} \]

---

Extension to quarks (hadron interactions)

\[
\begin{bmatrix}
  (\nu_e) & (e_R) & (\nu_e) \\
  (e) & (d_R) & (u_R) \\
  (d) & (u) & (d)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  (\nu_\mu) & (\mu_R) & (\nu_\mu) \\
  (\mu) & (s_R) & (c_R) \\
  (s) & (c) & (s)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  (\nu_\tau) & (\tau_R) & (\nu_\tau) \\
  (\tau) & (b_R) & (t_R)
\end{bmatrix}
\]

1st generation (or family) \hspace{2cm} 2nd family \hspace{2cm} 3rd family

same quantum numbers

before giving way to fermions as way of distinguishing the three replicas
INTERACTIONS OF THE GAUGE BOSONS AMONG THEMSELVES

The $SU(2) \times U(1)$ local invariance fixes the interactions of the gauge bosons among themselves:

For $F^a_\mu$: Given $F^a_\mu$ in the $A_\mu, Z^\mu, W$ basis they give rise to:

\[
\mathcal{L}_{\text{boson-boson}} = i g \cos \theta_W \left[ \left( W^+_\mu W^-_\mu - W^+_\nu W^-_\nu \right) \right. \\
\left. + (2 e W^-_\mu - e W^-_\nu) W^+ A^\mu \right] + i e \left[ (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) \right. \\
\left. + e W^+ A^\mu - W^+ A^\nu \right] \right] \quad \rightarrow \text{W-W-A-A}
\]

\[
+ g^2 \cos \theta_W \left[ W^+ \Delta^\mu - W^+ A^\mu \right] \rightarrow \text{W-W-Z-Z}
\]

\[
+ e^2 \left[ W^+ A^\mu A^\nu - W^+ A^\mu A^\nu \right] \rightarrow \text{W-W-W-W}
\]

\[
\rightarrow \text{CRUCIAL TEST OF NON-ABELIAN GAUGE THEORIES}
\]

\[
i g^2 \left[ 2 g^2 \gamma_5 g^5 - g^5 \gamma_5 g^5 - g^5 \gamma_5 g^5 \right] \ldots
\]
Before symmetry breaking

Once we choose the gauge symmetry group and the way fermions transform under it, all the fermion-gauge boson interactions and gauge boson interactions among themselves are completely determined in terms of only two unknown parameters $g, g'$.

However, the gauge bosons are massless after spontaneous symmetry breaking.

$H \rightarrow N_w + N_z$ related by $\theta_W$, $N_w \propto \nu$

Fermion-gauge sector interactions in terms of $g, g'$ and $\nu$

$G_f, e \rightarrow g, g', \nu$ are determined by $G_f$ and $e$, the remaining unknown to be determined by experiment for instance $\sin^22\theta_W$. 
**THE HIGGS SECTOR**

- SU(2) doublet 4 real fields
- $SU(2) \times U(1)_Y \to U(1)_{em}$
- 3 Goldstone bosons (2 el. charged, 1 neutral)
- If I choose a convenient gauge (unitary gauge) they disappear from the final Lagrangian
- They contribute the longitudinal components of $W^+, W^-, Z^0$
- 1 physical neutral (well) scalar field
  - The Higgs Boson *(WANTED!!!)*

**Interactions of the Higgs boson with itself and with the gauge bosons:**

- From $V(\phi \phi^*) \Rightarrow \mathcal{L}_{H-H} = -\frac{1}{4} \lambda H^4 - \lambda \phi H^3$
- From $(\partial_{\mu} \phi)^* (\partial_{\mu} \phi) \Rightarrow \mathcal{L} = \frac{1}{2} g^2 W_x^+ W_x^{-} H + \text{h.-gauge terms}$
  + $\frac{1}{4} g^2 W_x^+ W_x^{-} H^2 + \frac{\alpha g^2}{4 \sin^2 \theta_W} \frac{2}{2} \frac{Z_x^2 Z_x^2 H^2}{8 \cos^2 \theta_W} \frac{\partial_{\mu} \phi}{\partial_{\mu} \phi}$

- $\mathcal{L}_{H-gauge} \Rightarrow g \mathcal{G}, \partial_{\mu} \phi$
- $\mathcal{L}_{HH} \rightarrow$ additional parameter $\lambda$
THE PROBLEM OF FERMION Masses

\[ SU(2)_L \rightarrow \text{Chiral Invariance} \quad L^a = \tau^a (1 - i \gamma_5) \]

from the 1st lecture implementation of chiral invariance if \( m_{\text{ferm}} = 0 \)

\[ \Rightarrow \text{asking for SU}(2)_L \times U(1)_R \text{ to be an exact symmetry of the initial Lagrangian prevents any possible fermion mass term to be put before symmetry breaking. Let's see this point in more detail:} \]

Local group locally isomorphic to \( SU(2) \times SU(2) \)
two component spinors transform as \( (\frac{1}{2}, 0) \) or \( (0, \frac{1}{2}) \)
under the local group

\[ X (\frac{1}{2}, 0) \rightarrow \text{mass term} \quad \exp X^a X^b \quad (a, b = 1, 2) \]

\[ \Rightarrow \text{Majorana mass term in the standard model} \ X \text{ can be: } \]
\[ e_L (e^c)_L \quad \nu_L \quad d_L (u^c)_L \quad (d^c)_L \]

\[ X X \text{ carries } Y \neq 0 \text{ forbidden} \]

(only possible exception \( (\nu^c)_L \) - a term \( (\nu^c)_L \) \( (\nu^c)_L \)
is allowed by \( SU(2)_L \times U(1)_R \) since \( (\nu^c)_L \) does not transform under this group.)
\[ \chi (\frac{1}{2}, 0) \eta (0, \frac{1}{2}) \quad \chi \eta (\frac{1}{2}, \frac{1}{2}) \text{ it is not a Lorentz scalar} \]

\[ \eta^+ (\frac{1}{2}, 0) \quad \bar{\psi}_l \chi \eta^+ (0, 0) \text{ Lorentz scalar} \]

\[ \chi \rightarrow e_L \quad \eta \rightarrow e_R \]

\[ \eta^* (e^c)_L \quad e_L (e^c)_L \rightarrow \bar{e}_R e_L \]

\[ \text{SU(2)}_L \text{ doublet} \quad \text{SU(2)}_R \text{ singlet} \]

\[ \text{forbidden by SU(2)}_L \]

\[ \bar{e}_R e_L, \quad \bar{u}_R u_L, \quad \bar{d}_R d_L, \quad \bar{\nu}_R \nu_L \]

All these Dirac mass terms are forbidden by the gauge SU(2)_L invariance.

As long as SU(2)_L x U(1)_Y is a good symmetry there is no way to provide a mass for fermions which transform chirally under this symmetry (i.e. all the fermions of the SM - possible exception \( \nu_R \) it doesn't transform under SU(2)_L x U(1)_Y)
H - fermion interactions

\[
\bar{\psi}(x) \gamma^\mu \psi(x) \quad \gamma \text{ spinor}
\]

\[
\text{Yukawa interaction} \quad \phi \text{ scalar}
\]

\[
\mathcal{L}_Y = h_{e_i} \bar{L}_i \Phi e_R^j + h_{d_{ij}} \bar{Q}_i \Phi d_R^j + h_{u_{ij}} \Phi Q_R^i \Phi u_R^j + h.c.
\]

\[
\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_0 \end{pmatrix} \quad \bar{\Phi} = i (\Phi^T \tau_2) = \begin{pmatrix} \Phi_0^* \\ -\Phi_- \end{pmatrix}
\]

ij generation indices \( h_e, h_d, h_u \) 3x3 matrices in generation space

\[
L = (e) \quad Q = (d)
\]

\[
L, SU(2) \text{ doublet} \quad \Phi, SU(2) \text{ doublet or SU(2) singlet}
\]

Combine the two to form an SU(2) singlet

the Yukawa terms respect the \( SU(2)_L \times U(1)_Y \) invariance

when \( \langle \phi_0 \rangle \neq 0 \), i.e. after spontaneous symmetry breaking

\[
M_{\text{left}} = h_{u_{ij}} \langle \phi_0 \rangle, \quad M_u = h_u \langle \phi_0 \rangle, \quad M_d = h_d \langle \phi_0 \rangle
\]

(i.e. \( v_R \) is added \( \Rightarrow M_u = h_{v_{ij}} \langle \phi_0 \rangle \))

if \( v_R \) is \textbf{ABSENT} no \( \bar{\nu}_e v_R \) mass term

\[
\nu_L \phi \text{ is not } SU(2)_L \text{ invariant } \nu_L T_3 = +\frac{1}{2} \phi \text{ doublet}
\]

\[
\nu_L \phi \text{ is not isospin triplet}
\]

All the fermions, but the neutrinos, acquire a Dirac mass

\[ h \rightarrow \text{unknown (free) parameters} \]
measuring the fermion waves and from the knowledge of $<p_0>$ we can derive the values of the Yukawa couplings — but the SM does not PREDICT them — we must add all of them to the number of the free parameters of SO

In conclusion:

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_0 + \mathcal{L}_{\text{inter}}$$

$$\mathcal{L}_0 = \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$- \frac{1}{2} \bar{\psi} \gamma^\mu W^{\mu} + \frac{m^2}{2} \bar{\psi} \gamma^\mu W^{\mu}$$

$$- \frac{1}{4} \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \bar{\psi} + \frac{1}{2} \frac{m^2}{2} \bar{\psi} \gamma^\mu \gamma^\nu \bar{\psi}$$

$$+ \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} \frac{m^2}{2} H^2$$

$$\mathcal{L}_{\text{inter}} = \mathcal{L}_{\text{fermion-gauge bosons}} + \mathcal{L}_{\text{gauge boson-gauge boson}}$$

$$+ \mathcal{L}_{H-H} + \mathcal{L}_{H-gauge bosons} + \mathcal{L}_{H-fermions}$$
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