The Effect of Straight Sections on the Stability condition for the New Type of Synchrotron.

1. Introduction

In the memo. dated 22nd Aug., 1952, Govard has in eq.(11) given the stability condition for the new machine with no straight sections. Here the equivalent condition for a machine with straight sections will be given. It is then assumed that there is the same number of straight sections as of curved sections, and that each straight section is field free and has the length L. The effect of the large straight sections necessary for injection and ejection etc. is not considered, only the effect of the straight sections between each magnet.

It will be shown that the straight sections have a rather important effect on the stability condition, and in such a way that the number of sections must be increased or n decreased to get back to the best 'working point'.

2. Stability Condition

The solutions of the equation of motion for the magnet sections are given in eq.(5)–(7) in Govard's memo. We assume that e is measured from the beginning of the corresponding section. From Floquet's theorem it is then known that $A_1$ and $B_1$ can be chosen so that

\[
\begin{pmatrix}
A_3 \\
B_3
\end{pmatrix} = e^{ju} \begin{pmatrix}
A_1 \\
B_1
\end{pmatrix}
\]

(1)

The particular solutions satisfying (1) will in general not satisfy the starting conditions, but it is also known from Floquet's theorem that if $u$ in (1) is real, not only the particular solution used in (1), but also the general solution is bounded (cfr. e.g. Whittaker and Watson, 4th Edition, pp.412/6), so that the stability condition is also in the general case that $u$ is real.

There is also a linear relation between the constants,
which may be written

\[
\begin{pmatrix}
  a_{11}, a_{12} \\
  a_{21}, a_{22}
\end{pmatrix}
\begin{pmatrix}
  A_1 \\
  B_1
\end{pmatrix}
= \begin{pmatrix}
  A_3 \\
  B_3
\end{pmatrix}
\]  

(2)

which, combined with (1) gives

\[
\begin{pmatrix}
  a_{11}, a_{12} \\
  a_{21}, a_{22}
\end{pmatrix}
\begin{pmatrix}
  A_1 \\
  B_1
\end{pmatrix}
= e^{ju} \begin{pmatrix}
  A_1 \\
  B_1
\end{pmatrix}
\]  

(3)

which has non-trivial solutions only if

\[
\begin{vmatrix}
  a_{11} - e^{ju}, & a_{12} \\
  a_{21}, & a_{22} - e^{ju}
\end{vmatrix} = 0
\]  

(4)

or

\[
e^{ju} = \frac{1}{2} (a_{11} + a_{22}) \pm j \sqrt{a_{11} a_{22} - a_{12} a_{21} - (a_{11} + a_{22})^2 / 4}
\]  

(5)

It can be shown that

\[
a_{11} a_{22} - a_{12} a_{21} = 1
\]  

(6)

and the condition for stability can therefore be written

\[-1 < (a_{11} + a_{22})/2 < 1
\]  

(7)

So far the calculations are only a repetition of Goudard's memo.

3. Evaluation of the transformation matrix.

The boundary condition for \(dx/de\) is unchanged by the straight sections, so that

\[
B_2 = (n_1/n_2)^{1/2} (-A_1 \sin \gamma_1 + B_1 \cos \gamma_1)
\]  

(8)
where
\[ \gamma = (-n)^{1/2} 2\pi/N \quad N = \text{number of sections}. \]

Similarly
\[ B_3 = (n_2/n_1)^{1/2}(-A_2 \sin \gamma_2 + B_2 \cos \gamma_2) \quad (9) \]

If the length of each straight section is L, the deviation from the stable orbit increases by the following amount over one straight section
\[ \Delta x = \left[ \frac{dx}{de} \frac{L}{r_0} \right]_{e=2\pi/N} \quad (10) \]

From this we obtain, when we also introduce the abbreviation
\[ \eta = L/r_0 \quad \quad (11) \]

\[ A_2 = A_1 (\cos \gamma_1 - \eta \sqrt{-n_1} \sin \gamma_1) + B_1 (\sin \gamma_1 + \eta \sqrt{-n_1} \cos \gamma_1) \quad (12) \]
\[ A_3 = A_2 (\cos \gamma_2 - \eta \sqrt{-n_2} \sin \gamma_2) + B_2 (\sin \gamma_2 + \eta \sqrt{-n_2} \cos \gamma_2) \quad (13) \]

From (8), (12) and (13) we find
\[ a_{11} = \cos \gamma_1 \cos \gamma_2 - \frac{1}{\eta n_2} \sin \gamma_1 \sin \gamma_2 - \eta \left[ \frac{A_2}{\sqrt{-n_2}} \cos \gamma_1 \cos \gamma_2 \right] \]
\[ + \left( \frac{B_2}{\sqrt{-n_2}} \cos \gamma_1 \sin \gamma_2 - \eta \sqrt{n_1 n_2} \sin \gamma_1 \sin \gamma_2 \right) \quad (14) \]
\[ a_{12} = \sin \gamma_1 \cos \gamma_2 + \frac{1}{\eta n_2} \cos \gamma_1 \sin \gamma_2 + \eta \left[ \frac{A_2}{\sqrt{-n_2}} \cos \gamma_1 \cos \gamma_2 \right] \]
\[ - \left( \frac{B_2}{\sqrt{-n_2}} \sin \gamma_1 \sin \gamma_2 - \eta \sqrt{n_1 n_2} \cos \gamma_1 \sin \gamma_2 \right) \quad (15) \]

and from (8), (9) and (12) we find
\[ a_{21} = -\left[ \sin \gamma_1 \cos \gamma_2 + \frac{1}{\eta n_2} \cos \gamma_1 \sin \gamma_2 - \eta \sqrt{-n_2} \sin \gamma_1 \sin \gamma_2 \right] \quad (16) \]
\[ a_{22} = \cos \gamma_1 \cos \gamma_2 - \frac{1}{\eta n_1} \sin \gamma_1 \sin \gamma_2 - \eta \sqrt{-n_2} \cos \gamma_1 \sin \gamma_2 \quad (17) \]

and we obtain
\[ \cos u = \frac{(a_{11} + a_{22})}{2} = \cos \gamma_1 \cos \gamma_2 - \frac{1}{2} \left( \frac{Y_2}{Y_1} \right) \sin \gamma_1 \sin \gamma_2 \]
\[ -\sqrt{\frac{n_1}{n}} \left[ \sin \gamma_1 \cos \gamma_2 + \frac{Y_2}{Y_1} \cos \gamma_1 \sin \gamma_2 - \frac{\sqrt{\frac{n_1}{n}}}{2} \frac{Y_2}{Y_1} \sin \gamma_1 \sin \gamma_2 \right] \]

(18)

It is in general difficult to discuss this expression. However, in practice one would most probably use identical magnet sections for positive and negative \(n\) (cfr. also CERN-PS/ER9), which will always give \(n_2 = -n_1 = n\). This most important case gives the following condition for stability

\[ -1 < F(\gamma) < 1 \]

(19)

where

\[ \gamma = \gamma_1 = -\gamma_2 = 2\pi \sqrt{n} / N \]

(20)

and

\[ F(\gamma) = \cos \gamma \cosh \gamma - \sqrt{n} \left[ \sin \gamma \cosh \gamma - \cos \gamma \sinh \gamma \right] \]
\[ + \frac{n^{1/2}}{2} \sin \gamma \sinh \gamma \]

(21)

In Fig. 1 \(F(\gamma)\) has been plotted for various values of \(\sqrt{\eta/M}\).

It is noticed, as already mentioned in the introduction, that the straight sections have important effects. As an example we take the figures given by Dahl in CERN-PS drawing No. 6. Those figures give \(\gamma = \pi/2\) and \(\sqrt{n/M} = 0.3\), and from Fig. 1 we find for this particular example \(\cos u = F(\gamma) = 0.9\), which is rather close to the limit for stability. However, if the number of sections is increased to about 290 we obtain \(F(\gamma) = 0\).

The dependance on \(L\) can be seen more clearly from the following:

As before we assume \(n_1 = -n_2\). In addition we choose \(n\) to work in the middle of the stability region, i.e., we choose \(F(\gamma) = 0\). That gives a relation between \(\sqrt{n/M}\) and \(\gamma\). We write this relation as

\[ \frac{n}{M} = G(\gamma) \]

where \(G(\gamma)\) can easily be found from (21). This relation has
been plotted in Fig. 2, and this curve gives then the relation between the three parameters L, n and N to get to the middle of the stability region.

4. Examples.

a) \( n = 3600, \ r_0 = 100 \ \text{m} \).

From Fig. 2 we obtain

\[
\begin{array}{cc}
L & N \\
m & \\
0 & 240 \\
0.25 & 268 \\
0.50 & 292 \\
0.83 & 322 \\
1.00 & 342 \\
1.67 & 415 \\
\end{array}
\]

b) \( n = 400, \ r_0 = 10 \ \text{m} \) (Suitable for a model)

\[
\begin{array}{cc}
L & N \\
m & \\
0 & 30 \\
0.25 & 108 \\
0.50 & 140 \\
0.83 & 185 \\
\end{array}
\]

c) From discussions with Dahl it seems difficult to manage with substantially smaller straight sections than \( 0.5 \ \text{m} \). Straight sections of \( 0.5 \ \text{m} \) each modify the figures given in CERN-PS drawing No.6 in the following way:

<table>
<thead>
<tr>
<th>Final energy</th>
<th>30 GeV</th>
<th>30 GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit radius (( r_0 ))</td>
<td>100 m</td>
<td>130 m</td>
</tr>
<tr>
<td>n value</td>
<td>3600</td>
<td>6400</td>
</tr>
<tr>
<td>Length of each straight section (L)</td>
<td>0.5 m</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Number of magnet sections (N)</td>
<td>292</td>
<td>410</td>
</tr>
<tr>
<td>Length of one magnet section</td>
<td>2.15 m</td>
<td>1.53 m</td>
</tr>
<tr>
<td>Mean radius</td>
<td>123 m</td>
<td>133 m</td>
</tr>
</tbody>
</table>

In the first column I have used \( n=3600 \). But as shown
by Goward, we may lose phase stability before reaching 30 GeV for this value of $n$. To reach 30 GeV we must choose $n > 5000$. In the last column I have therefore given the corresponding figures for $n = 6400$.

d) As the figure 100 GeV has been mentioned in the press in connection with this new type of synchrotron I shall also give a few figures for that machine, based upon the assumption that $B_{\text{max}}$ at the orbit is about 10000 gauss, and that we cannot possibly pass the region with no phase stability, so that we must require $n > 56,000$, which looks prohibitive large.

<table>
<thead>
<tr>
<th>Final energy</th>
<th>100 GeV</th>
<th>100 GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit radius</td>
<td>333 m</td>
<td>333 m</td>
</tr>
<tr>
<td>n value</td>
<td>62500</td>
<td>62500</td>
</tr>
<tr>
<td>Length of each straight section</td>
<td>0.5 m</td>
<td>0</td>
</tr>
<tr>
<td>Number of magnet sections</td>
<td>1270</td>
<td>1000</td>
</tr>
<tr>
<td>Length of one magnet section</td>
<td>1.65 m</td>
<td>2.10 m</td>
</tr>
<tr>
<td>Mean radius</td>
<td>434 m</td>
<td>333 m</td>
</tr>
</tbody>
</table>

The weight of the magnet will be about three times the weight of the 30 GeV magnet. A tentative figure would be 2000 tons.

The increase in the number of sections the introduction of straight sections causes need not be a draw-back. In some respects it may even be an advantage. But one has in each case to make sure that the stability region is sufficiently wide. The width needed for the stability region again depends upon what changes one has to expect in $n$ during the working period. The width of the stability region decreases with increasing $L$.

The straight sections will in general tend to increase amplitude of the oscillations even if we work in the middle of the stability region. This important problem will be treated in more detail later.

Bergen, 18th September, 1952.
Kjell Johnsen.
Copies of this report have been sent to: Bakker, Bohr, Dahl, Fry, Gentner, Coward, Johnsen, Regenstreif, Schmelzer, Wideröe.
Fig. 1: $F(\theta)$ plotted against $\theta$ with $\phi/m$ as a parameter. The stability region is $-1 < \text{Re}(\lambda) < 1$.

$\theta = \frac{2\pi}{n}$, $n = 1/\rho$. 
Fig. 2: $\eta \sqrt{m}$ plotted against $v$ for $F(v) = 0$. 