SFS with Compensating Sections to make up for Changes in the Field Index.

Abstract. Two arrangements have been worked through, one with the compensating sections between + and - sections and one with the compensating sections in the middle of the main sections. The former arrangement seems to give insufficient compensation, whereas the latter one looks promising. Variations of 10-15 pct. in the n-value can (easily be adjusted for.

Introduction.

In a strong-focusing synchrotron it seems as if we will have to stay between main resonances of the free betatron oscillations. This means that the number of betatron oscillations per revolution must during the whole accelerating period stay between two neighbouring integral numbers. One can obtain that by keeping the n-value between very narrow limits. However, these limits are so narrow that this may prove impossible. Snyder, at Brookhaven National Laboratory, has proposed to introduce extra compensating lenses to compensate for n-variations. We shall in this report investigate this possibility a little more closely, and show that there exist arrangements that can compensate for fairly large variations in the field index.

Assumptions and Definitions.

As this report is intended to give only an indication of what can be expected from compensating sections, we shall make as many simplifying assumptions as possible.

We assume that the compensating sections are four-poles with no bending action on a particle following the 'ideal' orbit. Further, it is assumed that the field is linear in the compensating sections as well as in the main sections. All main sections are of the same length and the same applies to the compensating sections. It is also assumed in these calculations that all sections are perfectly made and perfectly mounted. At last, we assume that the compensating sections completely fill the space between the main sections, so that
there are no field-free straight sections.

We measure the amplitude of all oscillations from the 'ideal' orbit and get, with sufficient accuracy, the same equation for the radial or vertical oscillations in all sections. We get

$$\frac{d^2 x}{dy^2} = - \frac{e}{p} \frac{dB_z}{dx} x$$  \hspace{1cm} (1)$$

where $y$ is measured along the 'ideal' orbit. $p$ is the momentum of the particle. It is usually more convenient to write (1) as

$$\frac{d^2 x}{de^2} = - \frac{er_o^2}{p} \frac{dB_z}{dx} x$$  \hspace{1cm} (2)$$

where $r_o$ is the radius of curvature of the ideal orbit in the main sections, and $e - y/r_o$. It is readily seen that for the main sections the factor $-(er_o^2/p)dB_z/dx$ is the ordinary n-value, as e.g. defined in CERN-PS/576. And as (2) is also valid in the four-poled compensating sections, we employ this to define a field index also in these latter sections. We write

$$n' = -(er_o^2/p)dB_z/dx$$  \hspace{1cm} (3)$$

where dashes everywhere refer to compensating sections. (It is noticed that $r_o$ comes into the field index mainly to make it dimensionless. After we have, in our project, decided upon a fixed mean radius $r_m$ instead of a fixed $r_o$, it might have been advantageous to use $r_m$ instead of $r_o$ to make the field index dimensionless. This would, for instance, have the advantage of making $e$ equal to $2\pi$ after one revolution. However, as the definition used here is for the main sections equivalent to the definition used earlier, I stick to this definition.)

Example I: Compensating Sections Between + and - Sections.

The arrangement we shall deal with is indicated in Fig.1. We use the same method as in CERN-PS/K57, and also the same notation, with the exception of $L$, which in the present report means the length of a main section.
The transformation matrix from section 1 to section 3 can be written as a product of four matrices. We find after some calculation

\[
\begin{pmatrix}
 a_{11}, & a_{12} \\
 a_{21}, & a_{22}
\end{pmatrix} =
\begin{pmatrix}
 \cos \gamma_2', & \sin \gamma_2' \\
 -\frac{n_2'}{n_2} \sin \gamma_2', & \frac{n_2'}{n_2} \cos \gamma_2'
\end{pmatrix}
\begin{pmatrix}
 \cos \gamma_2, & \sin \gamma_2 \\
 \frac{n_2}{n_2} \sin \gamma_2, & \frac{n_2}{n_2} \cos \gamma_2
\end{pmatrix}
\begin{pmatrix}
 \cos \gamma_1', & \sin \gamma_1' \\
 -\frac{n_1'}{n_1} \sin \gamma_1', & \frac{n_1'}{n_1} \cos \gamma_1'
\end{pmatrix}
\begin{pmatrix}
 \cos \gamma_1, & \sin \gamma_1 \\
 \frac{n_1}{n_1} \sin \gamma_1, & \frac{n_1}{n_1} \cos \gamma_1
\end{pmatrix}
\]

\[\text{where}\]
\[\gamma_i = \sqrt{-n_i} L_i/r_o\]

This gives a complicated formula for \(\cos u\), and this general formula is also of rather little interest. In a practical case the field index will have the same magnitude in the + and - sections, and we write, therefore at once

\[-n_1 = n_2 = n\]
\[-n_1' = n_2' = n'\]

and we also write

\[\gamma = \gamma_1 = -j \gamma_2 = \sqrt{n' L}/r_o\]
\[\gamma' = \gamma_1' = -j \gamma_2' = \sqrt{n' L'}/r_o\]

As is known, condition (6) in CERN-PS/537 is always satisfied. We are therefore only interested in finding
\((a_1 + a_2)/2\), which is equal to \(\cos u\). We insert from (6) and (7) into (4) and obtain after some calculation

\[
\cos u = \cosh \gamma \cos \gamma \cosh \gamma' \cos \gamma' \\
- \frac{L}{L'} \gamma \left( \frac{L}{L'} \right)^2 \left( \cosh \gamma \sin \gamma - \sinh \gamma \cos \gamma \right) \left( \cosh \gamma' \sin \gamma' + \sinh \gamma' \cos \gamma' \right) \\
+ \left( \frac{L}{L'} \right)^2 \left( \cosh \gamma \sin \gamma + \sinh \gamma \cos \gamma \right) \left( \cosh \gamma' \sin \gamma' - \sinh \gamma' \cos \gamma' \right) \\
+ \frac{L}{L'} \gamma \left[ \left( \frac{L}{L'} \right)^2 + \left( \frac{L}{L'} \right)^2 \right] \sinh \gamma \sin \gamma' \sin \gamma' \sin \gamma' \]
\]

(8)

If we put \(n'=0\), it is easily seen that we come back to equation (21) in CERN-PS/777, as one would expect.

The problem is now to find if it is possible to change \(n'\) so that \(\cos u\) is constant when \(n\) changes during the accelerating period. If \(\cos u\) is constant, the betatron period \(\Delta \tau\) is also constant.

We assume that the normal values of \(n\) and \(n'\) are \(n=n_0\), \(n'=0\), and when \(n\) is changed to \(n+\Delta n\), the value of \(n'\) to compensate for this can be found from the relation

\[
\left( \cos u \right)_{\gamma=\gamma_0, \gamma'=0} = \left( \cos u \right)_{\gamma=\gamma_0 + \Delta \gamma, \gamma'=\gamma'} \]
\]

This can be worked out from (8). However, even in the case \(\Delta \gamma \ll 1\), we get complicated expressions. Fortunately, it turns out that the expressions are very little dependent on \(\gamma\). Approximate expressions found for \(\gamma \ll 0\) can be shown to be valid with a very high degree of accuracy in the whole region of interest. We, therefore, here only give this approximate expression. By inserting in (9) and letting \(\gamma\) tend to zero, we obtain the following relation between \(\Delta n\) and \(n'\), when both are small

\[
\frac{\Delta n}{n_0} = - \frac{3}{2} \left( \frac{L}{L'} \right)^2 \frac{1 + (L/L')^2}{1 + (L'/L')^2} \]

(10)

First of all, it is noticed from this formula that the arrangement in Fig. 1 gives quite insufficient compensation. If we, for instance, assume that we can make \(n'\) as
large as \( n_0 \) and that the length of the compensating sections is 10 pct of the main sections, we find that this compensates for a change in \( n \) of only -1.2 pct. Secondly, this arrangement gives compensation only for decreases in \( n \), so that \( n_0 \) must be the maximum value of \( n \) during the accelerating period.

In this example the correcting lenses are placed in the most insensitive place. If, for instance, straight sections are introduced on one side of the correcting sections, say the right side, and not on the other side, we get some improvement. I may come back to that in a later report. Instead we shall here consider the other extreme, that we place the compensating sections in the middle of the main sections, which seems to be the most sensitive arrangement.

Example II: Compensating Sections in the Middle of the Main Sections.

Fig. 2 shows an arrangement with the compensating sections in the middle of the main sections.

\[
\begin{array}{cccccccc}
L/2 & L/2 & L' & L/2 & L/2 & L' & L/2 & L/2 \\
n_2 & n_1 & n'_1 & n_1 & n_2 & n'_2 & n_1 & n_1
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 3
\end{array}
\]

Fig. 2

The calculations for this case are quite similar to the calculations in Example I, and we shall therefore only give the main results for the special case

\[
\begin{align*}
-n_1 &= n_2 = n \\
n'_1 &= n'_2 = n' \\
y &= y_1 = -jy_2 = \sqrt{n^2 L/r_0} \\
y' &= y'_1 = -jy'_2 = \sqrt{n'^2 L'/r_0}
\end{align*}
\]

We then get
\[
\cos u = \cosh \gamma \cos \gamma \cosh \gamma' \cos \gamma'
+ \frac{L'}{L} \left(1 + \left(\frac{L'}{L}\right)^2\right) \left(\sinh \gamma \cos \gamma \sinh \gamma' \cos \gamma' - \cosh \gamma \sin \gamma \cosh \gamma' \sin \gamma'\right)
+ \frac{L'}{L} \left(1 - \left(\frac{L'}{L}\right)^2\right) \left(\sinh \gamma \cosh \gamma' \sin \gamma' - \sin \gamma \sinh \gamma' \cosh \gamma'\right)
- \left(\frac{L'}{L}\right)^2 \left(1 + \left(\frac{L'}{L}\right)^2\right) \sinh \gamma \sin \gamma' \frac{\sinh \gamma' \sin \gamma'}{\gamma}
+ \left(\frac{L'}{L}\right)^2 \left(1 - \left(\frac{L'}{L}\right)^2\right) \left(\cosh \gamma - \cos \gamma\right) \frac{\sinh \gamma' \sin \gamma'}{\gamma}
\]
(12)

In the special case \(\gamma'=0\) this simplifies to

\[
\cos u = \cosh \gamma \cos \gamma + \frac{L'}{L} (\sinh \gamma \cos \gamma - \cosh \gamma \sin \gamma) + \frac{L'}{L} (\sinh \gamma' - \sin \gamma)
- \left(\frac{L'}{L}\right)^2 \sinh \gamma \sin \gamma' + \left(\frac{L'}{L}\right)^2 (\cosh \gamma - \cos \gamma)
\]
(13)

Also in this example we assume that the normal values are \(n=n_0\) and \(n'=0\). As in the previous example it can be shown that the \(n'\) necessary to compensate for a change of \(\Delta n\) in \(n\) is nearly independent of \(\gamma\) within the whole region of interest to us. We therefore get the following rather simple expression after some calculation

\[
\frac{\Delta n}{n_0} = -\frac{3}{2} \frac{n'}{n_0 L} \left(1 - \frac{n''}{n_0 L} \left(1 + (2/3) \frac{L'}{L}\right)\right)
\]
(14)

First of all it is noticed that this arrangement can compensate for both positive and negative variations in \(n\). Secondly, we see that we can compensate for rather large variations using this arrangement. For instance, if we consider the case \(L'/L\approx 0.1\), \(n'=10n_0\), this takes variations of \(\Delta n/n_0\approx \pm 0.15\).

**Conclusions.**

Only two examples have been treated here. The main conclusion to be drawn from these is that it seems possible to find arrangements that are able to compensate for
quite large $n$-variations during the accelerating cycle, and it should be possible to keep the period of a betatron oscillation within the required limits up to high field-intensities.

On the other hand, which arrangement should be chosen, is still open for discussion. Here we have shown that one arrangement (Fig.1) does not give sufficient compensation, whereas the period of betatron oscillation is quite sensitive to compensating sections arranged as indicated in Fig.2.

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