Solitons and Black Holes

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Abstract

We explore the relationship between black holes in Jackiw-Teitelboim(JT) dilaton gravity and solitons in sine-Gordon field theory. Our analysis expands on the well known connection between solutions of the sine-Gordon equation and constant curvature metrics. In particular, we show that solutions to the dilaton field equations for a given metric in JT theory also solve the sine-Gordon equation linearized about the corresponding soliton. Since the dilaton generates Killing vectors of the constant curvature metric, it is interesting that it has an analogous interpretation in terms of symmetries of the soliton solution. We also show that from the Bäcklund transformations relating different soliton solutions, it is possible to construct a flat SL(2,R) connection which forms the basis for the gauge theory formulation of JT dilaton gravity.

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1 Introduction

Black holes emerge from classical gravity theory, but present paradoxes which challenge our understanding of fundamental (quantum) physics. It seems that the resolution of this will likely lead to a new and deeper understanding of such issues as the unification of interactions and the origin of elementary particles and their symmetries. Indeed, the recent progress in understanding black hole thermodynamics from the behaviour of string and membrane physics has opened new doors (‘M-theory’) for understanding difficult problems in the relation of different string theories to each other [1].

In this paper, we begin a discussion which we hope will clarify some of the issues raised in the attempt to understand higher dimensional black holes in terms of non-perturbative string theory. We will examine the simplest low dimensional black holes in terms of an underlying non-linear, but integrable, field theory. In particular, we will show that constant negative curvature two (or three) dimensional black holes can be realized as solitons of the sine-Gordon equation and provide the details of this realization. In subsequent papers we will discuss the consequences for black hole thermodynamics and for such problems as black hole collisions.

It has been known for a long time [2] that the solutions of the sine-Gordon equation
\[ -\partial_t^2 u + \partial_x^2 u = m^2 \sin u, \]  
(1)
determine Riemannian geometries with constant negative curvature \(-2m^2\) whose metric is given by the line-element
\[ ds^2 = \sin^2 \left( \frac{u}{2} \right) dt^2 + \cos^2 \left( \frac{u}{2} \right) dx^2. \]  
(2)
The angle \(u\) describes the embedding of the manifold into a three dimensional Euclidean space. [2]

It is also easy to show [3] that Lorentzian geometries with line-element
\[ ds^2 = -\sin^2 \left( \frac{u}{2} \right) dt^2 + \cos^2 \left( \frac{u}{2} \right) dx^2, \]  
(3)
have constant negative curvature if and only if
\[ \Delta u := \partial_t^2 u + \partial_x^2 u = m^2 \sin u. \]  
(4)

The sine-Gordon equation is a well-studied system in mathematical physics [2]. In particular, it is known that it is an integrable equation, admitting
solitonic solutions. It has a rich dynamical structure, both classically and quantum mechanically.

Recently, there has been a great deal of interest in constant curvature black holes in two and three spacetime dimensions. The BTZ black hole is obtained from anti-deSitter space in 2+1 dimensions by doing suitable identifications [5] that give the spacetime the global structure of a black hole with a single bifurcative horizon. By imposing axial symmetry on 2+1 gravity, one obtains Jackiw-Teitelboim (JT) dilaton gravity [4]: a theory of gravity in 1+1 dimensions in which a scalar field (the dilaton) is non-minimally coupled to the spacetime metric. The dilaton is essentially the circumference of the circle through a given point in the direction of axial symmetry. The black hole solutions in JT gravity are therefore dimensionally reduced BTZ black holes. In the following we will explore the relationship between the dynamics of JT dilaton/gravity and sine-Gordon field theory.

The relationship between constant curvature metrics and sine-Gordon solitons has been known for some time. Moreover, our analysis is very similar in spirit to the original one by Jackiw in Ref. [4]. In that work, the connection between Liouville theory and constant curvature metrics in conformal gauge was used to motivate the study of what is now called Jackiw-Teitelboim gravity. However, besides the obvious difference of dealing with sine-Gordon theory as opposed to Liouville theory, the present paper goes beyond previous work in two important respects, motivated by the current context in which JT gravity is considered. In particular, given the importance of black hole solutions and their relationship to the BTZ black hole [5], the dilaton field takes on new physical importance, beyond being simply a Lagrange multiplier. We will show that the dilaton field has a natural counterpart in the sine-Gordon theory: it is a zero-mode of the corresponding linearized sine-Gordon equation. Secondly, we will reveal a deep connection between Bäcklund transformations and the gauge theory formulation of JT gravity. It therefore appears that there is a duality between JT black holes and sine-Gordon solitons that might be used to shed light on the field theory origin of black holes and the dynamical source of black hole entropy.

The paper is organized as follows. In Section 2 we review Jackiw-Teitelboim gravity, with emphasis on the nature of the black hole solutions and the form of the gauge theory formulation of the theory [6]. In Section 3 we outline the properties of Euclidean sine-Gordon solitons, while the relationship between the one soliton solution and JT black holes is derived in Section 4. Section 5 shows how the gauge theory form of JT gravity arises naturally from a con-
sideration of the Bäcklund transformations of sine-Gordon solitons. Finally, Section 6 closes with conclusions and prospects for future work.

2 Jackiw-Teitelboim Dilaton/Gravity

The simplest theory of two dimensional gravitation was first discussed in 1984 by Jackiw and Teitelboim [4]. The equation of motion was $R(g) = -2m^2$, i.e. that the spacetime $M_2$ had a Lorentzian metric $g_{\mu\nu}$ with constant negative (Ricci scalar $R(g)$) curvature $-2m^2$. In order to derive this from an action principle, one needs to introduce an additional spacetime scalar field $\tau$, with action functional

$$I_{JT}[\tau, g] = \frac{1}{2G} \int_{M_2} d^2x \sqrt{-g} \left( R + 2m^2 \right),$$

(5)

where $G$ is the gravitational coupling constant, which in two dimensional spacetime is dimensionless. Sufficient conditions that this functional be stationary under arbitrary variations of the dilaton and metric fields are, respectively

$$R + 2m^2 = 0;$$

$$\left( \nabla_\mu \nabla_\nu - m^2 g_{\mu\nu} \right) \tau = 0.$$  

(6)

(7)

Since the constant curvature metrics are maximally symmetric, there are three Killing vector fields. This follows rather directly from the dilaton equations of motion [7] in that the three functionally independent solutions $\tau_{(i)}$, $i = 0, 1, 2$ of Eq.(7) determine three functionally independent vector fields $k^\mu_{(i)}$ via

$$k^\mu_{(i)} = \frac{\epsilon^{\mu\nu}}{m\sqrt{-g}} \partial_\nu \tau_{(i)},$$

(8)

where $\epsilon^{\mu\nu}$ is the permutation symbol. These three vector fields satisfy the Killing equations by virtue of Eq.(7). Associated with each solution and corresponding Killing vector there is a conserved charge for the system, namely [7]:

$$M_{(i)} = -\frac{1}{m^2} \left| \nabla \tau_{(i)} \right|^2 + \tau_{(i)}^2,$$

(9)

When the Killing vector is timelike, it can be shown that this corresponds to the ADM energy of the solution.
Although all the solutions of Jackiw-Teitelboim gravity are locally diffeomorphic to two-dimensional anti-DeSitter spacetime, one may obtain distinct global solutions, some of which display many of the attributes of black holes [7, 8]. For example, consider a solution where the metric is given by the line-element

\[ ds^2 = -\left( m^2 r^2 - M \right) dt^2 + \left( m^2 r^2 - M \right)^{-1} dr^2, \]

where \( M \) is a constant and the dilaton is

\[ \tau = mr. \]

This metric is a dimensionally truncated three dimensional BTZ black hole [5]. Clearly there is an event horizon located at \( r = \sqrt{M/m} \). It is important to note here that though this fact can be easily read off from the metric, since the latter is in manifestly static form, it also follows from solving for the variable \( r \) in the equation

\[ |k^\mu|^2 := g_{\mu\nu}k^\mu k^\nu = 0, \]

where \( k^\mu \) is the Killing vector field determined by the dilaton \( \tau \) above via Eq.(8). It is straightforward to show that the ADM energy of the black hole solution Eq.(10) and Eq.(11) is

\[ E_{BH} = mM/2G, \]

We note for later reference that there is a conical singularity located in the BTZ black hole at \( r = 0 \). In the context of JT gravity there is no conical singularity, but the vanishing of the dilaton field gives rise to an infinite effective Newton’s constant. Surfaces for which \( \tau = 0 \) should therefore be excluded from the manifold.

It was mentioned above that for any given metric, there exist three linearly independent solutions to the dilaton field equations. For the metric given in Eq.(10) the simplest solution is the standard one (Eq.(11)). The other two solutions for this metric are:

\[ \tau_{(2)} = \sqrt{m^2 r^2 - M} \sinh \sigma \text{ + constant} \]
\[ \tau_{(3)} = \sqrt{m^2 r^2 - M} \cosh \sigma \text{ + constant} \]

where \( \sigma \equiv m\sqrt{Mt} \). The corresponding Killing vectors are:

\[ \vec{k}_{(2)} = \left( \frac{mr}{\sqrt{m^2 r^2 - M}} \sinh \sigma, -\sqrt{M}\sqrt{m^2 r^2 - M} \cosh \sigma \right) \]
\[ \vec{k}_{(3)} = \left( \frac{mr}{\sqrt{m^2 r^2 - M}} \cosh \sigma, -\sqrt{M}\sqrt{m^2 r^2 - M} \sinh \sigma \right) \]
A straightforward calculation shows that the conserved charge Eq.(9) is in fact the same for all values of the dilaton:

\[ M_{(1)} = M_{(2)} = M_{(3)} = M \]  

(17)

providing that the constants in Eq.(13) and Eq.(14) are set to zero.

There is also a gauge theory version of Jackiw-Teitelboim gravity [6]. The action functional is

\[ I_{BF}[\phi, A] = \frac{1}{2G} \int_{M_2} Tr(\phi F(A)), \]  

(18)

where \( \phi \) is a spacetime scalar with values in the Lie algebra \( sl(2, \mathbb{R}) \), \( A \) is an SL(2, \( \mathbb{R} \)) connection on a principal bundle over spacetime \( M_2 \), whose curvature is denoted by \( F(A) := dA + \frac{1}{2}[A, A] \). The trace is over the adjoint representation of SL(2, \( \mathbb{R} \)).

The stationary configurations of \( I_{BF} \) are

\[ F(A) = 0, \]  

(19)

\[ D_A \phi := d\phi + [A, \phi] = 0. \]  

(20)

It is well-known [6] that if we identify two of the components, say \( \frac{1}{2} A^a, a = 0, 1 \) with the spacetime frame-field \( e^a \), and the remaining component, \( A^2 \) with the spin-connection \( \omega \), then the zero-curvature condition for \( A \) implies that the connection \( \omega \) is torsion-free,

\[ D_\omega e^a := de^a - e^b \omega \wedge e^b = 0, \]  

(21)

and has constant negative curvature

\[ d\omega - m^2 \epsilon_{ab} e^a \wedge e^b = 0. \]  

(22)

Note that our convention here is that \( A := A^i T_i \) where the \( T_i \) are the generators of SL(2, \( \mathbb{R} \)) obeying

\[ [T_i, T_j] = \epsilon_{ijk} \eta^{kl} T_l, \]  

(23)

where \( \eta^{ij} \) is the Minkowski metric with signature +2 and \( \eta^{00} = -1 \) and \( \epsilon_{ijk} \) is the permutation symbol.

If we define the metric tensor

\[ g_{\mu\nu} := \eta_{ab} e^a_\mu e^b_\nu, \]  

(24)
then the Ricci scalar curvature satisfies \( R(g) = -2m^2 \). Hence every solution of the geometrodynamic version is a solution of the gauge theory version, but it is easy to see that there are solutions of the gauge theory, (e.g. \( A = 0 \)) which do not immediately yield non-degenerate Lorentzian geometries of constant negative curvature.

We note here that the information about the dilaton and Killing vector fields is encoded in the covariant constancy of the Lie-algebra valued scalar field \( \phi = \phi^i T_i \). Up to a constant multiple, one may identify the dilaton with \( \phi^2 \), and the frame-field components of the corresponding Killing vector field with the \( \phi^a \).

### 3 The Sine-Gordon Equation and Constant Curvature Geometry

We have seen that when the Lorentzian metric is parametrized as in Eq.(3), it has constant negative curvature \(-2m^2\) if the function \( u(t, x) \) satisfies the Euclidean sine-Gordon equation Eq.(4). In the following we summarize the properties of this non-linear partial differential equation, specializing to the slightly less well-studied case of Euclidean signature [2].

The only obvious solution of the sine-Gordon equation is the trivial solution \( u_n(t, x) = 2\pi n \). These solutions are important because, though locally trivial, they can be used, in conjunction with Bäcklund transformations, to generate an infinite family of non-trivial soliton solutions. The Bäcklund transformations are a one parameter family of first order non-linear partial differential equations in two real-valued functions \( u, u' \), whose integrability conditions are precisely the sine-Gordon equations for \( u \) and \( u' \). We write the Bäcklund transformations in the form [9, 2]:

\[
\Gamma, z &= \frac{1}{2} \left( 1 + \Gamma^2 \right) u, z + \frac{1}{2} mk \Gamma, \\
\Gamma, \bar{z} &= \frac{m}{2k} \left[ \cos \Gamma - \frac{1}{2} \sin u \left( 1 - \Gamma^2 \right) \right].
\]

(25)

(26)

In the above, \( \Gamma \) is defined by

\[
\Gamma := \tan \left( \frac{u + u'}{4} \right);
\]

(27)
the quantity \( z := x + it \), with \( \bar{z} \) its complex conjugate. Finally, \( k \) is a complex parameter, in general. One can verify that the integrability condition \( u_{xt} = u_{tx} \) implies that \( u' \) satisfy Eq.\((4)\), and similarly exchanging \( u \) with \( u' \).

Starting with the trivial seed solution \( u = 0 \), it follows from Eq.\((26)\) that
\[
    u' = 4 \tan^{-1} \exp \left\{ \pm m \gamma [x - x_0 - v(t - t_0)] \right\},
\]
with \( \gamma := (1 + v^2)^{-\frac{1}{2}} \), is also a solution of Eq.\((4)\). In the above \( t_0, x_0 \) are integration constants. In order to obtain a real solution \( u' \), the parameter \( k \) must have modulus one, i.e. \( k = e^{i\alpha} \), and we have \( \cos \alpha = \gamma, \sin \alpha = v\gamma \).

The solution with the + sign in the exponent is the 1-soliton solution; the opposite sign is the anti-soliton solution. Upon ‘Wick rotation’ to the Lorentzian signature, (and in this case \( v \rightarrow iv \)), one sees that the soliton(anti-soliton) propagates through space with constant velocity \( v (-v) \). Hence we may think of the soliton as being located at \( x = vt \) at time \( t \).

The Böcklund transformation may be used to generate multi-soliton solutions. For example, if we use the 1-soliton solution for \( u \), then the Eq.\((26)\) yields the 2-soliton solution for \( u' \). This iteratively defines an infinite family of independent solutions of the sine-Gordon equation. In the sense to be described below, these solutions are topologically distinct.

The following vector field
\[
    j^\mu := \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu u,
\]
(on \( E_2 \)) is trivially conserved, i.e. \( \partial_\mu j^\mu = 0 \) identically for any smooth function \( u \). The soliton number of \( u \) is defined to be
\[
    Q[u] := \int_S ds n^\mu j^\mu,
\]
where \( S \) is a surface in \( E_2 \) such that \( n^\mu \), its normal, is non-spacelike. It is easy to see that \( Q[u] = \pm 1 \) if \( u \) is respectively, the 1-soliton (anti-soliton). An \( n \)-soliton \( u \) has \( Q[u] = n \).

Besides the \( n \)-solitons, there are topologically trivial (\( Q[u] = 0 \)), but locally non-trivial, solutions. We will not discuss these here, except in the following general sense. Obviously a given solution \( u \) of the sine-Gordon equation can not be constructed as a linear superposition of some basis of solutions. In fact, the best one can do is the ‘inverse scattering picture’. Here one looks at a solution \( u(t, x) \) as a series of snapshots taken at various times \( t \). The data at \( t = -\infty \) (or at \( +\infty \)) at least partially determines the structure, including its topological structure, of a given \( u \).
4 The 1-Soliton as a Black Hole

In this Section, we indicate the connection between the one soliton in sine-Gordon theory, and JT black holes. When the metric is parametrized as in Eq.(3), the action Eq.(5) takes the form:

\[ I_{JT}[\tau, u] = \frac{1}{2G} \int_{M^2} d^2x \tau (\nabla u - m^2 \sin u), \]  

(31)

where the \( \nabla \) denotes the flat space Euclidean Laplacian. The field equations that result from a variation of this “reduced” action are the sine-Gordon equation Eq.(4) for the field \( u \) and the linearized sine-Gordon equation for the dilaton:

\[ (\Delta - m^2 \cos u) \tau = 0. \]  

(32)

Note that Eq.(32) can also be derived directly from the equations of motion Eq.(7) for the unreduced theory. It is interesting that the dilaton fields which generate the Killing vectors of the constant curvature metric in JT gravity also correspond to symmetries of the sine-Gordon model. The dilaton field infinitesmally maps solutions of the sine-Gordon equation on to other solutions. That is, the field \( u' = u + \epsilon \tau \), where \( u \) and \( \tau \) obey Eq.(4) and Eq.(32), also solves Eq.(4) to first order in \( \epsilon \). However, it should be noted that there are more solutions to the linearized sine-Gordon equation than there are to the dilaton equations.

We shall now demonstrate that the 1-soliton solution Eq.(28) of the sine-Gordon equation determines a metric in a coordinate patch on \( M^2 \) in which there is a Killing vector field which is timelike in the asymptotic region and becomes null at an interior point of the patch. In other words, it determines a black hole metric. Indeed, when Eq.(28) is used in the Lorentzian metric Eq.(3), the latter simplifies to:

\[ ds_{1-sol}^2 = -\text{sech}^2 \rho dt^2 + \tanh^2 \rho dx^2, \]  

(33)

where

\[ \rho := m \gamma (x - vt), \]  

(34)

and we have chosen for simplicity \( x_0 = t_0 = 0 \). We perform successive coordinate transformations: first from \((t, x) \rightarrow (T, \rho)\), with \( \rho \) as defined above and with

\[ dT = dt - v \frac{\tanh^2 \rho}{m \gamma (\text{sech}^2 \rho - v^2 \tanh^2 \rho)} d\rho. \]  

(35)
Next we transform from \((T, \rho) \to (T, r)\) with
\[
r := \frac{1}{m\gamma} \text{sech}\rho.
\]
Both of these transformations have non-zero Jacobian for all \((t, x)\) in \(R^2\). The result is the metric with line-element:
\[
d_{bh}^2 = - \left( m^2 r^2 - v^2 \right) dT^2 + \left( m^2 r^2 - v^2 \right)^{-1} dr^2.
\]
This is the metric of a Jackiw-Teitelboim black hole with total energy \(mv^2/2G\) and event horizon at \(r = v/m\).

We see that the metric Eq.(33) is Kruskal-like in that there is no coordinate singularity at the horizon, but is singular (in fact degenerate) at \(\rho = 0\), which is the location of the soliton, and at \(r = 0\), i.e. where \(\rho \to \infty\).

5 From Sine-Gordon to Geometry

So far we have seen that if the coordinates of spacetime are fixed so that the metric is of the form Eq.(3), then \(u\) must satisfy the sine-Gordon equation and the dilaton \(\tau\) must satisfy the linearized sine-Gordon equation. So we have ‘derived’ sine-Gordon from carefully dressed Jackiw-Teitelboim gravity theory. We now address the question of reversing this and deriving gravity from the structure of sine-Gordon theory. In fact, the work over twenty years ago by Ablowitz et. al. [9] and others [2] provides a partial answer to this.

Consider the covariant constancy condition
\[
D_A w := dw + Aw = 0,
\]
where \(A\) is an SL(2,R) connection of the form \(A = A_i T_i\) with
\[
A^0 = \frac{m}{2} \left( kdz + k^{-1} \cos ud\bar{z} \right),
\]
\[
A^1 = \frac{m}{2k} \sin ud\bar{z},
\]
\[
A^2 = u_z dz,
\]
and
\[
w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},
\]
Then if we perform the Ricatti transformation

$$\Gamma = w_2/w_1,$$  \hspace{1cm} (41)

it follows from Eq.(38) that the $u, u'$, with $\Gamma = \tan[(u + u')/4]$ are related by a Bäcklund transformation Eq.(26). It is also straightforward to check that the connection $A$ is flat, i.e., that $F(A) = 0$.

There is a natural metric on $R^2$ induced by the flat connection $A$. This comes from the Casimir invariant metric $\eta_{ij} := 2 Tr (T_i T_j)$, where the generators $T_i$ of SL(2,R) are defined so that $\eta_{00} = \eta_{11} = -\eta_{22} = 1$. To get a real Lorentzian metric on $R^2$, we must choose the parameter $k$ in the connection to have modulus one, i.e. $k = e^{i \alpha/2}$. The complex frame-field is

$$\{E^0, E^1\} := \frac{1}{m} \{A^0, A^1\},$$  \hspace{1cm} (42)

and the spin-connection is

$$\Omega := A^2 = \frac{i}{2} (u, dt - u, dx) + \frac{1}{2} du.$$  \hspace{1cm} (43)

The line-element is real:

$$ds^2 := \left( E^0 \right)^2 + \left( E^1 \right)^2$$

$$= -\frac{1}{2}(\cos a - \cos u) dt^2 + \frac{1}{2}(\cos a + \cos u) dx^2 - \sin a \ dt \ dx,$$  \hspace{1cm} (44)

and has constant negative curvature given by $Im(\Omega)$ if $u$ obeys the sine-Gordon equation.

6 Conclusion and Speculations

We have seen that the sine-Gordon theory may be used to provide an alternative classical description of the simplest two dimensional gravity theory. The latter necessarily contains, besides the metric tensor, a scalar field (the dilaton), and we have seen that such a field also emerges naturally from the sine-Gordon theory. Finally, we have shown that the black hole solutions of JT dilaton/gravity originate from the 1-soliton solutions of sine-Gordon theory.
At this level, what remains for us to understand is, first, the relation between the integrability of sine-Gordon theory (which has the concommitant of the existence of an infinite number of conserved quantities) and the zero-modes of the linearized sine-Gordon equation. Second, we would like to examine the role of multi-solitons and other non-trivial solutions of the sine-Gordon equation in JT dilaton/gravity.

Our ultimate goal is to understand the relation between quantum JT dilaton/gravity and sine-Gordon theory. In particular, we speculate that the microstates of the black hole responsible for the black hole entropy originate in the contributions to the path integral from topologically trivial ‘components’ in the one-soliton sector, e.g. the so-called breather solutions. Thus, the black hole entropy may be given by the number of ways of preparing quantum sine-Gordon states with fixed total energy and soliton number $Q = 1$.

Work along these lines is in progress.

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References


