a correction to $R_0$ has an influence on the first-order change in the bound-state energy, $\Delta E_0$, which was seen to depend on $R_0^2$.

It should be pointed out, in conclusion, that the approximations used here in solving the second Born approximation are based on the fact that in the region of electron energy considered, the nucleons remain nonrelativistic. For incoming electrons with energies of the order of the nucleon rest mass energy, or more, this method will fail.

Anticommutator for a Nonlinear Field Theory

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The anticommutator for the Thirring model is computed by ordering the operator $\psi(x)\psi^*(x')$ and evaluating its renormalized vacuum expectation value. The infrared divergence is defined by introducing an $ad$ hoc cutoff. The final expression does not agree with the approximations obtained by using perturbation theory or by using expansion over intermediate states (with the same cutoff). It is also found that Heisenberg's procedures cannot be applied to this two-dimensional problem.

I

In recent years, there has been much discussion concerning the form of the anticommutator for bare-particle spinor operators which satisfy nonlinear equations of motion such as

$$i\gamma^\mu \partial_\mu \psi + 2g \{\bar{\psi} \gamma^\mu \psi\} \psi = 0.$$  

(1)

It has been suggested that

$$S_{ab'}(x_1,x_2) = i \langle 0 \mid \{\psi_a(x_1), \bar{\psi}_{b'}(x_2)\} \mid 0 \rangle$$

resembles a classical solution of Eq. (1) modified by the addition of a mass term near the light cone, and that (in four dimensions) it is "effectively" more regular in this region than $S_{ab}(x_1,x_2)$, the corresponding free-field function. The solvable two-dimensional Thirring model allows one to check on the first of these speculations for a nonlinear theory, but since $g$ is dimensionless in Thirring's case, $S'_f(x) = L^{-\frac{1}{2}}$ in Heisenberg's, the actual forms of $S'(x)$ in the two problems cannot be directly compared.

At first sight it would seem that the calculation is trivial for the two-dimensional example. Since no lengths are present, it has been pointed out that the most general form for Lehmann's spectral function is $\rho(m) = a(g)b(m) + b(g)/m$, giving $S'(x) = S(x)[a(g) + b(g)]$, $I = \int_F d\xi$ $dx/x(1-x)$. Perturbation theory and expansion over intermediate states also yield this expression, which does not resemble a $c$-number solution of Eq. (1). However, the infrared divergence in $I$ leads to an ill-defined space-time dependence for $S'$. If one tries to specify the divergent term precisely by introducing a cutoff, $k_{min} = K$, the dimensional argument fails and singular contributions to $\rho(m)$ such as $m^{-\frac{1}{2}}\sin(K/m)$, etc., cannot be excluded, even as $K \rightarrow 0$.

In order to resolve any ambiguity, it is desirable to write the operator $\psi(x)\psi^*(x')$ as an ordered functional of the free-field operators $\phi_r, \phi_r^*$, as in Glaser's treatment for $\psi_r$. This is done in Secs. II, III. In Sec. IV, the renormalized vacuum expectation value is computed and a "covariant" infrared cutoff is introduced. The result is

$$S_{12}(x_1,x_2) \equiv S_{12}(v_1-v_2) \exp[\{gg'/(2\pi)^2\} \ln(L/|v_1-v_2|)]$$

$$\int_{v_1,v_2} \exp \{\frac{1}{2\pi} \int F d\xi \}$$

(2)

where $v = x - t, u = x + t, g' = g + 2\pi n$, so that $|g'/2\pi| < 1$, and $L$ is a constant (with dimension of length) which transforms as $L' = \gamma L(1 - \beta)$ under a Lorentz transformation.

The functional dependence of Eq. (2) does not agree with the predictions of perturbation theory (there is, in fact, an essential singularity at $u/L = 0$) or of the intermediate state expansion. Furthermore, it is shown in the last section that Heisenberg's techniques cannot be applied to this two-dimensional example.

II

In Thirring's two-component representation $(\gamma(\xi) = \eta_1, \gamma(\xi) = \eta_2)$, Eq. (1) becomes

$$\frac{\partial \psi_1}{\partial u} = ig_1 \eta_2 \psi_2, \quad \frac{\partial \psi_2}{\partial u} = -ig_1 \eta_2 \psi_2,$$

(3)

and the general c-number solutions are
\[ \psi_1 = \phi_1(v) U_1(u, -\infty) = \phi_1(v) \exp \left( ig \int_{-\infty}^{u} \rho_1(u') du' \right), \]
\[ \psi_2 = \phi_2(u) U_2(v, \infty) = \phi_2(u) \exp \left( -ig \int_{-\infty}^{v} \rho_2(v') dv' \right), \]
where \( \rho_1(v) = \rho^*_2 \phi_1 \phi_1^* \), \( \rho_2(u) = \rho^*_1 \phi_2 \phi_2^* \). Glaser has quantized the problem by treating \( \phi \) as an incoming field with free-field commutation relations
\[ \{ C_r(p), C_{r'}^*(p') \} = \delta_{rr'} \delta(p-p'), \quad \{ C_r(p), C_{r'}(p') \} = 0. \]

The physical vacuum is defined by
\[ C_{r'}^{(+)} |0\rangle = C_{r'}^{(-)} |0\rangle = 0, \quad C_{r'}^{(+)} |0\rangle = C_{r'}^{(-)} |0\rangle = 0. \]

Then, since \( \{ \rho_1(x), \rho_{r'}(x') \} = 0 \) for all \( x, x' \), it is possible to treat the \( \rho_r \) as c-numbers so that Eqs. (4), (5) remain valid as operator equations relating \( \psi \) to the incoming fields (it is convenient, however, to replace \( \rho \) by \( \rho_1 \) in these equations; this eliminates a physically meaningless infinite phase factor).

Because of the Hermitian nature of \( \rho_1 \), it follows that
\[ \psi_1(x_1) \psi_2^*(x_2) = \phi_1(x_1) \phi_2^*(x_2) U_1(u_1, u_2) \]
and
\[ U_1(u_1, u_2) = U_1 = \exp \left( ig \int_{-\infty}^{u_2} \phi_2^*(u') \phi_2(u') du' \right). \]

In order to evaluate matrix elements of this operator, we would like to have \( U \) in the form suggested by Glaser’s work:
\[ U_1(u_1, u_2) = \exp \left( ig \int_{-\infty}^{u_2} \phi_2^*(u') \phi_2(u') du' \right) U. \]

H, assume that
\[ [U_1, \phi_2(x)] = G; \phi_2(x) U_1 = \int dy \times S(x, y) \phi_2(x') \left[ \phi_2^{(+)}(y) U_1 + \phi_2^{(-)}(y) \right] \]
has been evaluated. Then, by considering \( \left( 0 | \{ C_{r'}^{(+)}(p), \right. \left. C_{r'}^{(-)}(p') \} | 0 \right) \) and inserting in turn Eq. (8) for \( U_1 \) and Eq. (11) for the commutator, it can be verified that \( H(x, y) = -G(x, y) \). Thus, the calculation of \( U_1 \) in the ordered exponential form is reduced to evaluation of \( G(x, y) \). We shall order \( \phi_1 \phi_2^* \) below; the corresponding result for \( \phi_2 \phi_1^* \) can be obtained from this by letting \( x \rightarrow -x \).

III

Glaser has shown that
\[ \phi_2(x) U_1(u, -\infty) = U_1(u, -\infty) \phi_2(x) \exp ig \theta(u-x). \]
This equation and the relationship
\[ U_1(u_1, u_2) = U_1^{-1}(u_2, -\infty) U_1(u_1, -\infty) \]
lead to
\[ U \phi = \phi U - (\varepsilon^{q-1}) \psi \phi(\psi^{-1}) \]
and
\[ \phi U = \psi U - (\varepsilon^{q-1}) \psi^{-(\psi^{-1})} \]
where the indices on \( \phi, U \) are suppressed. The positive and negative-frequency parts of \( \phi \) are defined by \( \phi^{(+)} + \phi^{(-)} = \phi \),
\[ \phi^{(\pm)}(x) = P^{(\pm)} \phi(x) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dy}{y - x + i\epsilon}. \]

When \( P^{(\pm)} \) is applied to (12) and \( P^{(\pm)} \) to (13), they become
\[ [U_1, \phi^{(\pm)}] = K^{(\pm)} \phi U = \int_{u_1}^{u_2} dy \times [\phi^{(\pm)}(y) U^{(\pm)}(y) U^{(\pm)}(x, y)] K^{(\pm)}(x, y), \]
\[ \phi^{(\pm)} U = K^{(\pm)} U \phi = \int_{u_1}^{u_2} dy \]
\[ \times [U^{(\pm)}(y) + U^{(\pm)}(y)] K^{(\pm)}(x, y), \]

8 In the original calculation, \( U_1 \) was ordered by transforming Eq. (11) to momentum space. This led to an integral equation,
\[ f(p) + \lambda \int d^3 k \sin(p-k) f(k) \sin(p-k) = g(p), \]
which was solved by letting \( \sin x/x \rightarrow e^{im|x|} \sin x/x \) and using the Wiener-Hopf technique for the modified equation. The method of solution is nonrigorous and the solution and consequent integrations have an extremely complicated form which could not be simplified appreciably. I am greatly indebted to Dr. V. Glaser for suggesting the present rigorous and relatively simple configuration-space approach and especially for pointing out that the latter method naturally leads to a reduction of the number of integrations in Eq. (10) [see Eq. (28), below]. Subsequent comparison shows that the two calculations give equivalent expressions for \( H(x, y) \) but different values for \( U_1 \) (unrenormalized).
with
\[ \mathfrak{K}(x,y) = \frac{(1 - e^{i\xi y})}{2\pi i} \frac{1}{y - x + i\epsilon}, \quad u_3 > u_1. \] (17)

In symbolic notation, Eqs. (15) and (16) are
\[ [1 + K^{(+)}] U \phi^{(+)} = \phi^{(+)} U - K^{(+)} U \phi^{(-)}, \]
\[ [1 + K^{(-)}] \phi^{(-)} U = U \phi^{(-)} - K^{(-)} \phi^{(+)} U. \]

Inverse operators, defined by
\[ [1 + G^{(+)})] [1 + K^{(+)}] = 1, \]
\[ [1 + G^{(-)})] K^{(-)} = -G^{(-)}, \]
so that
\[ [U, \phi^{(+)}] = G^{(+)} \phi^{(+)} U; \quad \text{and} \quad [\phi^{(-)} U] = G^{(-)} \phi^{(-)} U. \]
Thus \[ [U, \phi^{(+)}] = G^{(+)} \phi^{(+)} U; \] with \( G = G^{(+)} - G^{(-)} \). The inverse operators are constructed in the Appendix by solving the integral equations \( [1 + K^{(+)}] f = g \) in the form \( f = [1 + G^{(+)}] g \). The results are
\[ G^{(+)} f(x) = \int_{u_1}^{u_2} dy \mathcal{G}^{(+)}(x,y) f(y) = \frac{1 - e^{i\xi y}}{2\pi i} \int_{u_1}^{u_2} dy \]
\[ \times \left( \frac{x - u_1}{x_0 - u_1} \right)^{\xi/2y} \left( \frac{u_2 - y}{x_0 - u_1} \right)^{-\xi/2y} \frac{f(y)}{x_0 - x_2} \right) \]
\[ \mathcal{G}(x,y) = (1 - e^{-i\lambda})[\delta(x-y) + \mathcal{G}^{(+)}(x,y)], \]
\[ u_1 \leq x, y \leq u_2, \]
so that Eq. (10) can be written as
\[ 1 \partial (U_1) = \frac{1 - e^{-i\lambda}}{(2\pi i)^2} \int_{u_1}^{u_2} dy \int_{u_1}^{u_2} dx \int_{u_1}^{u_2} \mathcal{G}^{(+)}(x,y) \mathcal{G}(x,y) \delta(x-y) \]
\[ \times \int_{u_1}^{u_2} dy \delta(x-y) + \mathcal{G}^{(+)}(x,y) \]
\[ \times \frac{1}{(u_0 - u_1)(u_0 - u_2)} \mathcal{G}(x,y) \mathcal{G}(x,y) \]
\[ \times \mathcal{G}^{(+)}(x,y). \]

As before, \( [1 + G^{(-)}] K^{(-)} = -G^{(-)} \) or
\[ \int_{u_1}^{u_2} du' \int_{u_1}^{u_2} dx \mathcal{G}^{(-)}(x,u') \mathcal{G}^{(-)}(u',x) f(x) \]
\[ = \int_{u_1}^{u_2} du' [\mathcal{G}^{(-)}(x,u') + \mathcal{G}^{(-)}(u',x)] f(x), \]
and furthermore,
\[ \mathcal{G}^{(+)}(x,u') = [e^{i\lambda} \mathcal{G}^{(-)}(x,u') + (e^{i\lambda} - 1) \delta(x-u')]. \]

When \( \mathcal{G}^{(+)} \) in Eq. (21) is written in terms of \( \mathcal{G}^{(-)} \), and \( \mathcal{G}^{(-)} \) is reduced using Eq. (22) \( \lim f(x) = \delta(x-z) \), an indeterminate form is obtained for the last integrand. Using L'Hôpital's rule, this gives
\[ \frac{1}{i} \frac{\partial g}{\partial g} = \frac{2}{2(\pi i)^2} \int_{u_1}^{u_2} du' \left[ \frac{1}{u_0 - u_1} + \frac{1}{u_0 - u_2} \right] \mathcal{G}(x,y) \]
\[ \exp \left[ -\frac{g}{2(\pi i)^2} \int_{u_1}^{u_2} du' \left[ \frac{1}{u_0 - u_1} + \frac{1}{u_0 - u_2} \right] \right]. \]

The final expression \( \mathcal{G}(x,y) \) has \( (U_0)_{\infty} = 1 \) for \( u_1 = u_2; (U_0)_{\infty} = 0 \), otherwise. The result is not meaningful because it is a matrix element of unrenormalized operators. Although Glaser has calculated \( Z^1 \) in momentum space, it is necessary to repeat the calculation in configuration space (the two computations give different values for the renormalization constant). In the latter case, \( Z^0 = \langle U_1^{-1}(u_0, -\infty) \rangle \) may be obtained by setting \( u_1 = -\infty \) in Eqs. (18), (20):
\[ Z^1 = \exp \left[ -\frac{g}{2(\pi i)^2} \int_{u_1}^{u_2} du' \left[ \frac{1}{u_0 - u_1} + \frac{1}{u_0 - u_2} \right] \right]. \]

A corresponding substitution gives \( \langle U_1(u_0, -\infty) \rangle _\infty \). Note that \( \lim u_1 = -\infty \Rightarrow \langle U_1(u_1, u_2) \rangle _\infty \) is not equal to \( \langle U_1(u, u) \rangle \). The renormalized matrix element, defined as \( \langle U \rangle_R = Z^1 Z^{-\frac{1}{2}} \mathcal{G}(x,y) \) has the form
\[ \langle U \rangle_R = \exp \left[ \frac{g}{2(\pi i)^2} \int_{u_1}^{u_2} du' \left[ \frac{1}{u_0 - u_1} + \frac{1}{u_0 - u_2} \right] \right]. \]

The divergent quantity \( \langle U \rangle_R \) is independent of \( u_1, u_2 \). In this sense, the results of perturbation theory,

\[ \text{[Ref. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 29, 12 (1955).]} \]
expansion over intermediate states, and arguments based on Lehmann's representation are reproduced; \( S^{(+)} \) is simply \( S^{(+)} \) multiplied by an infinite constant. However, inspection of Eq. (26) shows that the divergence is entirely of the infrared type. There is no singular contribution for \( u' \sim u_1, u_2 \) and it should be possible to introduce a covariant cutoff into the definition of \( Z^1 \).

Since an infrared divergence is associated with massless virtual particles, one can try to account for a finite mass, \( m \), in a phenomenological manner. We observe that the integral in Eq. (25) can be interpreted as a sum of contributions from all space-time points \( u' = (x' + t') \sim u_2 \) along the past light cone, weighted by the propagator, \( \langle u_2 | u' \rangle^{-1} \). For massive particles, it would then be expected that propagation over distances \( \Delta x \gg \lambda_{\text{Compton}} \) or \( \Delta t \gg m^{-1} \) is attenuated. Furthermore, virtual particles would not exist for times \( \Delta t \gg E^{-1} \), or \( \Delta t \gg m^{-1} \). However, \( Z^1 \) contains contributions only from points on the light cone and all trajectories have the same proper distance and proper time; an exact calculation with \( m = 0 \) at all stages is required to eliminate the infrared divergence in this manner.

Another, more successful, \textit{ad hoc} cutoff is introduced by considering the system placed in a box of length \( L \). Then the maximum contribution to the integral comes from the space-time point \( (-L/2, -L/2) \), or \( u_{\text{max}} = x_{\text{max}} + t_{\text{max}} = -L \). \[ \text{Note: } u_{\text{max}} \text{ transforms like } x \times t \text{ so that } u_{\text{max}}' = \gamma u_{\text{max}} (1 - \beta) \text{ under a Lorentz transformation.} \]

Of course, the exact solution with box normalizations should be used in the construction, but if the box is sufficiently large, the cutoff can be inserted directly into Eq. (26). Negligible error results from

\[ \langle U_1(u_1,u_2) \rangle^R = \exp \left( \frac{ig^2}{2 \pi^2} \int \frac{dL}{|u_2 - u_1|} \right), \]

\[ u_1, u_2 \ll L. \]  

(27)

The \( L \) dependence of Eq. (27) differs strongly from the logarithmic divergence predicted by perturbation theory with the same cutoff. In fact, \( \langle U_1 \rangle^R \) can be expanded in a power series in \( g \) only if \( (g^2/8 \pi^2) \times [\ln (1 - u_2 - u_1)] < 1 \) and the radius of convergence tends to zero as \( L \to \infty \).

Furthermore, \( \langle U_1(u_2,u_2) \rangle^R = \infty \) whether or not an infrared cutoff is present [this follows from \( \langle \psi^R \psi^R \rangle = Z^1 Z^2 \delta(x - x', t = t') \)]. However,

\[ \langle U_1(u_2,u_2) \rangle^R = \sum_n \sum_p \langle 0 | \psi^R(u_2) | nt \rangle \langle nt | \psi^R(u_2) | 0 \rangle \]

\( (n \) is the number of particles and \( p \) refers to their momenta), and with an infrared cutoff, each term in the series over \( n \) is finite. Thus, the expansion over intermediate states must diverge.

Therefore, to the extent that this cutoff is meaningful, one can conclude that both expansions yield poor approximations to the exact function, \( S' \). These results also indicate that the spectral density \( \rho(m) \) for a system in a box may be considerably different from that predicted on dimensional grounds when no length is present.

V

Heisenberg considers the quantity

\[ \chi(x,x') = \{ \exp[iR(x')] \} \psi_a(x) \{ \exp[-iR(x')] \}, \]

(28)

where \( R(x') = [\alpha^a x^a (x') + \sigma^a \psi^* (x')] \), \( \psi \) is a solution of Eq. (1), and \( \sigma^a \) is constant spinor with

\[ \{ \sigma^a, \psi \} = \{ a^a, \psi \} = 0. \]

Clearly, \( \chi \) satisfies Eq. (1) as a function of \( x \), but not as a function of \( x' \). A further property of \( \chi \) is

\[ \lim_{a^a \to 0} \partial \chi_a / \partial a^a = i \{ \psi^*_a(x'), \psi_a(x) \}. \]

(29)

In the four-dimensional case, Heisenberg assumes \( \chi = \chi^2 + c \) where \( c \) is a singular c-number function and \( \chi^2 \) is an operator regular near the relative light cone. It is then argued that \( (\hat{\chi} x) \psi \to \delta(x) \psi \) near the light cone. Setting \( \chi \ll c \) in this region, Heisenberg examines classical solutions of

\[ i \gamma^a \partial_a + 2 \gamma \partial \psi \psi c + \psi \psi c, \]

(30)

for \( \chi \ll c \) and takes

\[ \lim_{a^a \to 0} \partial \chi_a / \partial a^a = i \{ \psi^*_a(x'), \psi_a(x) \} | 0 \]  

(31)

This procedure is not valid in the two-dimensional case. First, consider Eq. (1): the most general solution for \( \chi \) consistent with \( [\chi_a(x)]^2 = 0 \) or \( [\chi_a(x,x')]^2 = 0 \) is

\[ \chi_1(x,x') = \chi_1^\in(z)(x,x') \]

\[ \times \exp \left( \int_{x_0}^{x} \chi_1^\in(y,x') \chi_1^\in(y,x') dy \right), \]

\[ \chi_2(x,x') = \chi_2^\in(u,x') \]

\[ \times \exp \left( - \int_{y_0}^{y} \chi_2^\in(y,x') \chi_1^\in(y,x') dy \right), \]

(32)

where the operator \( \chi^\in \) is defined by setting \( \psi_a(x) \to \psi_a^\in(x) \) in Eq. (28). These expressions cannot be decomposed into sums of regular operators plus singular c-numbers: one can, of course, set \( \chi = \hat{\chi} + (x - c) \) but then \( \chi^\in = (x - c) \) is also singular.\(^{10} \)

Secondly, although one component of Eq. (30) is

\[ \partial \hat{c}_a / \partial u - i [\hat{c}_a, c_1 - c_1^* c_2] c_2 + ic_2 c_2 \to 0, \]

(33)

the exact function satisfies

\[ \partial S_1 / \partial u - i (g^2/2 \pi^3) 1 / |u' - u| S_1 \]

(34)

\[ \to 0. \]

\(^{10} \) A more significant observation is that \( \{ \psi_a(x), \psi^*_a(x') \} = \{ \psi_a(x), \psi^*_a(x') \} \}

\( U_1(u,u') \) is a singular operator. Thus, the anti-commutator itself cannot be written as a singular c-number plus a bounded operator.
The anticommutator does indeed obey a differential equation which differs from \( \gamma^a \partial_a S = 0 \) but it is a singular, linear equation with no mass term [as classical equations, (1) and (33) contain terms of the form \( \phi \phi^* \phi \phi^* \), but since \( \phi \bar{\phi} = 0 \) for a spinor operator such a term does not appear in the operator equations]. It may also be noted that Eq. (33) would seem to predict oscillatory behavior for \( S' \), however, as \( a' \to 0 \), \( c_a \to 0 \) \([X_a \to \psi_a \) which is not singular] and Eqs. (31), (33) may be combined to give
\[
\partial S_{x'x} / \partial u + ic S_{x'x} = 0.
\]
(35)
Since \( S_{x'x} = 0 \), it can be seen that Heisenberg's method is not applicable to the Thirring model.

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APPENDIX

The equations \([1 + K^{(+)}]f = g\) may be solved using the methods of reference 7. For instance, if one lets \( h(x) = f(x) - g(x) \), the \( K^{(+)} \) equation is
\[
h(x) + \frac{(e^{-iu} - 1)}{2\pi i} \int \frac{dy}{y - x} [h(y) + g(y)] = 0,
\]
(A.1)
which implies that \( h(x) \) is the limiting value of a func-

† Note added in proof.—We have recently determined the exact solutions for the Thirring model with \( g = g(x,f) \). No infrared divergence appears if \( g \to 0 \) as \( u, r \to \infty \). A variation \( g = g_0 (L + u) \bar{g}(L + u) \) gives the same results as those obtained with the ad hoc cutoff of Sec. IV, and all of the conclusions contained here are rigorously valid for this modified field theory.

tion \( h(x) \), analytic in the entire complex plane, except for a branch cut along the real axis between \( u_1 \) and \( u_2 \). Then,
\[
h(x_+) - h(x_-) = -(e^{-iu} - 1)[h(x) + g(x)],
\]
(A.2)
\[
= 0, \quad \text{otherwise.}
\]

Let \( h(x) = (x - u_1)^{-\nu' \gamma} (u_2 - x)^{-\nu' \gamma} S(x) \), where \( g' = g + 2\pi i \), and define the factors to be real for \( x = |x| + i \epsilon \), \( u_1 < |x| < u_2 \). If \( |g'/2\pi| \) is always chosen to be less than unity, there is a single-valued continuation from \( x \) to \( x_- \) and
\[
e^{-iu} h(x_+) - h(x_-)
\]
\[
e^{-iu} (x_+ - u_1)^{-\nu' \gamma} (u_2 - x_+) S(x_+) - S(x_-),
\]
(A.3)

Equations (A.2) and (A.3) yield
\[
S(x_+) - S(x_-) = -(1 - e^{iu}) (x_+ - u_1)^{-\nu' \gamma}
\]
\[
\times (u_2 - x_+)^{-\nu' \gamma} g(x),
\]
(A.4)
which finally leads to
\[
f(x) = g(x) + \int \frac{dy}{u_1} \xi^{(+)}(x,y) g(y)
\]
\[
= g(x) + \frac{(1 - e^{iu})}{2\pi i} \int \frac{dy}{u_1} \xi^{(+)}
\]
\[
\frac{(y_+ - u_1)^{-\nu' \gamma}}{(x_+ - u_1)} \frac{(u_2 - y_+)^{-\nu' \gamma}}{(u_2 - x_+)} g(y)
\]
(A.5)

The equation corresponding to (A.2) for the \( K^{(-)} \) operator is just the complex conjugate of (A.2) for real valued \( h, g \). Thus, \( \xi^{(-)} \) is defined by taking the complex conjugate of (A.5), assuming \( f, g \) real. The factor \( (x_+ - u_1)^{-\nu' \gamma} (u_2 - x_+)^{-\nu' \gamma} \) is chosen to be real when \( x \) is real and between \( u_1 \) and \( u_2 \).